SOME ALGEBRA-FLAVORED EXERCISES

Abstract. In this evolving note, we record some algebra-flavored exercises relevant for the Oberwolfach seminar.

1. Cyclic and regular matrices

Let F be a field, let V be a finite-dimensional vector space over F, and let $M := \mathrm{End}(V)$ denote the space of linear maps $V \to V$.

Definition 1. Let $\tau \in M$ and $v \in V$. We denote by $F[\tau]v$ the set of elements of V that may be written as a polynomial in τ applied to v, or equivalently, the span of the elements

$$v, \quad \tau v, \quad \tau^2 v, \quad (\dots).$$

We say that a vector $v \in V$ is *τ-cyclic*, or that v is a *cyclic vector* for τ, if

$$F[\tau]v = V.$$

We say that τ is *cyclic* if it admits a cyclic vector.

Exercise 1. Show that τ is cyclic if and only if there is a basis with respect to which it is of the form, e.g., for $\dim(V) = 4$,

$$\begin{pmatrix} 0 & 0 & 0 & * \\ 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}.$$

Exercise 2. Show that the set of conjugacy classes consisting of cyclic elements is in bijection with the set of characteristic polynomials, that is to say:

(i) For each monic polynomial of degree $\dim(V)$, there exists a cyclic element $\tau \in M$ whose characteristic polynomial is that polynomial.
(ii) Two cyclic elements with the same characteristic polynomial are conjugate.

Exercise 3. Show that a matrix given in Jordan form is cyclic precisely when the eigenvalues pertaining to different Jordan blocks are distinct. In particular, a diagonal matrix is cyclic precisely when its diagonal entries are distinct.

Definition 2. We say that $\tau \in M$ is *regular* if $\dim M_\tau = \dim V$, where

$$M_\tau := \{x \in M : x\tau = \tau x\}$$

denotes the *centralizer* of τ in M.

Exercise 4. Show that for $\tau \in M$, the following are equivalent.

(i) τ is cyclic.
(ii) $M_\tau = F[\tau]$.
(iii) τ is regular.

Definition 3. We recall that $\tau \in M$ is *nilpotent* if some power of τ vanishes.

Exercise 5. Suppose that $F = \mathbb{R}$. Fix a norm $\|.\|$ on M. Let

$$\mathcal{O} \subseteq M$$

be a conjugacy class consisting of regular (equivalently, cyclic) elements. Let $x_j \in \mathcal{O}$ be a sequence whose matrix norms $\|x_j\|$ tend to infinity.

(i) Show that, after passing to a subsequence if necessary, the normalized limit

$$x := \lim_{j \to \infty} \frac{x_j}{\|x_j\|}$$

exists, and is nilpotent.

(ii) Show that for each $\mathcal{O}$ as above, there exists a sequence x_j as above for which the normalized limit x is regular nilpotent.

Exercise 6. Suppose that a matrix of the form

$$\tau = \begin{pmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 1 & * & * & * \end{pmatrix}$$

has the property that the standard basis vector e_4 is cyclic. Show that τ is invertible.

Exercise 7. Show that if a matrix is cyclic, then so is its transpose.

Exercise 8. Let $\mathcal{O}$ be a conjugacy class of $n \times n$ matrices. Show that the intersection

$$\mathcal{O}_\psi := \mathcal{O} \cap \begin{pmatrix} * & * & * & * \\ 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{pmatrix}$$

is nonempty if and only if $\mathcal{O}$ consists of regular (equivalently, cyclic) elements, in which case the group

$$N := \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

acts simply-transitively on $\mathcal{O}_\psi$ via conjugation.

References