

# 1. SOME THINGS TO THINK ABOUT IN THE AFTERNOONS AND BEYOND

- (1) Understand the details of the consequences of subconvexity mentioned in lecture (reference: [19, §5]):

- Subconvexity vs. geometry of numbers.
- Distinguishing modular forms.
- Duke theorem. Reduction of supersingular elliptic curves.
- Quantum unique ergodicity. One exercise here is to understand how this follows from subconvexity in the special case of Eisenstein series, as in [17, §2].

- (2) Some “classic” papers (non-exhaustive):

- Bounds for Fourier coefficients of half-integral weight, applications to quadratic forms: [15], [9], [11]
- Subconvexity for  $GL_2$ : [10] [12] [13].
- Moments and amplification via periods: [27], [20], [16], [24].
- Shifted convolution sums:
  - via  $\delta$ -symbol: [10]
  - via periods: [25], [4], [5].
- Papers emphasizing variation of the test vector: [23, 2, 27].

One exercise is draw parallels, e.g., between

- [20, Thm 5.1] and [10],
- [27, §4] and [12], or
- [20, Thm 5.2] and [13].

Another is to reprove some results using different methods, e.g., by working out a “classical” proof in the style of [13] for subconvexity for Maass forms at special points, namely, for  $L(1/2 + it_f, f)$  with  $f$  on  $SL_2(\mathbb{Z})$  of eigenvalue  $1/4 + t_f^2$ , by estimating an amplified fourth moment, e.g.,

$$\sum_{f: t_f \in [T, T+1]} \left| \sum_{\ell \lesssim L} c_\ell \lambda_f(\ell) \right|^2 |L(\tfrac{1}{2} + it_f, f)|^4.$$

- (3) Study the proof of the convexity bound. There are two steps:

- The Phragmen–Lindelöf convexity principle, to reduce estimates for  $\Re(s) = 1/2$  to estimates for  $\Re(s) = 1 + \varepsilon$  and  $\Re(s) = -\varepsilon$ .
- The functional equation, to reduce further to estimates for  $\Re(s) = 1 + \varepsilon$ .
- Establishing the necessary bounds for  $\Re(s) = 1 + \varepsilon$ , for which see <https://www.math.wsu.edu/faculty/scliu/papers/Convexity.pdf> and references.

- (4) Some recent papers, concerning subconvexity or related problems, that haven’t been fully explored (e.g., interpreted via integral representations):

- $\delta$ -method papers such as [26] and [1]
- Higher moments over very large families, as in [7], [8]
- Rankin–Selberg when the rank difference is larger than one, as in [6]

- (5) Higher rank subconvex bounds [3], [18], [22], [21], [14]. There are many “exercises” implicit in these papers; for instance, a half-dozen are suggested in [22, Remark 1.4]. Some other questions:

- These have all proceeded via arithmetic amplification. Is it possible to succeed in some cases via “family shortening” (as in, e.g., [25])? A natural case to try would be the  $t$ -aspect. Some experiments with  $GL_2$

suggest this is difficult (see [https://ultronozm.github.io/math/20230522T174726\\_\\_shrinking-archimedean-families-second-moment-gl2.html](https://ultronozm.github.io/math/20230522T174726__shrinking-archimedean-families-second-moment-gl2.html)).

“Purely horizontal” aspects remain open, e.g., twists by Dirichlet characters of prime conductor on  $GL_4$ .

- (6) (Extra credit) Create a song of thematic relevance to the lectures. Examples:

- <https://suno.com/playlist/a5ddede1-8601-4cba-acb3-dea7abdc1b81>
- <https://suno.com/song/4b267a84-2268-4fe7-a499-42e56b34526b>,
- <https://suno.com/song/001a15e5-29d0-4fab-98c2-3aaf2257fe92>

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