SOME NOTES ON RECENT RESULTS ON LANDAU–SIEGEL ZEROS

Contents

1. Introduction 1
2. Spaces of polynomials 4
3. Evaluation matrices 5
4. Divisibility 5
5. Local estimates 6
6. The global inequality 7
7. Proof of the main estimate 8
8. Variables and derivatives 10
Appendix A. Prime sums 14
Appendix B. Interpolation 15
Appendix C. Points of small height 16
Appendix D. Comparison with Bombieri–Stepanov 18
Appendix E. Explicit estimates 20
Appendix F. Excluding real zeros 21
References 23

1. Introduction

A recent paper of OpenAI [12] established the following result, known as the *exclusion of Landau–Siegel zeros*:

Theorem 1. *There is an effective absolute constant $c > 0$ such that, for every primitive real Dirichlet character χ of conductor q, we have*

$$L(s,\chi) \neq 0 \qquad \text{for real } s > 1 - \frac{c}{\log(2q)}.$$

Theorem 1 has significant consequences in analytic number theory, including for the distribution of primes in arithmetic progressions and the sizes of class groups of quadratic fields. The book by Davenport [6] is a standard reference for that context. The result was immediately superseded by another result of OpenAI [11] showing that $L(s,\chi) \neq 0$ for $\Re s > 7/8$ for any Dirichlet character χ, but the method of proof of Theorem 1 is of independent interest.

The proof uses the following standard criterion, relating zeros to prime sums:

Lemma 2. *Suppose there are effective constants $0 < \kappa < 1$ and $B \geq 0$ such that for every $x \geq 2$ and every primitive real Dirichlet character χ of conductor q, we have*

$$\sum_{\substack{p \leq x \\ \chi(p) = -1}} \frac{\log p}{p} \leq \kappa \log x + B \log(2q). \tag{1}$$

Then the conclusion of Theorem 1 holds.

Proof. See Appendix F. □

Mertens's theorem gives $\sum_{p\le x}\frac{\log p}{p}\le \log x+O(1)$, so (1) for $\kappa<1$ represents a saving over the trivial bound, uniformly for $\log q=o(\log x)$.

These notes were written to help me understand the method of proof of Theorem 1. They are based on notes of Alexandru Pascadi [13] and a blog post by Tao [16], which was in turn based on a note by Green [7]. There are many related notes, including those of Banks [2, 1], Barreto [3], van Dijk [18], Guo [8], Kerr [9] and Kominers [10]. We used Codex (GPT-6.1 Sol on "Low" reasoning effort) extensively in both preparing and revising these notes (particularly for the estimates of §E).

The new method for establishing (1) may be understood as a way to control the distribution of Frobenius elements. Let F/F_0 be a finite Galois extension of number fields (we lose little by taking $F_0=\mathbb{Q}$). We write

$$d_0:=[F_0:\mathbb{Q}],\qquad d:=[F:F_0],\qquad \Delta:=|\operatorname{disc}F|. \tag{2}$$

We denote by p a prime of F_0, i.e., a nonzero prime ideal of the ring of integers $\mathcal{O}_{F_0}$. We denote by $|p|:=|\mathcal{O}_{F_0}/p|$ its absolute norm.

Suppose p is unramified in F. For each prime $\mathfrak{p}$ of F above p, we write $\mathrm{Frob}_{\mathfrak{p}}\in \mathrm{Gal}(F/F_0)$ for the arithmetic Frobenius element, characterized by

$$\mathrm{Frob}_{\mathfrak{p}}(u)\equiv u^{|p|}\pmod{\mathfrak{p}} \tag{3}$$

for all $u\in\mathcal{O}_F$. We write $\mathrm{Frob}_p\subseteq\mathrm{Gal}(F/F_0)$ for the Frobenius conjugacy class, which consists of the $\mathrm{Frob}_{\mathfrak{p}}$ for all $\mathfrak{p}$ above p. Throughout these notes, sums or products involving $\mathrm{Frob}_{\mathfrak{p}}$ or Frob_p are understood to be restricted to primes p that are unramified in F.

Example 3. Take $F_0:=\mathbb{Q}$, so that $d_0=1$, p is a rational prime and $|p|=p$. Let $D\neq 1$ be a fundamental discriminant. Let χ_D denote the corresponding primitive real Dirichlet character (the Kronecker symbol). Every such character except the principal character arises in this way. Take for $F:=\mathbb{Q}(\sqrt{D})$ the corresponding quadratic extension, thus $d=2$. Write $\mathrm{Gal}(F/F_0)=\{1,\sigma\}$, and take for $C:=\{\sigma\}$ the nontrivial class. Then p is unramified in F precisely when $p\nmid D$, and

$$\mathrm{Frob}_p=\sigma \iff \chi_D(p)=-1. \tag{4}$$

In view of Example 3, the sums appearing in (1) are controlled by the following more general result:

Theorem 4. *Let $C\subseteq\mathrm{Gal}(F/F_0)$ be a conjugacy class. For every real $x\ge 2$,*

$$\sum_{\substack{|p|\le x\\ \mathrm{Frob}_p=C}}\frac{\log|p|}{|p|}\le\frac{|C|}{d}\left(\log x+\frac{1}{2}\log\Delta\right)+d_0\log\log(3x)+50d_0|C|. \tag{5}$$

Theorem 4, specialized as in Example 3, implies the hypotheses of Lemma 2 for any fixed $1/2<\kappa<1$, such as $\kappa=3/4$, with an absolute constant B. Theorem 1 follows. The coefficient $|C|/d$ is the density occurring in the Chebotarev density theorem (see Stevenhagen–Lenstra [15] for background, and Thorner–Zaman [17, §1] for a discussion of effective versions).

The idea of the proof of Theorem 4 is as follows. Let $\mathfrak{p}\mid p$ be as in the definition (3), and suppose that $\mathrm{Frob}_{\mathfrak{p}}=\sigma\in C$. Then for each $u\in\mathcal{O}_F$, we have $\sigma(u)\equiv u^{|p|}$

(mod $\mathfrak{p}$); in other words, the pair $(u, \sigma(u)) \in \mathcal{O}_F^2$ lies in the vanishing set modulo $\mathfrak{p}$ of the polynomial $X^{|p|} - Y$. This vanishing forces many relations, modulo $\mathfrak{p}$ and its powers, between the values taken by polynomials at the pairs $(u, \sigma(u))$. For example, for $f \in \mathcal{O}_F[X, Y]$, we have the implication

$$(X^{|p|} - Y)^k \mid f \implies f(u, \sigma(u)) \in \mathfrak{p}^k. \tag{6}$$

To control these relations as $\mathfrak{p}$ varies, it is natural (at least in retrospect) to consider the determinants δ obtained by specializing f and u to finite sets. To fix ideas, take a finite subset $\{u_i\} \subset \mathcal{O}_F$ and a finite collection of monomials $\{X^{a_j} Y^{b_j}\}$, where $1 \le i, j \le r$, and consider

$$\delta := \det\big(u_i^{a_j} \sigma(u_i)^{b_j}\big)_{i,j} \in \mathcal{O}_F. \tag{7}$$

If $\delta \neq 0$, then it satisfies the product formula

$$\sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(\delta) \log|\mathfrak{p}| = \sum_{v|\infty} \log|\delta|_v. \tag{8}$$

The integrality of δ makes each term on the left-hand side nonnegative, so by dropping some terms, we obtain the inequality

$$\sum_{\substack{\mathfrak{p}: \\ \operatorname{Frob}_{\mathfrak{p}} = \sigma}} \operatorname{ord}_{\mathfrak{p}}(\delta) \log|\mathfrak{p}| \le \sum_{v|\infty} \log|\delta|_v. \tag{9}$$

We might expect relations like (6) to force $\operatorname{ord}_{\mathfrak{p}}(\delta)$ to be large whenever $\operatorname{Frob}_{\mathfrak{p}} = \sigma$. After estimating the archimedean norms of δ, we might then hope to use (9) to give upper bounds for the set $\{\mathfrak{p} : \operatorname{Frob}_{\mathfrak{p}} = \sigma\}$.

Without derivatives, this strategy gives no improvement over the trivial bound in the quadratic case $d = 2$, as we explain in §8. In that case, a nontrivial bound was obtained in [12] using the following idea. Take a biquadratic extension F'/F_0 containing F, and let $\sigma_1, \sigma_2 \in \operatorname{Gal}(F'/F_0)$ denote the two lifts of the nontrivial element $\sigma \in \operatorname{Gal}(F/F_0)$. If $\operatorname{Frob}_{\mathfrak{p}} = \sigma_1$, then $u^{|p|} \equiv \sigma_1(u) \pmod{\mathfrak{p}}$; if $\operatorname{Frob}_{\mathfrak{p}} = \sigma_2$, then $u^{|p|} \equiv \sigma_2(u) \pmod{\mathfrak{p}}$. Thus the triples $(u, \sigma_1(u), \sigma_2(u))$ allow one determinant

$$\det\big(u_i^{a_j} \sigma_1(u_i)^{b_j} \sigma_2(u_i)^{c_j}\big)_{i,j}, \tag{10}$$

with $u_i \in \mathcal{O}_{F'}$, to detect both families of primes. The product formula over F' turns out to give a nontrivial bound. We explain the numerics in Example 25. One can do better numerically by passing to further extensions and introducing additional variables (Example 26).

A refined approach, observed in [7], is to consider the determinants defined using not just point evaluations, but also derivatives in the first variable:

$$\det\big(\partial_X^k X^{a_j} Y^{b_j} \mid_{X = u_i,\, Y = \sigma(u_i)}\big)_{(i,k),j}. \tag{11}$$

Derivatives preserve the divisibility in (6) (Lemma 8), while supplying more evaluation conditions at the same points. This method readily yields the sharp bounds noted in Theorem 4; we use it to prove the theorem in §7, then discuss the various approaches in §8.

2. Spaces of polynomials

We begin by constructing spaces of polynomials in which "many elements are divisible by large powers of $X^q - Y$". We do so by taking spans of sets of monomials closed under the substitution $X^q \mapsto Y$.

Lemma 5. *Let $q \geq 1$ be a natural number. Let $\mathcal{T}$ be a finite subset of $\mathbb{Z}^2_{\geq 0}$. Assume that $\mathcal{T}$ is closed under $(a,b) \mapsto (a-q, b+1)$ whenever $a \geq q$. For $(a,b) \in \mathcal{T}$, define*

$$f_{a,b} := X^{a-qk} Y^b (X^q - Y)^k, \qquad k := \left\lfloor \frac{a}{q} \right\rfloor.$$

These polynomials define a basis for

$$\bigoplus_{(a,b)\in\mathcal{T}} \mathbb{Z} X^a Y^b \subset \mathbb{Z}[X,Y].$$

Proof. By the binomial expansion,

$$f_{a,b} - X^a Y^b = \sum_{j=1}^{k} \binom{k}{j} (-1)^j X^{a-qj} Y^{b+j}.$$

By our closure assumption concerning $\mathcal{T}$, each monomial on the right-hand side has its exponents in $\mathcal{T}$. Moreover, each such monomial is strictly less than $X^a Y^b$ in the lexicographic ordering (first by a, then by b). The $f_{a,b}$ are thus related to the standard basis $X^a Y^b$ via an upper-triangular substitution with ones on the diagonal, hence form a basis. □

Example 6. For $H \geq 1$ and $t \geq 2$, set

$$\mathcal{T} := \{(a,b) : a + Hb < t\}.$$

This selects monomials of weighted degree less than t, where X has weight 1 and Y has weight H. The closure assumption of Lemma 5 is satisfied if $H \leq q$ (in other words, if $X^q \mapsto Y$ does not increase the weight).

Lemma 5 generalizes to more variables, with the same proof:

Lemma 7. *Let $q \geq 1$ and $r \geq 1$ be natural numbers, and let $\ell \in \{1, \dots, r\}$. Let $\mathcal{T}$ be a finite subset of $\mathbb{Z}^{r+1}_{\geq 0}$ that is closed under $a = (a_0, a_1, \dots, a_r) \mapsto a - qe_0 + e_\ell$ whenever $a_0 \geq q$. For $a \in \mathcal{T}$, define*

$$f_a := X_0^{a_0 - qk} X_1^{a_1} \cdots X_r^{a_r} (X_0^q - X_\ell)^k, \qquad k := \left\lfloor \frac{a_0}{q} \right\rfloor.$$

These polynomials define a basis for

$$\bigoplus_{a\in\mathcal{T}} \mathbb{Z} X_0^{a_0} X_1^{a_1} \cdots X_r^{a_r} \subset \mathbb{Z}[X_0, \dots, X_r].$$

Given weights $H_1, \dots, H_r \geq 1$, Example 6 generalizes to

$$\mathcal{T} := \{(a_0, a_1, \dots, a_r) : a_0 + H_1 a_1 + \cdots + H_r a_r < t\}.$$

The substitution $X_0^q \mapsto X_\ell$ preserves this set whenever $q \geq H_\ell$.

3. Evaluation matrices

For the rest of these notes, we assume given the following data:

- A commutative ring R. In practice, it will be a ring of integers, a local field or a global field.
- An integer $r \geq 1$. We eventually specialize to $r = 1$.
- A finite, nonempty subset $\mathcal{T}$ of $\mathbb{Z}_{\geq 0}^{r+1}$.
- An integer $I \geq 1$ and, for each $1 \leq i \leq I$, an element $z_i = (z_{i,0}, \dots, z_{i,r}) \in R^{r+1}$.
- An integer $J \geq 1$.

We impose further conditions as needed.

Given the above data, we introduce some notation:

- $T := |\mathcal{T}|$.
- For each exponent tuple $a \in \mathbb{Z}_{\geq 0}^{r+1}$, we define the corresponding monomial $X^a := X_0^{a_0} \cdots X_r^{a_r}$ and its total degree $|a| := a_0 + \cdots + a_r$.
- We denote by
$$\tau : \bigoplus_{a \in \mathcal{T}} RX^a \to R^{I \times J}$$
the map assigning to each polynomial f with exponents in $\mathcal{T}$ its values and derivatives at the given elements z_i:
$$\tau(f) := \big((\partial_{X_0}^j f)(z_i)\big)_{i,j},$$
where $1 \leq i \leq I$ and $0 \leq j < J$.
- M denotes the matrix of τ with respect to the standard bases:
$$M_{(i,j),a} := (\partial_{X_0}^j X^a)(z_i). \tag{12}$$

When $r = 1$, we also write X, Y for X_0, X_1.

4. Divisibility

Suppose that R has an ideal $\mathfrak{p}$ containing the image of an integer $q \geq 1$. For example, take $R = \mathcal{O}_F$, $q = |p|$, and $\mathfrak{p}$ a prime above p.

If $u, v \in R$ satisfy $v \equiv u^q \pmod{\mathfrak{p}}$, then
$$(X^q - Y)^k \mid f \implies f(u, v) \in \mathfrak{p}^k.$$

A useful observation [7, §5] is that this remains true after taking any number of derivatives with respect to X:

Lemma 8. Let $f \in R[X, Y]$ and $u, v \in R$. Assume that f is divisible by $(X^q - Y)^k$ and that $v \equiv u^q \pmod{\mathfrak{p}}$. Then for every $j \geq 0$,
$$(\partial_X^j f)(u, v) \in \mathfrak{p}^k.$$

Proof. Set $g := X^q - Y$. We write $f = g^k h$ and apply the product rule to the k factors of g and the factor h. By hypothesis, $g(u, v) \in \mathfrak{p}$. On the other hand, every derivative $g', g'', \dots$ of g belongs to $qR[X, Y] \subseteq \mathfrak{p}R[X, Y]$. Thus, after evaluating at (u, v), each of the k factors contributes either $g(u, v) \in \mathfrak{p}$ or a positive derivative of g, also in $\mathfrak{p}$. Thus every term lies in $\mathfrak{p}^k$. □

We now apply this observation to the evaluation matrix of §3.

Lemma 9. *Fix $\ell \in \{1, \dots, r\}$. Suppose that $\mathcal{T}$ is closed under $a \mapsto a - qe_0 + e_\ell$ whenever $a_0 \geq q$, and that $z_{i,\ell} \equiv z_{i,0}^q \pmod{\mathfrak{p}}$ for every i. Then every $T \times T$ minor of M belongs to*

$$\mathfrak{p}^{\sum_{a \in \mathcal{T}} \lfloor a_0/q \rfloor}.$$

Proof. The ideal generated by the $T \times T$ minors is unchanged by an invertible change of basis. We may therefore use the basis f_a of Lemma 7. Since f_a is divisible by $(X_0^q - X_\ell)^{\lfloor a_0/q \rfloor}$, Lemma 8 gives

$$(\partial_{X_0}^j f_a)(z_i) \in \mathfrak{p}^{\lfloor a_0/q \rfloor}$$

for every i, j. Thus the ath column in the new basis is divisible by $\mathfrak{p}^{\lfloor a_0/q \rfloor}$. Each term in a $T \times T$ determinant has one factor from each column, giving the claim. □

5. Local estimates

We take $R = F$, where F is a local field equipped with its normalized absolute value $|\cdot|$. In the following estimates, we denote by δ any $T \times T$ minor of M and by

$$\beta := \max_{\substack{1 \leq i \leq I \\ 0 \leq \ell \leq r}} \{1, |z_{i,\ell}|\}$$

the "local height" of the given set of points z_i.

We first record some crude bounds:

Lemma 10. *If F is nonarchimedean, then*

$$|\delta| \leq \beta^{\sum_{a \in \mathcal{T}} |a|}.$$

Proof. Each entry in the ath column of M is an integer times a monomial of total degree $|a| - j$ for some $j \geq 0$. Since integers have absolute value at most one and $\beta \geq 1$, each such entry has absolute value at most $\beta^{|a|}$. The claim follows by expanding the determinant and applying the ultrametric inequality. □

Lemma 11. *Suppose that F is archimedean, and set $a_* := \max(\{1\} \cup \{a_0 : a \in \mathcal{T}\})$. Then*

$$|\delta| \leq \beta^{\sum_{a \in \mathcal{T}} |a|} \left(T^T a_*^{T(J-1)} \right)^{[F:\mathbb{R}]}.$$

Proof. We write $|\cdot|_{\mathrm{E}}$ for the Euclidean absolute value, so that $|\cdot| = |\cdot|_{\mathrm{E}}^{[F:\mathbb{R}]}$. The determinant is a sum of $T! \leq T^T$ products, each involving one entry from each column. Each entry in the ath column of M has magnitude

$$|(\partial_{X_0}^j X^a)(z_i)|_{\mathrm{E}} \leq a_*^j \beta^{|a|/[F:\mathbb{R}]}$$

for some $j \leq J - 1$. The claim follows. □

We may improve the crude nonarchimedean bound using the Frobenius congruences (Lemma 9):

Lemma 12. *Suppose that F/F_0 is a finite unramified Galois extension of nonarchimedean local fields. Write p and $\mathfrak{p}$ for their maximal ideals, and σ for the arithmetic Frobenius. Fix $\ell \in \{1, \dots, r\}$. Assume that $z_{i,\ell} = \sigma(z_{i,0})$ for all i, and that $\mathcal{T}$ is closed under $a \mapsto a - |p|e_0 + e_\ell$ whenever $a_0 \geq |p|$. Then*

$$|\delta| \leq \beta^{\sum_{a \in \mathcal{T}} |a|} |\mathfrak{p}|^{-\sum_{a \in \mathcal{T}} \lfloor a_0/|p| \rfloor}.$$

Proof. Suppose first that all coordinates are integral. Then $z_{i,\ell} \equiv z_{i,0}^{|p|} \pmod{\mathfrak{p}}$, and Lemma 9 gives the claim with $\beta = 1$.

In general, since F/F_0 is unramified, we may choose $\lambda \in F_0$ with $|\lambda| = \beta$. The scaled points z_i/λ are then integral, and since $\sigma(\lambda) = \lambda$, we still have $z_{i,\ell}/\lambda = \sigma(z_{i,0}/\lambda)$. If $\widetilde{\delta}$ is the minor defined like δ, but using the scaled points, then

$$\delta = \lambda^{\sum_{a\in\mathcal{T}} |a| - \sum_{(i,j)\in\mathcal{M}} j} \widetilde{\delta},$$

where $\mathcal{M}$ is the set of rows used to define the minors. Since $|\lambda| = \beta \geq 1$, the bound for the scaled points implies the required bound for the original points. □

6. The global inequality

We take $R = F$, where F is a global field. We denote by v a place of F and by $|\cdot|_v$ the corresponding normalized absolute value. We define the global height of the set of points z_i to be

$$\beta := \prod_v \beta_v, \qquad \beta_v := \max_{\substack{1\leq i\leq I\\ 0\leq \ell\leq r}} \{1, |z_{i,\ell}|_v\}. \tag{13}$$

We will use the following identity to pass from primes of F to primes of F_0.

Lemma 13. *Let F/F_0 be a finite Galois extension of global fields of degree d, and let C_σ be the conjugacy class of $\sigma \in \mathrm{Gal}(F/F_0)$. If $\mathrm{Frob}_p = C_\sigma$, then*

$$\sum_{\substack{\mathfrak{p}\mid p\\ \mathrm{Frob}_{\mathfrak{p}}=\sigma}} \log|\mathfrak{p}| = \frac{d}{|C_\sigma|}\log|p|. \tag{14}$$

Proof. We write f_p for the residue degree and g_p for the number of primes above p, so that $f_p g_p = d$. The Galois group acts transitively on these primes, and the map $\mathfrak{p} \mapsto \mathrm{Frob}_{\mathfrak{p}}$ is equivariant for conjugation. Thus its fibers have equal size $g_p/|C_\sigma|$. Each prime above p satisfies $\log|\mathfrak{p}| = f_p \log|p|$, giving the claim. □

We now combine the local estimates using the product formula.

Proposition 14. *Let F/F_0 be a finite Galois extension of global fields, of degree d. Let $\sigma_1, \dots, \sigma_r$ be distinct elements of $\mathrm{Gal}(F/F_0)$, with conjugacy classes $C_1, \dots, C_r$. Take*

$$z_i := (u_i, \sigma_1(u_i), \dots, \sigma_r(u_i))$$

for some $u_1, \dots, u_I \in F$. Let $J \geq 1$ be an integer. Retain the notation $\mathcal{T} \subset \mathbb{Z}_{\geq 0}^{r+1}$, $T = |\mathcal{T}|$ and $\tau : \bigoplus_{a\in\mathcal{T}} RX^a \to R^{I\times J}$ from §3. Assume that τ is injective. Let $t \geq 2$ be such that $a_0 < t$ for all $a \in \mathcal{T}$. For each $\ell \in \{1, \dots, r\}$, let $H_\ell \geq 1$. Assume that for all integers $q \geq H_\ell$, the set $\mathcal{T}$ is closed under $a \mapsto a - qe_0 + e_\ell$ whenever $a_0 \geq q$. Set

$$A_\infty := \begin{cases} [F_0 : \mathbb{Q}]\big(T\log T + T(J-1)\log t\big), & \text{for number fields,}\\ 0, & \text{for function fields.}\end{cases} \tag{15}$$

Then

$$\sum_{\ell=1}^{r} \frac{1}{|C_\ell|} \sum_{\substack{|p|\geq H_\ell\\ \mathrm{Frob}_p = C_\ell}} \left(\sum_{a\in\mathcal{T}} \left\lfloor \frac{a_0}{|p|} \right\rfloor\right) \log|p| \leq \frac{\log\beta}{d} \sum_{a\in\mathcal{T}} |a| + A_\infty. \tag{16}$$

Proof. Since τ is injective, we may choose a nonzero $T \times T$ minor δ. At every finite place $\mathfrak{p}$, Lemma 10 gives the bound $\beta_{\mathfrak{p}}^{\sum_a |a|}$. If $\mathrm{Frob}_{\mathfrak{p}} = \sigma_\ell$ and $|p| \geq H_\ell$, then Lemma 12, applied to $F_{\mathfrak{p}}/F_{0,p}$, improves this bound by the factor

$$|\mathfrak{p}|^{-\sum_{a\in\mathcal{T}} \lfloor a_0/|p| \rfloor}.$$

Since the σ_ℓ are distinct, each prime $\mathfrak{p}$ occurs for at most one ℓ. At the archimedean places, Lemma 11 and the inequality $a_* \leq t$ give an additional factor of at most $\exp(dA_\infty)$, using here that $\sum_{v|\infty}[F_v : \mathbb{R}] = d[F_0 : \mathbb{Q}]$ in the number field case; in the function field case, there are no archimedean places. The product formula then gives

$$1 = \prod_v |\delta|_v \leq \beta^{\sum_{a\in\mathcal{T}} |a|} \exp(dA_\infty) \prod_{\ell=1}^{r} \prod_{|p|\geq H_\ell} \prod_{\substack{\mathfrak{p}|p \\ \mathrm{Frob}_{\mathfrak{p}}=\sigma_\ell}} |\mathfrak{p}|^{-\sum_{a\in\mathcal{T}} \lfloor a_0/|p| \rfloor}.$$

Taking logarithms, applying Lemma 13, and dividing by d gives the claim. □

Corollary 15. *Retain the extension F/F_0, the distinct elements $\sigma_1, \dots, \sigma_r$, and the points z_i of Proposition 14. Assume that F is a number field. Let $2 \leq H \leq t$, and take*

$$\mathcal{T} := \{a : a_0 + H(a_1 + \cdots + a_r) < t\}.$$

For each conjugacy class C, set

$$w(C) := \frac{\#\{\ell : \sigma_\ell \in C\}}{|C|}, \qquad \|w\|_\infty := \max_C w(C).$$

Define

$$\begin{aligned} \mathcal{B} := {} & \frac{\sum_{a\in\mathcal{T}} |a|}{d \sum_{a\in\mathcal{T}} a_0} \log\beta + \frac{A_\infty}{\sum_{a\in\mathcal{T}} a_0} \\ & + \|w\|_\infty \left(\frac{T}{\sum_{a\in\mathcal{T}} a_0} \sum_{|p|\leq t} \log|p| + \sum_{|p|<H} \frac{\log|p|}{|p|} \right). \end{aligned} \tag{17}$$

If τ is injective, then

$$\sum_{|p|\leq t} w(\mathrm{Frob}_p) \frac{\log|p|}{|p|} \leq \mathcal{B}. \tag{18}$$

Proof. We apply Proposition 14 with $H_\ell = H$ for all ℓ, and restrict the prime sum to $H \leq |p| \leq t$. Using

$$\sum_{a\in\mathcal{T}} \left\lfloor \frac{a_0}{|p|} \right\rfloor \geq \frac{1}{|p|} \sum_{a\in\mathcal{T}} a_0 - T,$$

we obtain the claim by dividing by $\sum_{a\in\mathcal{T}} a_0 > 0$, restoring the primes below H, and bounding each weight by $\|w\|_\infty$. □

7. Proof of the main estimate

We state the inputs needed to prove Theorem 4, deferring their proofs to the appendices. We then complete the proof.

7.1. **Prime sums.**

Lemma 16. *Let F_0 be a number field of degree d_0. For every real $y \geq 2$, we have*

$$\sum_{|p|\leq y} \log|p| \leq 2d_0 y, \qquad \sum_{|p|\leq y} \frac{\log|p|}{|p|} \leq d_0(\log y + 2),$$

where the sums run over all primes of F_0.

This estimate reduces to the case $F_0 = \mathbb{Q}$, where it is established in sharper form by Rosser–Schoenfeld [14, (3.24), (3.32)]. We record a self-contained proof in Appendix A.

7.2. **Interpolation.** Let F be a field of characteristic zero, and let σ be an automorphism of F. Let $t \geq 2$, $H \geq 1$ and $I, J \geq 1$ be integers. Let $u_1, \dots, u_I \in F$. We consider (as in §3) the map

$$\bigoplus_{a+Hb<t} FX^aY^b \longrightarrow F^{I\times J}$$

sending a polynomial f to its tuples of derivatives of order less than J at the points $(u_i, \sigma(u_i))$:

$$f \mapsto \big((\partial_X^j f)(u_i, \sigma(u_i))\big)_{1\leq i\leq I, 0\leq j<J}.$$

Lemma 17. *If the u_i are distinct and*

$$I > t - 1 + \left\lfloor \frac{t-1}{H} \right\rfloor, \qquad J \geq 2\left\lfloor \frac{t-1}{H} \right\rfloor + 1,$$

then τ is injective.

This was established in a note of Kevin Barreto [3]. We record a proof in Appendix B.

7.3. **Points of small height.** Let F be a number field of absolute discriminant Δ.

Proposition 18. *Let $\sigma_1, \dots, \sigma_r \in \mathrm{Aut}(F)$, and let $I \geq 1$ be an integer. There exist distinct $u_1, \dots, u_I \in F$ such that the elements*

$$z_i = (u_i, \sigma_1(u_i), \dots, \sigma_r(u_i)) \in F^{r+1}$$

have global height (see (13)*) satisfying*

$$\beta \leq I\sqrt{\Delta}.$$

This follows (in slightly sharper form) from Blichfeldt's theorem; see Cassels [5, Chapter III, §3, Theorem IV]. We record an adelic proof in Appendix C.

7.4. **A numerical estimate.** Finally, we require estimates for the right-hand side of Corollary 15.

Lemma 19. *Let F/F_0, d, d_0 and Δ be as in the introduction, and let $C \subseteq \mathrm{Gal}(F/F_0)$ be a conjugacy class. Let $x \geq 16$ be an integer satisfying*

$$\frac{\log(2x\sqrt{\Delta})}{d} < d_0 \log x.$$

In (17)*, take* $t = x$*,* $r = 1$*,* $\sigma_1 \in C$*, and*

$$\mathcal{T} := \{(a,b) : a + Hb < x\}, \qquad H := \lceil \log x \rceil, \qquad J := 2\left\lfloor \frac{x-1}{H} \right\rfloor + 1. \tag{19}$$

If $\beta \le 2x\sqrt{\Delta}$*, then*

$$|C|\mathcal{B} \le \frac{|C|}{d}\left(\log x + \frac{1}{2}\log \Delta\right) + d_0 \log\log(3x) + 50 d_0 |C|.$$

We prove this estimate in Appendix E.

7.5. **Completion of the proof.**

Proof of Theorem 4. We must verify the inequality (5). Since its left-hand side is unchanged on replacing x by $\lfloor x \rfloor$, and its right-hand side is increasing in x, we may assume that x is an integer. If $x < 16$, then Lemma 16 gives the bound $d_0(\log 16 + 2) < 5d_0$ for its left-hand side, which is less than the term $50 d_0|C|$ appearing on its right-hand side. We may thus assume that $x \ge 16$.

We may also assume that

$$\frac{\log(2x\sqrt{\Delta})}{d} < d_0 \log x. \tag{20}$$

Otherwise,

$$\frac{|C|}{d}\left(\log x + \frac{1}{2}\log \Delta\right) \ge d_0 |C| \log x - \frac{|C|}{d}\log 2,$$

and the theorem follows from Lemma 16, since $|C| \ge 1$ and $2d_0 + (|C|/d)\log 2 < 3d_0|C|$.

Choose $\sigma \in C$, and take H, J as in (19) and $I := 2x$. Proposition 18 gives distinct $u_1, \dots, u_I \in F$ for which the pairs $(u_i, \sigma(u_i))$ have height $\beta \le 2x\sqrt{\Delta}$. For $\mathcal{T}$ as in (19), Lemma 17, with $t = x$, implies that τ is injective. Corollary 15, again with $t = x$ and $r = 1$, now gives

$$\sum_{\substack{|p| \le x \\ \mathrm{Frob}_p = C}} \frac{\log |p|}{|p|} \le |C|\mathcal{B}.$$

The required bound then follows from Lemma 19. □

8. Variables and derivatives

The proof of Theorem 4 is complete, modulo the inputs proved in the appendices. We now estimate the right-hand side of Corollary 15 more simply to explain the choices of variables and derivatives. We first compare these choices under optimistic interpolation assumptions, then use the actual interpolation bound to give a softer proof of a weaker bound.

We take F to be a number field and write $d_0 := [F_0 : \mathbb{Q}]$.

8.1. **Estimating $\mathcal{B}$.** In this section, we regard r as fixed – any implied constants may depend upon it.

Lemma 20. *Let $r \geq 1$ and $2 \leq H \leq t$. For*

$$\mathcal{T} := \{a \in \mathbb{Z}_{\geq 0}^{r+1} : a_0 + H(a_1 + \cdots + a_r) < t\},$$

we have

$$T \asymp \frac{t^{r+1}}{H^r}, \qquad \sum_{a \in \mathcal{T}} a_0 \asymp \frac{t^{r+2}}{H^r}, \qquad \sum_{a \in \mathcal{T}} \sum_{\ell=1}^{r} a_\ell \ll \frac{t^{r+2}}{H^{r+1}}. \tag{21}$$

Proof. The set $\mathcal{T}$ is contained in the rectangle given by $0 \leq a_0 < t$ and $0 \leq a_\ell < t/H$ for $\ell \geq 1$, and contains the rectangle given by

$$0 \leq a_0 \leq t/2, \qquad 0 \leq a_\ell < t/(2rH) \quad (1 \leq \ell \leq r).$$

Counting the integer points in these rectangles and summing a_0 gives the first two estimates. For the last, we use $\sum_{\ell=1}^{r} a_\ell < t/H$ on $\mathcal{T}$ and the upper bound for T. □

Lemma 21. *With the notation of Corollary 15, we have*

$$\begin{aligned} \mathcal{B} \leq \frac{1 + \mathrm{O}(1/H)}{d} \log \beta + d_0 \|w\|_\infty \log H \\ + \mathrm{O}\left(d_0 \left(1 + \frac{(J-1)\log t}{t} \right) \right). \end{aligned} \tag{22}$$

All implied constants depend only on r and are effective.

Proof. By Lemma 20,

$$\frac{\sum_{a \in \mathcal{T}} |a|}{\sum_{a \in \mathcal{T}} a_0} = 1 + \mathrm{O}(1/H), \qquad \frac{T}{\sum_{a \in \mathcal{T}} a_0} \ll \frac{1}{t}.$$

Since $\log T \ll \log t$, the archimedean term satisfies

$$\frac{A_\infty}{\sum_{a \in \mathcal{T}} a_0} \ll d_0 \frac{\log t + (J-1)\log t}{t}.$$

Lemma 16 bounds the remaining terms by

$$\mathrm{O}(d_0) + d_0 \|w\|_\infty (\log H + 2).$$

Using $\log t/t \ll 1$ and $\|w\|_\infty \leq 1$ gives the claim. □

8.2. **Optimistic interpolation.** For any I, Proposition 18 supplies distinct points $u_i \in F$ for which $(u_i, \sigma_1(u_i), \ldots, \sigma_r(u_i))$ have height $\beta \leq I\sqrt{\Delta}$. The remaining question is whether the evaluation map τ is injective. Its matrix has T columns and IJ rows, so injectivity requires $IJ \geq T$. We first ask what bound we would obtain if this necessary condition were sufficient up to a fixed factor. Below, we first estimate $\mathcal{B}$ in this range (Proposition 22) and then apply the resulting estimate when τ is injective (Corollary 23). We restrict to $J \leq t$ because our polynomials have degree less than t in X_0.

Proposition 22. *Let the setting of Corollary 15 vary with an integral parameter $t \to \infty$. Suppose that, for some $K \geq 1$,*

$$IJ \leq KT, \qquad \beta \leq I\sqrt{\Delta}, \qquad 2 \leq H \leq t, \quad 1 \leq J \leq t. \tag{23}$$

Assume that r, d, d_0, K remain bounded and that

$$H \to \infty, \qquad J = o(t), \qquad \log \Delta = o(\log t). \tag{24}$$

If moreover

$$\log H = o(\log t) \text{ and } J = t^{\theta + o(1)} \text{ for some } 0 \le \theta \le 1, \tag{25}$$

then

$$\mathcal{B} \le \left(\frac{r+1-\theta}{d} + o(1) \right) \log t + \frac{1}{2d} \log \Delta. \tag{26}$$

Proof. By the upper bounds on IJ and β, and Lemma 20, we have

$$\log \beta \le \log \frac{KT\sqrt{\Delta}}{J} = \log \frac{t^{r+1}\sqrt{\Delta}}{H^r J} + \mathrm{O}_{r,K}(1).$$

Substituting into Lemma 21 gives

$$\begin{aligned} \mathcal{B} \le \frac{1 + \mathrm{O}(1/H)}{d} \log \frac{t^{r+1}\sqrt{\Delta}}{H^r J} + d_0 \|w\|_\infty \log H \\ + \mathrm{O}\left(d_0 \left(1 + \frac{(J-1)\log t}{t} \right) \right). \end{aligned} \tag{27}$$

This estimate holds for finite t, with effective implied constants depending only on r, K.

Under (24) and (25), we have

$$\log \frac{t^{r+1}\sqrt{\Delta}}{H^r J} = (r+1-\theta)\log t + \frac{1}{2}\log \Delta + o(\log t).$$

The error terms in (27) are $o(\log t)$, giving (26). □

Corollary 23. *In the setting of Proposition 22, suppose that τ is injective. Then*

$$\boxed{\sum_{\ell=1}^{r} \frac{1}{|C_\ell|} \sum_{\substack{|p| \le t \\ \mathrm{Frob}_p = C_\ell}} \frac{\log |p|}{|p|} \le \left(\frac{r+1-\theta}{d} + o(1) \right) \log t + \frac{1}{2d} \log \Delta.} \tag{28}$$

Proof. Combine Corollary 15 with Proposition 22. □

We give examples to illustrate the range of parameters and the resulting bounds. Each of these should be understood as conditional on the injectivity of τ in the "optimistic" range of parameters specified in Proposition 22.

Example 24. Let $F := \mathbb{Q}(\sqrt{D})$ be a quadratic field and let σ denote its nontrivial automorphism. We have $d = 2$. Take $r := 1$ and $J := 1$ (thus $\theta = 0$); this amounts to working with the determinants (7) in two variables without derivatives. Then (28) specializes to

$$\sum_{\substack{p \le t \\ \chi_D(p) = -1}} \frac{\log p}{p} \le \left(\frac{2}{2} + o(1) \right) \log t,$$

which fails to improve upon the trivial bound.

Example 25. Let $F := \mathbb{Q}(\sqrt{D}, \sqrt{2})$ with $\mathbb{Q}(\sqrt{D}) \neq \mathbb{Q}(\sqrt{2})$ be a biquadratic extension of $\mathbb{Q}$, with Galois group $G \cong (\mathbb{Z}/2\mathbb{Z})^2$. We have $d = 4$. Take $r := 2$ and $J := 1$ (thus $\theta = 0$); this amounts to working with the determinants (10) in three variables without derivatives, as in OpenAI's original paper [12]. Let σ_1 and σ_2 denote the two automorphisms that restrict to the nontrivial automorphism of $\mathbb{Q}(\sqrt{D})$. Observe that if p is unramified in F and $\chi_D(p) = -1$, then $\mathrm{Frob}_p \in \{\sigma_1, \sigma_2\}$. Under the hypotheses of Corollary 23, we obtain from (28)

$$\sum_{\substack{p \le t \\ \chi_D(p) = -1}} \frac{\log p}{p} \le \sum_{\substack{p \le t \\ \mathrm{Frob}_p \in \{\sigma_1, \sigma_2\}}} \frac{\log p}{p} + \mathrm{O}(1) \le \left(\frac{3}{4} + o(1)\right) \log t,$$

which improves upon the trivial bound. We note that OpenAI's original argument establishes surjectivity rather than injectivity of the evaluation map τ with $J = 1$, giving nonzero minors of size $I \times I$ rather than $T \times T$.

Example 26. Choose an auxiliary finite set $p_1, \dots, p_k$ of fundamental discriminants. Assume that $D, p_1, \dots, p_k$ are independent in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. Take

$$F = \mathbb{Q}(\sqrt{D}, \sqrt{p_1}, \dots, \sqrt{p_k}),$$

with Galois group $G \cong (\mathbb{Z}/2\mathbb{Z})^{k+1}$. Then $d = 2^{k+1}$. Take $r := 2^k$ and $J := 1$ (thus $\theta = 0$), and let $\sigma_1, \dots, \sigma_r$ denote the automorphisms that restrict to the nontrivial automorphism of $\mathbb{Q}(\sqrt{D})$. Under the same interpolation assumptions, (28) gives, for each fixed k,

$$\sum_{\substack{p \le t \\ \chi_D(p) = -1}} \frac{\log p}{p} \le \left(\frac{2^k + 1}{2^{k+1}} + o(1)\right) \log t.$$

Note that the coefficient of $\log t$ approaches $1/2$ as $k \to \infty$.

8.3. **A qualitative bound.** We now take $r := 1$ and $J \asymp t/H$, corresponding to the determinants (11) in two variables, as in [7]. In that setting, Lemma 17 gives the injectivity of τ in the required range, and we obtain the following qualitative form of Theorem 4.

Theorem 27. *Let F/F_0, d, d_0 and Δ be as in the introduction, and let $C \subseteq \mathrm{Gal}(F/F_0)$ be a conjugacy class. For every $\varepsilon > 0$ and every real $t \ge 2$, we have*

$$\frac{1}{|C|} \sum_{\substack{|p| \le t \\ \mathrm{Frob}_p = C}} \frac{\log |p|}{|p|} \le \frac{1 + \varepsilon}{d} \log(t\sqrt{\Delta}) + \mathrm{O}_{\varepsilon, d, d_0}(1), \tag{29}$$

with an effective implied constant depending only on ε, d, d_0.

Proof. We may assume that t is an integer. Choose $\sigma \in C$, and let $H \ge 2$ be an integer to be chosen in terms of ε, d, d_0. For $t \le H$, the result follows from Lemma 16, so assume that $t > H$.

By Proposition 18, we can choose $I := 2t$ distinct points $u_i \in F$ such that the pairs $(u_i, \sigma(u_i))$ have height $\beta \le 2t\sqrt{\Delta}$. Take

$$\mathcal{T} := \{(a, b) : a + Hb < t\}, \qquad J := 2\left\lfloor \frac{t-1}{H} \right\rfloor + 1,$$

so that by Lemma 17, τ is injective. Since $J-1 \le 2t/H$, Corollary 15 and Lemma 21, with $r=1$ and $\|w\|_\infty = 1/|C|$, give

$$\frac{1}{|C|}\sum_{\substack{|p|\le t\\ \mathrm{Frob}_p = C}} \frac{\log|p|}{|p|} \le \left(\frac{1}{d} + \mathrm{O}\left(\frac{1+dd_0}{dH}\right)\right)\log(t\sqrt{\Delta}) + \mathrm{O}\big(d_0(1+\log H)\big).$$

Here we have used that $\log t \le \log(t\sqrt{\Delta})$. Taking H large enough in terms of ε, d, d_0 gives the claim. □

Taking $d=2$, $d_0=1$ and $\varepsilon = 1/2$ in Theorem 27 gives the saving required by Lemma 2. Thus the softer estimates of this section also suffice to exclude Landau–Siegel zeros.

Appendix A. Prime sums

Proof of Lemma 16. We first take $F_0 = \mathbb{Q}$ and write

$$\vartheta(N) := \sum_{p\le N} \log p, \qquad M(N) := \sum_{p\le N} \frac{\log p}{p}.$$

We claim that $\vartheta(N) \le 2N$ for every integer $N \ge 1$. Indeed, primes in $(m, 2m]$ divide $\binom{2m}{m}$, while primes in $(m+1, 2m+1]$ divide $\binom{2m+1}{m}$. Thus

$$\vartheta(2m) - \vartheta(m) \le \log\binom{2m}{m} \le 2m\log 2,$$
$$\vartheta(2m+1) - \vartheta(m+1) \le \log\binom{2m+1}{m} \le 2m\log 2.$$

For the last inequality, we use that the two middle coefficients of $(1+1)^{2m+1}$ are equal. Since $\log 2 < 1$, induction from $\vartheta(1) = 0$ gives the claim.

We estimate $M(N)$ via an argument in the spirit of the determinant method applied elsewhere in these notes, observed by Pascadi [13] and Tao [16]. For $N \ge 2$, consider the Vandermonde determinant

$$V_N := \det\big(u^a\big)_{1\le u\le N,\, 0\le a<N} = \prod_{1\le i<j\le N} (j-i).$$

For each prime p, the polynomials

$$X^{a-pk}(X^p - X)^k, \qquad k := \lfloor a/p \rfloor$$

give a basis for the space $\bigoplus_{0\le a<N} \mathbb{Z}X^a$ of polynomials of degree less than N. The change of basis is triangular with ones on the diagonal, so it leaves V_N unchanged. Since $u^p - u$ is divisible by p for every integer u, the ath column is now divisible by $p^{\lfloor a/p\rfloor}$. Writing $N = mp + r$ with $0 \le r < p$, we obtain

$$\mathrm{ord}_p(V_N) \ge \sum_{a=0}^{N-1} \left\lfloor \frac{a}{p} \right\rfloor = \frac{N^2}{2p} - \frac{N}{2} + \frac{r(p-r)}{2p} \ge \frac{N^2}{2p} - \frac{N}{2}.$$

Comparing divisibility and size gives

$$\frac{N^2}{2} M(N) - \frac{N}{2}\vartheta(N) \le \log V_N \le \frac{N(N-1)}{2}\log N.$$

Consequently,

$$M(N) \le \left(1 - \frac{1}{N}\right) \log N + \frac{\vartheta(N)}{N} \le \log N + 2.$$

Taking $N = \lfloor y \rfloor$ proves both bounds for real $y \ge 2$.

For general F_0, write $|p| = \ell^{f_p}$ when $p \mid \ell$, and use $\sum_{p|\ell} f_p \le d_0$. The contribution above ℓ to the first sum is at most $d_0 \log \ell$, and that to the second is at most $d_0 \log \ell/\ell$, so we reduce to the bounds over $\mathbb{Q}$. □

Appendix B. Interpolation

We return to two variables X, Y and points (u_i, v_i). Take $R = F$, where F is a field of characteristic zero. For integers $r, s \ge 0$, set

$$\mathcal{T}_{r,s} := \left\{(a,b) \in \mathbb{Z}^2_{\ge 0} : a \le r,\, b \le s\right\},$$

and take $\mathcal{T} = \mathcal{T}_{r,s}$. We record a criterion for the injectivity of

$$\tau : \bigoplus_{\substack{0 \le a \le r \\ 0 \le b \le s}} F X^a Y^b \longrightarrow F^{I \times J}.$$

Proposition 28. *Suppose that the u_i are pairwise distinct and that the v_i are pairwise distinct. If*

$$I > r + s, \qquad J \ge 2s + 1, \tag{30}$$

then τ is injective.

We use the Wronskian argument of Barreto [3, proof of Theorem 1], also used by Kominers [10, proof of Theorem 1].

Proof. Suppose that a nonzero polynomial $P \in F[X,Y]$ of X-degree at most r and Y-degree at most s lies in the kernel of τ. In other words, each specialization $P_i(X) := P(X, v_i)$ vanishes to order at least J at u_i.

At most s of the specializations P_i are identically zero. Indeed, write $P = \sum_a c_a(Y) X^a$ and choose a nonzero coefficient c_a. Since $\deg c_a \le s$, it has at most s roots. Since the v_i are distinct, there are at most s indices i for which $c_a(v_i) = 0$. For the remaining indices i, we have $c_a(v_i) \ne 0$, hence $P_i \ne 0$, as required.

We now write $P = \sum_{b=0}^{s} f_b(X) Y^b$ and let $W \subseteq F[X]$ denote the span of the coefficients f_b. Write $\ell := \dim_F W$, so that

$$1 \le \ell \le s + 1. \tag{31}$$

Each P_i belongs to W, and each element of W has degree at most r.

Choose a basis $g_1, \dots, g_\ell$ of W, and form its Wronskian

$$\mathcal{W}(X) := \det\left(\partial_X^j g_b(X)\right)_{0 \le j < \ell,\ 1 \le b \le \ell}.$$

For example, when $\ell = 3$,

$$\mathcal{W}(X) = \det \begin{pmatrix} g_1(X) & g_2(X) & g_3(X) \\ g_1'(X) & g_2'(X) & g_3'(X) \\ g_1''(X) & g_2''(X) & g_3''(X) \end{pmatrix}$$

. Changing the basis multiplies $\mathcal{W}$ by a nonzero constant. We observe that

$$\mathcal{W} \ne 0, \qquad \deg \mathcal{W} = \sum_b d_b - \binom{\ell}{2} \le \ell r. \tag{32}$$

Indeed, by canceling leading terms, we may choose the g_b monic with strictly increasing degrees $d_1 < \cdots < d_\ell$. The coefficient of degree $\sum_b d_b - \binom{\ell}{2}$ in $\mathcal{W}$ is

$$\det\big(d_b(d_b-1)\cdots(d_b-j+1)\big)_{0\le j<\ell,\ 1\le b\le \ell} = \prod_{a<b}(d_b-d_a).$$

Since F has characteristic zero, this product is nonzero, so it is the leading coefficient of $\mathcal{W}$, giving (32).

Now fix i with $P_i \neq 0$, and extend P_i to a basis of W. The Wronskian in this new basis contains the column

$$\begin{pmatrix} P_i \\ \partial_X P_i \\ \vdots \\ \partial_X^{\ell-1} P_i \end{pmatrix}.$$

Each entry in that column is divisible by $(X-u_i)^{J-\ell+1}$. The same divisibility holds for the new Wronskian, hence also for $\mathcal{W}$. Since the u_i are distinct and at least $I-s$ of the P_i are nonzero, we obtain

$$\deg \mathcal{W} \ge (I-s)(J-\ell+1).$$

Our hypotheses (30) and (31) give

$$I-s > r, \quad J-\ell+1 \ge 2s+2-\ell \ge \ell,$$

so we obtain $\deg \mathcal{W} > r\ell$, giving the required contradiction to (32). □

Proof of Lemma 17. We must verify the hypotheses of Proposition 28. The points $v_i = \sigma(u_i)$ are distinct. For $(a,b) \in \mathcal{T}$, we have $a < t$ and $Hb < t$, hence

$$a \le r := t-1, \qquad b \le s := \lfloor (t-1)/H \rfloor. \tag{33}$$

Thus the domain of τ is contained in the span of the monomials with X-degree at most r and Y-degree at most s. The assumed bounds on I and J give (30), so Proposition 28 applies. □

Appendix C. Points of small height

We record a proof of Proposition 18.

C.1. **Haar measures.** Let F be a global field, and let $\mathbb{A}$ denote its adele ring. We equip $\mathbb{A}$ with product Haar measure $\mu := \prod_v \mu_v$, normalized as follows:

- at finite places, μ_v assigns volume one to the ring of integers;
- at real places, μ_v is Lebesgue measure;
- at complex places, μ_v is twice the Euclidean area.

On $\mathbb{A}/F$, we use the quotient measure induced by μ and counting measure on F. When F is a number field, we continue to write $\Delta := |\operatorname{disc} F|$. When F is a function field with constant field $\mathbb{F}$ of cardinality q, we set

$$g := \dim_{\mathbb{F}} \mathbb{A}/\left(F + \prod_v \mathfrak{o}_v\right), \qquad \Delta := q^{2g-2}.$$

Lemma 29. We have

$$\mu(\mathbb{A}/F) = \sqrt{\Delta}.$$

Proof. For number fields, see Weil [19, §2.1.3]. For function fields, $F \cap \prod_v \mathfrak{o}_v = \mathbb{F}$, so the image of $\prod_v \mathfrak{o}_v$ in $\mathbb{A}/F$ has measure $1/q$. Its index is q^g, giving total volume q^{g-1}. □

We denote the probability Haar measure on $\mathbb{A}/F$ by dt.

C.2. **Averaging.**

Lemma 30. *For every compact measurable subset $\Omega \subseteq \mathbb{A}$, we have*

$$\int_{\mathbb{A}/F} \#\big((t+\Omega)\cap F\big)\, dt = \frac{\mu(\Omega)}{\sqrt{\Delta}}.$$

Proof. Call the left-hand side $\nu(\Omega)$. It defines a Haar measure ν on $\mathbb{A}$, so the two sides are proportional. To compute the constant of proportionality, we take Ω to be a measurable fundamental domain for $\mathbb{A}/F$. The integrand on the left-hand side is then identically one, so $\nu(\Omega) = 1$. We conclude using Lemma 29. □

C.3. **Producing points of small height.**

Lemma 31. *Let $I \geq 1$ be an integer. For each place v, choose $B_v > 0$, with $B_v = 1$ at almost all places and $B_v \in |F_v^\times|_v$ at each place. If*

$$\prod_v B_v \geq I\sqrt{\Delta},$$

then there is a subset $\{u_1, \dots u_I\} \subset F$ of cardinality I such that

$$|u_i|_v \leq B_v$$

for all i and v.

Proof. At finite places, set $\Omega_v := \{z \in F_v : |z|_v \leq B_v\}$. At real places, set $\Omega_v = [0, B_v]$. At complex places, let Ω_v be a square of side $\sqrt{B_v/2}$. These choices satisfy

$$\mu_v(\Omega_v) = B_v, \qquad \Omega_v - \Omega_v \subseteq \{z \in F_v : |z|_v \leq B_v\}.$$

Thus $\Omega := \prod_v \Omega_v$ is compact and $\mu(\Omega) = \prod_v B_v$. By Lemma 30, we have

$$\int_{\mathbb{A}/F} \#\big((t+\Omega)\cap F\big)\, dt = \frac{\mu(\Omega)}{\sqrt{\Delta}} \geq I.$$

Thus some translate $t + \Omega$ contains I distinct elements $x_1, \dots, x_I$ of F. The differences $u_i := x_i - x_1$ lie in $F \cap (\Omega - \Omega)$ and satisfy the required bounds. □

Remark 32. Over number fields, using disks instead of squares at the complex places improves the factor $\sqrt{\Delta}$ in the hypothesis of Lemma 31 to $(2/\pi)^{r_2}\sqrt{\Delta}$, where r_2 is the number of complex places.

Proof of Proposition 18. We set $H := (I\sqrt{\Delta})^{1/[F:\mathbb{Q}]}$ and apply Lemma 31 with

$$B_v := \begin{cases} 1, & v \nmid \infty, \\ H^{[F_v:\mathbb{R}]}, & v \mid \infty. \end{cases}$$

These radii are at least one and have product $I\sqrt{\Delta}$. They are preserved by every automorphism of F: if σv is the transported place, characterized by $|\sigma\alpha|_{\sigma v} = |\alpha|_v$,

then $B_{\sigma v} = B_v$. The resulting u_i are integral at all finite places, and every conjugate has Euclidean modulus at most H. The same bounds hold for each $\sigma_\ell(u_i)$, so

$$\beta \le \prod_v B_v = I\sqrt{\Delta}.$$

□

Appendix D. Comparison with Bombieri–Stepanov

The "small points" estimate (Lemma 31), and the product formula employed in §6 are also relevant for the Bombieri–Stepanov method [4] for proving the Riemann hypothesis for curves over finite fields. We give a brief sketch of that argument.

Let F be a global function field, thus F is the field of rational functions on a curve C over a finite field $\mathbb{F}$ of some cardinality q. We always assume that $\mathbb{F}$ is the full constant field of F. We retain the notation $\mathbb{A}$ and $\Delta = q^{2g-2}$ from the preceding appendix. We assume that F has a place ∞ of degree one, which can be arranged after passing to a finite extension of $\mathbb{F}$. Take

$$R := F \cap \prod_{v\neq\infty} \mathfrak{o}_v, \qquad R_\ell := \{f \in R : \operatorname{ord}_\infty(f) \ge -\ell\} \quad (\ell \ge 0).$$

The R_ℓ are finite-dimensional $\mathbb{F}$-vector spaces. They are special cases of the sets considered in the proof of Lemma 31.

We have $R_0 = \mathbb{F}$. By considering the leading coefficient at ∞, whose residue field is $\mathbb{F}$, we see that

$$0 \le \dim_{\mathbb{F}} R_{\ell+1} - \dim_{\mathbb{F}} R_\ell \le 1.$$

In particular, $\dim R_\ell \le \ell + 1$, and we may choose bases of the R_ℓ with distinct pole orders at ∞.

For $\ell \ge g - 1$, we apply Lemma 31 with

$$I := q^{\ell+1-g}, \qquad B_\infty := q^\ell, \qquad B_v := 1 \quad (v \neq \infty).$$

Since $\prod_v B_v = I\sqrt{\Delta}$, the lemma supplies $q^{\ell+1-g}$ distinct elements of R_ℓ. As R_ℓ is an $\mathbb{F}$-vector space, we obtain the Riemann–Roch estimate

$$\ell + 1 - g \le \dim_{\mathbb{F}} R_\ell \le \ell + 1. \tag{34}$$

The lower bound is automatic for $\ell < g - 1$.

For $m \ge 1$, write

$$N_m := \sum_{\deg v \mid m} \deg v, \qquad \deg v := [\mathfrak{o}_v/\mathfrak{p}_v : \mathbb{F}].$$

This is the number of points over $\mathbb{F}_{q^m}$ on C. For a place $\mathfrak{p}$ of F, write $|\mathfrak{p}| = q^{\deg \mathfrak{p}}$ for its absolute norm. If a nonzero $f \in R_L$ vanishes at every place $\mathfrak{p} \neq \infty$ with $\deg \mathfrak{p} \mid m$, then the product formula gives

$$(N_m - 1)\log q \le \sum_{\mathfrak{p}\neq\infty} \operatorname{ord}_{\mathfrak{p}}(f) \log|\mathfrak{p}| = \log|f|_\infty \le L \log q. \tag{35}$$

Thus $N_m \le L + 1$. The equality in (35) is the analogue of (8), while comparisons like (35) are analogous to (9).

Proposition 33. *For even $m \ge 2$, we have*

$$N_m \le q^m + \mathrm{O}_F(q^{m/2}).$$

Proof. Write $Q := q^{m/2}$. Let $\ell_1, \ell_2 \geq 0$ be integers with $\ell_1 < Q$. Choose bases $\{x_i\}$ of R_{ℓ_1} and $\{y_j\}$ of R_{ℓ_2} with distinct pole orders at ∞.

Suppose we can find coefficients $c_{ij} \in \mathbb{F}$, not all zero, such that

$$\widetilde{f} := \sum_{i,j} c_{ij} x_i^Q y_j = 0.$$

For such coefficients, we set

$$f := \sum_{i,j} c_{ij} x_i y_j^Q \in R_{\ell_1 + Q\ell_2}.$$

The pole orders at ∞ of the $x_i y_j^Q$ are distinct: those of the x_i lie in $[0, \ell_1] \subset [0, Q)$, while those of the y_j^Q are distinct multiples of Q. Since the coefficients are not all zero, we have $f \neq 0$.

Now, for each place $v \neq \infty$ with $\deg v \mid m$, we have $u^{Q^2} \equiv u \pmod{\mathfrak{p}_v}$ for all $u \in R$ (an analogue of the Frobenius condition employed in the rest of this note). Since also $c_{ij}^Q = c_{ij}$ (a condition involving the field of constants $\mathbb{F}$, which admits no obvious analogue in the number field setting), we obtain

$$f^Q \equiv \sum_{i,j} c_{ij} x_i^Q y_j = \widetilde{f} = 0 \pmod{\mathfrak{p}_v}.$$

Hence f vanishes at all the required places, and the product formula (35) gives

$$N_m \leq 1 + \ell_1 + Q\ell_2.$$

It remains to find the coefficients c_{ij}. The map $(c_{ij}) \mapsto \widetilde{f}$ takes values in $R_{Q\ell_1 + \ell_2}$, so in view of the dimension estimates (34), it has a nonzero kernel provided that

$$(\ell_1 + 1 - g)(\ell_2 + 1 - g) > Q\ell_1 + \ell_2 + 1,$$

assuming $\ell_1, \ell_2 \geq g - 1$. Take

$$\ell_1 := Q - 1, \qquad \ell_2 := Q + 2g.$$

For sufficiently large Q, the required inequality holds, since

$$(Q - g)(Q + g + 1) - (Q^2 + 2g + 1) = Q - g^2 - 3g - 1 > 0.$$

We conclude that $N_m \leq Q^2 + (2g + 1)Q$ for m large enough. For small m, we conclude via trivial estimation. □

For the Galois argument that passes from these upper bounds to the Riemann hypothesis, see Bombieri [4, §III].

Remark 34. The above argument is similar to that of the rest of this note in the use of the product formula, the existence of points of small height, and a Frobenius condition to detect certain primes (here those whose degree divides m, elsewhere those whose Frobenius is some given automorphism σ). The arguments differ in how the elements fed into the product formula are constructed: in Bombieri–Stepanov, they are nonzero elements of the kernel of a linear map defined over the constant field $\mathbb{F}$, while in the rest of the note, they are nonzero determinantal minors of an injective map defined using auxiliary polynomials. The Bombieri–Stepanov argument also uses the exact additivity $(\sum a_{ij})^Q = \sum a_{ij}^Q$, which has no analogue in the number field setting.

Appendix E. Explicit estimates

We now prove Lemma 19. The estimates recorded here were done in "one shot" by Codex (GPT-6.1 Sol on "Low" reasoning effort) after we asked it to use Corollary 15 to prove any completely effective estimate in the setting of Theorem 4.

E.1. **Monomial sums.**

Lemma 35. *Let $x \geq 6$ and $1 \leq H \leq x/2$ be integers, and take*

$$\mathcal{T} := \{(a,b) : a + Hb < x\}.$$

Then

$$T \leq \frac{x^2}{H}, \qquad \sum_{\mathcal{T}} a \geq \frac{x^3}{8H}, \qquad \sum_{\mathcal{T}} b \leq \frac{8}{H} \sum_{\mathcal{T}} a.$$

Proof. For each integer b with $0 \leq b \leq \lfloor (x-1)/H \rfloor$, the possible values of a are $0, \dots, x - Hb - 1$. Comparing the resulting sums with integrals gives

$$T = \sum_{b=0}^{\lfloor (x-1)/H \rfloor} (x - Hb) \leq x + \frac{1}{H} \int_0^x u \, du = x + \frac{x^2}{2H} \leq \frac{x^2}{H}$$

and

$$\begin{aligned} \sum_{\mathcal{T}} a &= \sum_{b=0}^{\lfloor (x-1)/H \rfloor} \frac{(x - Hb)(x - Hb - 1)}{2} \\ &\geq \frac{1}{H} \int_1^x \frac{u(u-1)}{2} \, du = \frac{2x^3 - 3x^2 + 1}{12H} \geq \frac{x^3}{8H}. \end{aligned}$$

Here the first comparison uses that u is increasing, and the second uses that $u(u-1)/2$ is increasing and nonnegative for $u \geq 1$. The last inequality holds for $x \geq 6$. Finally, $b < x/H$ for every $(a,b) \in \mathcal{T}$, so

$$\sum_{\mathcal{T}} b \leq \frac{xT}{H} \leq \frac{x^3}{H^2} \leq \frac{8}{H} \sum_{\mathcal{T}} a.$$

□

E.2. **Dependence on H.**

Proposition 36. *Retain the notation of Lemma 19, but allow any integer $2 \leq H \leq x/2$, still taking $J = 2\lfloor (x-1)/H \rfloor + 1$. Then*

$$|C|\mathcal{B} \leq \frac{|C|}{d} \log(2x\sqrt{\Delta}) + \frac{8|C|}{H} \left(\frac{\log(2x\sqrt{\Delta})}{d} + 2d_0 \log x \right) + d_0 \log H + 21 d_0 |C|.$$

Proof. Since $r = 1$ and $\|w\|_\infty = 1/|C|$, the definition (17) gives

$$\begin{aligned} |C|\mathcal{B} = \frac{|C|}{d} \frac{\sum_{\mathcal{T}} (a+b)}{\sum_{\mathcal{T}} a} \log \beta + \frac{|C| A_\infty}{\sum_{\mathcal{T}} a} \\ + \frac{T}{\sum_{\mathcal{T}} a} \sum_{|p| \leq x} \log |p| + \sum_{|p| < H} \frac{\log |p|}{|p|}. \end{aligned} \tag{36}$$

We estimate these four terms in order.

By Lemma 35, the height term is at most

$$\frac{|C|}{d}\left(1+\frac{8}{H}\right)\log(2x\sqrt{\Delta}).$$

Next, the same lemma gives $T/\sum_{\mathcal{T}} a \le 8/x$. Using the definition (15) of A_∞, together with $T \le x^2$ and $J+1 \le 2x/H+2$, we find

$$\begin{aligned}\frac{|C|A_\infty}{\sum_{\mathcal{T}} a} &\le \frac{8d_0|C|}{x}\big(\log T + (J-1)\log x\big)\\ &\le 16d_0|C|\left(\frac{\log x}{H}+\frac{\log x}{x}\right)\\ &\le \frac{16d_0|C|\log x}{H} + 3d_0|C|.\end{aligned}$$

Here we used that $16\log x/x \le \log 16 < 3$ for $x \ge 16$. Finally, Lemma 16 gives

$$\frac{T}{\sum_{\mathcal{T}} a}\sum_{|p|\le x}\log|p| \le 16d_0.$$

The omitted primes with $|p| < H$ contribute at most $d_0(\log H+2)$, again by Lemma 16. Combining these bounds and using $|C| \ge 1$ proves the claim. □

E.3. **Choosing H.**

Proof of Lemma 19. Take

$$H := \lceil \log x \rceil.$$

Then $2 \le H \le x/2$, and the hypothesis of the lemma gives

$$\frac{8|C|}{H}\left(\frac{\log(2x\sqrt{\Delta})}{d}+2d_0\log x\right) \le 24d_0|C|\frac{\log x}{H} \le 24d_0|C|.$$

Moreover, $H \le \log x + 1 \le \log(3x)$, so $\log H \le \log\log(3x)$. Proposition 36 now gives

$$\begin{aligned}|C|\mathcal{B} \le \frac{|C|}{d}\left(\log x+\frac{1}{2}\log\Delta\right) + d_0\log\log(3x)\\ + \frac{|C|}{d}\log 2 + 45d_0|C|.\end{aligned}$$

Since $|C|/d \le d_0|C|$ and $45+\log 2 < 50$, this proves the lemma. □

Appendix F. Excluding real zeros

We now prove Lemma 2, following some auxiliary lemmas. Let χ be a primitive real nonprincipal character of conductor q. The idea is that a zero of $L(s,\chi)$ very close to 1 forces almost all the logarithmic prime mass onto $\chi(p) = -1$, but the hypothesis bounds that mass by a fixed proportion $\kappa < 1$. In what follows, all implied constants are absolute and effective.

We first record the two facts about L-functions that we need. Write $\mathcal{L} := \log(2q)$.

Lemma 37. Let $1 < s \le 2$. We have

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + \mathrm{O}(1).$$

If $L(s,\chi)$ has a real zero $\beta \in (0,1)$, then

$$-\frac{L'}{L}(s,\chi) \le -\frac{1}{s-\beta} + \mathrm{O}(\mathcal{L}).$$

Proof. The first estimate follows from the fact that ζ has a simple pole at $s=1$. As for the second estimate, the partial fraction expansion of L'/L [6, §12, (17)–(18)] gives

$$-\frac{L'}{L}(s,\chi) = -\sum_{\rho} \frac{1}{s-\rho} + \frac{1}{2}\log q + \mathrm{O}(\mathcal{L}),$$

where ρ runs over the nontrivial zeros with multiplicity and the sum is performed symmetrically, grouping the zeros ρ into conjugate pairs. For each such ρ, we have $\Re \frac{1}{s-\rho} \ge 0$ for s in the indicated range, so taking real parts and dropping all terms but one $\rho = \beta$ gives the required estimate. □

Lemma 38. *For $\Re s > 1$, we have*

$$-\frac{\zeta'}{\zeta}(s) = \sum_{k=1}^{\infty}\sum_{p} \frac{\log p}{p^{ks}} = \sum_{p} \frac{\log p}{p^s} + \mathrm{O}(1)$$

and

$$-\frac{L'}{L}(s,\chi) = \sum_{k=1}^{\infty}\sum_{p} \frac{\chi(p)^k \log p}{p^{ks}} = \sum_{p} \frac{\chi(p)\log p}{p^s} + \mathrm{O}(1).$$

Proof. These estimates follow from the Euler product expansions of ζ and $L(s,\chi)$, together with the fact that $\sum_{k\ge 2}\sum_p \frac{\log p}{p^k}$ converges. □

Lemma 39. *If $L(s,\chi)$ has a real zero $\beta \in (0,1)$, then for $1 < s \le 2$,*

$$\sum_{p:\chi(p)=-1} \frac{\log p}{p^s} \ge \frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s-\beta}\right) + \mathrm{O}(\mathcal{L}).$$

Proof. Away from primes dividing q, we have $1_{\chi(p)=-1} = (1-\chi(p))/2$. The primes dividing q contribute $\mathrm{O}(\mathcal{L})$, so after some rearrangement, we deduce from Lemma 38 that

$$\sum_{p:\chi(p)=-1} \frac{\log p}{p^s} = \frac{1}{2}\left(-\frac{\zeta'}{\zeta}(s) + \frac{L'}{L}(s,\chi)\right) + \mathrm{O}(\mathcal{L}).$$

The desired estimate then follows from Lemma 37. □

We now complete the proof of Lemma 2. Suppose $L(s,\chi)$ has a real zero $\beta \in (0,1)$. Take $R \ge 1$. Applying Lemma 39 with $s := 1 + 1/R$ and multiplying by $1/R$ gives

$$P_R := \frac{1}{R}\sum_{p:\chi(p)=-1} \frac{\log p}{p}\exp\left(-\frac{\log p}{R}\right) \ge \frac{1}{2}\left(1 + \frac{1}{1+R(1-\beta)}\right) - C\mathcal{L}/R$$

for some absolute effective constant C. With the notation

$$S(t) := \sum_{\substack{\chi(p)=-1 \\ \log p \le t}} \frac{\log p}{p},$$

our hypothesis reads

$$S(t) \le \kappa t + B\mathcal{L}.$$

We see using the identities

$$e^{-x} = \int_x^\infty e^{-t}\,dt, \quad \int_0^\infty e^{-t}\,dt = 1, \quad \int_0^\infty te^{-t}\,dt = 1$$

that

$$P_R = \int_0^\infty e^{-t}\frac{S(Rt)}{R}\,dt \le \int_0^\infty e^{-t}\left(\kappa t + B\mathcal{L}/R\right)\,dt = \kappa + B\mathcal{L}/R.$$

Combining the lower and upper bounds for P_R gives

$$\frac{1}{2}\left(1 + \frac{1}{1+R(1-\beta)}\right) - C\mathcal{L}/R \le P_R \le \kappa + B\mathcal{L}/R. \tag{37}$$

This implies that $\mathcal{L}(1-\beta) \ge c$ for some absolute effective $c > 0$. Informally, suppose $\mathcal{L}(1-\beta)$ is small, so that $\mathcal{L}$ is much smaller than $1/(1-\beta)$. We may then choose a scale R, intermediate between $\mathcal{L}$ and $1/(1-\beta)$, so that both $\mathcal{L}/R$ and $R(1-\beta)$ are small. The two sides of (37) are then arbitrarily close to 1 and κ, respectively, contradicting our assumption that $\kappa < 1$.

To carry out this last argument effectively, write $\delta := 1-\beta$ and $D := B + C$. The bound (37) rearranges to

$$1 - \kappa \le \frac{R\delta}{2(1+R\delta)} + D\mathcal{L}/R.$$

Take $R := \sqrt{\mathcal{L}/\delta}$. Then $R\delta$ and $\mathcal{L}/R$ are both equal to $\sqrt{\mathcal{L}\delta}$, so we obtain

$$1 - \kappa \le \sqrt{\mathcal{L}\delta}(\tfrac{1}{2} + D),$$

i.e.,

$$\mathcal{L}\delta \ge c := \left(\frac{1-\kappa}{\frac{1}{2}+D}\right)^2,$$

as required.

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