WINDOWS OF SCALES

Prerequisites. Chapter 52, Section 52.4: window pairs, Theorem 52.4.1, Theorem 52.4.2 and Problem 52.4.1; there the page exponent is written e and the twist i. Chapter 30: the lengths $h_{\ell,e}(F)$ and the sum $V(E,t)$ of Section 30.3. Chapter 54: layers, Lemma 54.3.2, Theorems 54.3.1, 54.4.1 and 54.4.2, with the constant C_H, and Problems 54.1.5 and 54.4.1. Chapter 55: the fields $E(\psi,a,q,d)$, the two signs, and Theorem 55.2.3. Chapter 56: the bad primes $\mathcal{B}_\psi$ and Theorem 56.4.1.

Throughout, ψ is a primitive Dirichlet character of conductor N_ψ and order o, $0<\varepsilon<1/52$, $b=\exp(x^\varepsilon)$, and $a(x)$ is a *page exponent*: an integer with $x^{2+8\varepsilon}\le 2^{a(x)}\le 2x^{2+8\varepsilon}$ for $x\ge 1$. We write $E_x=E(\psi,a(x),q,d)$, and N_x for its modulus $\operatorname{lcm}(2^{a(x)+2},N_\psi)qd$.

57.1. The statement

Chapter 52 states the bound that the argument needs as Theorem 52.4.1; in [Adr] it is a hypothesis. In the Lean formalization it is a theorem, and this chapter outlines its proof.

Theorem 57.1.1 (Selmer bound on windows)**.** *For ψ, ε and a as above there are $J\ge 1$ and X_0 such that for every $X\ge X_0$ and every window pair (q,d) at $X^{1/J}$ there is Y with $X^{1/J}\le Y$ and $Y^2\le X$ such that, for every $x\in[Y,Y^2]$ and $t=\lfloor x^{401}\rfloor$, all groups $\mathrm{Sha}_\ell(E_x,e)$ with $1\le e<t$ are finite and $V(E_x,t)\le\sqrt{b}$.*

The auxiliary primes q, d are fixed through the window while the field E_x changes with x. By Problem 55.1.1,

$$V(E_x,t)=B_2\log 2+B_-+B_+,$$

with

$$B_2=\sum_{1\le e<t}h_{2,e}(E_x),\qquad B_\pm=\sum_{\ell\ \text{odd}}\log\ell\sum_{1\le e<t}h^{\pm}_{\ell,e}(E_x).$$

Theorem 55.2.3 bounds B_- at every scale. Theorem 56.4.1 bounds B_+, given a bound on the layers at the bad primes. The prime 2 and those layers are where the window is needed.

Problem 57.1.1. For $x\ge 1$, $a(x)+2\le 4(1+\log x)$.

Problem 57.1.2. If (q,d) is a window pair at Y_0, then $\log(qd)\le Y_0^{\varepsilon/4}$ and, for each fixed $J\ge 1$, $q,d>2Y_0^{401J}+1$ for all large Y_0, uniformly in the pair.

57.2. The prime two and the bad primes

At $\ell=2$ the field E_x is compared with a layer over a base field that does not depend on x. Let $L_0=A_\psi(i,\sqrt{-q},\sqrt{-d})$.

Problem 57.2.1. $i\in L_0$, $[L_0:\mathbb{Q}]\le 8o$, $L_0\subseteq\mathbb{Q}(\zeta_{4N_\psi qd})$, and $E_x\subseteq L_0(\zeta_{2^{a(x)+2}})$.

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Lemma 57.2.1 (descent at two)**.** *For every $a \geq 2$ there is a layer L of degree 2^m over L_0, with $m \leq a+2$, such that $h_{2,e}(E(\psi,a,q,d)) \leq 1 + h_{2,e}(L)$ for every $e \geq 1$.*

Lemma 57.2.2 (uniform towers)**.** *There is R_2, depending only on ψ, such that for every pair of primes $q \equiv 3 \pmod 8$, $d \equiv 7 \pmod{16}$ prime to N_ψ the hypothesis of Lemma 54.3.2 holds over L_0 with $\ell = 2$ and $R = R_2$. For each odd $\ell \in \mathcal{B}_\psi$ there is R_ℓ such that it holds over $\mathbb{Q}(\zeta_{\ell N_\psi})$ with $R = R_\ell$.*

Status: both lemmas are proved in the Lean formalization, the second from Theorem 54.3.1; we have read their statements only. We have not followed how the formalization makes R_2 independent of q and d; the congruences on q and d are among its hypotheses.

Problem 57.2.2. Let $j_0 = \lfloor \log_2 R_2 \rfloor + 3$ and L' the layer of degree 2^{j_0} over L_0. Then $[L' : \mathbb{Q}] \leq 8o\,2^{j_0}$, and $\log(2|\Delta_{L'}|) \leq c_1(1 + \log(qd))$ with c_1 depending only on ψ.

57.3. The pigeonhole on scales

By Theorem 54.4.2, for $\ell = 2$ over L_0, and for each $\ell \in \mathcal{B}_\psi$ over $\mathbb{Q}(\zeta_{\ell N_\psi})$, the twists $e \leq Z$ split into at most R_ℓ exceptional ones and the rest. At the exceptional twists only Theorem 54.4.1 is available, and by Problem 54.4.1 it is affordable only for $e \leq x^\eta$, with $\eta = \varepsilon/(2C_H)$. The twists that matter at scale x are those with $e \leq x^{C_0}$. So one needs scales at which no exceptional twist lies in $(x^\eta, x^{C_0}]$.

Problem 57.3.1. Let $R \geq 0$, $B \geq 1$, $U > 0$, and T a set of at most R real numbers. There is y with $U/(2^{R+2}B^R) \leq y \leq U/2$ such that for every $\alpha \in T$ and $u \in [y, 2y]$, $\alpha \leq u$ implies $B\alpha \leq u$.

Problem 57.3.2. Let $0 < \eta \leq C_0$, $R \geq 0$, $X > 1$, $\mathcal{E}$ a set of at most R positive integers, and $J = 2^{R+2}(C_0/\eta)^R$. There is Y with $X^{1/J} \leq Y$ and $Y^2 \leq X$ such that for every $x \in [Y, Y^2]$ and $e \in \mathcal{E}$, $e \leq x^{C_0}$ implies $e \leq x^\eta$.

Problem 57.3.3. In Problem 57.3.2 let $L = \log X$ and $u_i = (L/2)(\eta/(2C_0))^i$ for $0 \leq i \leq R$. The intervals $(\eta u_i, 2C_0 u_i]$ are disjoint, one of them contains no $\log e$ with $e \in \mathcal{E}$, and $Y = e^{u_i}$ for that i satisfies the conclusion with J replaced by $J/2$.

Problem 57.3.4. For $R = 1$, $\mathcal{E} = \{1000\}$, $C_0 = 2$, $\eta = 1/2$ and $X = e^{20}$, the proof of Problem 57.3.1 by induction on R gives $Y = e^{5/4}$; here $J = 32$, and $x^2 \leq e^5 < 1000$ for every $x \in [Y, Y^2]$.

Problem 57.3.5. One window serves all the primes at once: if $\mathcal{E}_2$ and $\mathcal{E}_\ell$, $\ell \in \mathcal{B}_\psi$, have at most R_2 and R_ℓ elements, Problem 57.3.2 applies to their union with $R = R_2 + \sum_{\ell \in \mathcal{B}_\psi} R_\ell$. With $\eta = \varepsilon/(2C_H)$ the exponent is $J = 2^{R+2}(2C_H C_0/\varepsilon)^R$.

57.4. The bounds on a window

Fix $C_0 \geq 1$ and a function t with $2 \leq t(x) \leq x^{C_0}$ for $x \geq 2$. Let X be large, $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$ primes prime to N_ψ with $\log(qd) \leq X^{\varepsilon/(4J)}$, $Z = \lfloor X^{C_0} \rfloor$, and Y as in Problem 57.3.5 for the exceptional sets of Theorem 54.4.2 with this Z. Let $x \in [Y, Y^2]$.

Problem 57.4.1. Let $e \leq t(x)$ be non-exceptional for $\ell = 2$, and L a layer of degree 2^m over L_0 with $m \leq a(x) + 2$. Then $h_{2,e}(L) \leq C(1 + \log x)$, with C independent of x, X, q and d.

Problem 57.4.2. Let $e \le t(x)$ be exceptional for $\ell = 2$, and L as in Problem 57.4.1. Then $h_{2,e}(L) \le C(1+\log x) + 2\exp(Cx^{3\varepsilon/4})$.

Problem 57.4.3. For every C and all large x, $x^{C_0}\big(1 + C(1+\log x) + 2\exp(Cx^{3\varepsilon/4})\big) \le \exp(x^{4\varepsilon/5})$.

The same bounds hold, more easily, at $\ell \in \mathcal{B}_\psi$ over the fixed layer of degree ℓ^{j_ℓ}, $j_\ell = \lfloor \log_\ell R_\ell \rfloor + 3$, over $\mathbb{Q}(\zeta_{\ell N_\psi})$, whose discriminant does not depend on q and d. Problems 57.1.1, 57.2.2 and 57.3.1–57.4.3, Lemmas 57.2.1 and 57.2.2, and Theorem 54.4.2 prove the following.

Theorem 57.4.1 (windows at two and at the bad primes)**.** *Let $C_0 \ge 1$, $A_1 > 0$, $R = R_2 + \sum_{\ell \in \mathcal{B}_\psi} R_\ell$ and $J = 2^{R+2}(2C_H C_0/\varepsilon)^R$. There is X_0 such that for all functions a, t with $2 \le a(x)$, $2^{a(x)} \le x^{A_1}$ and $2 \le t(x) \le x^{C_0}$ for $x \ge 2$, every $X \ge X_0$ and all primes $q \equiv 3 \pmod 8$, $d \equiv 7 \pmod{16}$ prime to N_ψ with $\log(qd) \le X^{\varepsilon/(4J)}$, there is Y with $X^{1/J} \le Y$ and $Y^2 \le X$ such that for every $x \in [Y, Y^2]$*

$$\sum_{1 \le e < t(x)} h_{2,e}(E_x) \le \exp(x^{4\varepsilon/5}),$$

and the same bound holds for $\sum_{1\le e<t(x)} h_{\ell,e}$ of the layer of degree ℓ^{j_ℓ} over $\mathbb{Q}(\zeta_{\ell N_\psi})$, for every $\ell \in \mathcal{B}_\psi$.

Status: proved in the Lean formalization, as just described. Problem 57.1.1 uses the page exponent; for general a the bound $m \le a + 2 \le C(1+\log x)$ follows from $2^{a(x)} \le x^{A_1}$ in the same way. Zhang states the result as [Zh, Prop. 5.3], with the window lemma [Zh, Lemma 5.5].

57.5. The assembly

Let $t = \lfloor x^{401} \rfloor$, and let x lie in the window of Theorem 57.4.1 with $C_0 = 401$ and $A_1 = 4$; these are admissible since $2x^{2+8\varepsilon} \le x^4$ for $x \ge 2$. By Problem 52.4.1, $\log(N_x t) \le 2x^{\varepsilon/4}$ for large x.

Problem 57.5.1. The right side of Theorem 55.2.3 for E_x and t is at most $\exp(Cx^{\varepsilon/4})$, at every large scale x.

Problem 57.5.2. Theorem 56.4.1 applies on the window, with $B = \exp(x^{4\varepsilon/5})$, and gives $B_+ \le \exp(2Cx^{\varepsilon/4}) + C\exp(x^{4\varepsilon/5})$.

Problem 57.5.3. On the window, all $\mathrm{Sha}_\ell(E_x, e)$, $1 \le e < t$, are finite and $V(E_x, t) \le \sqrt{b}$ for all large x.

Problems 57.1.2, 57.5.1–57.5.3 and Theorem 57.4.1 prove Theorem 57.1.1. *Status*: this is how the Lean formalization proves Theorem 57.1.1; the inputs it uses are those listed in Chapters 54, 55 and 56, with their statuses there. In [Adr] the statement is assumed as a hypothesis, and Chapter 52 records it as such.

Why windows. The minus part and the plus part away from $\mathcal{B}_\psi$ are bounded at every scale. At $\ell = 2$ and at the finitely many bad primes, the vanishing of μ makes the lengths along a cyclotomic tower linear in the layer, except at boundedly many twists ℓ-adically close to fixed centres (Theorem 54.4.2). At those twists only the doubly exponential bound for H^2 is known, which is affordable only for $e \le x^{\varepsilon/(2C_H)}$ (Problems 54.4.1 and 54.4.2). The exceptional twists do not move with x, so the scale is moved instead: a pigeonhole on logarithmic scales finds $[Y, Y^2]$ free of them,

at the price of the exponent J, which grows like $(C/\varepsilon)^R$. Chapter 52 then passes from $[Y, Y^2]$ to ranges $[Y, Y^{3/2}]$ (Theorem 52.4.3); in the formalization a similar passage, to families of scales with parameter $3/2$, is made by a step we have not followed.

In the Lean formalization. In [Lean] Theorem 57.1.1 is `VWin` (`VWinAt` ψ 401 for every ψ), with `VAt`, `WinPair`, `PageN` and `IsChiField`; it is proved by `UpperTheory.One.vwin_of_row` through `HV.Core.of_four_alt`, `vwin_of_counts` and `vwinAt_of_joint`, whose last steps are `vvle_of_parts`, `admissible_of_pageN`, `log_mul_le_rpow_of_winPair`, `eventually_winPair_gt_power` and `error_terms_le_sqrt_exp_rpow`. Theorem 57.4.1 is `ThmBab`, proved by `Win.thmBab_of` from `partA` ($\ell = 2$) and `partB` ($\ell \in \mathcal{B}_\psi$); Problems 57.3.1 and 57.3.2 are `Win.window1` and `Win.window`, the conclusion being `Good`; Problem 57.4.3 is `Win.final`. Lemma 57.2.1 is `TwoBase`, and Lemma 57.2.2 is `TwoIn` and `FixedIn`, from `towerTwo` and `towerFixed`. The bound is passed on by `supplyChi`, as `HostExW` $3/2$.

Hints

§57.1.

57.1.1: $\varepsilon < 1/52$ gives $(2 + 8\varepsilon)/\log 2 < 3.2$.

57.1.2: The size conditions in the definition of a window pair; $Y_0^{\varepsilon/20}$ eventually exceeds $2(1 + \log^2 Y_0)$.

§57.2.

57.2.1: $q \equiv d \equiv 3 \pmod 4$, so $\sqrt{-q} \in \mathbb{Q}(\zeta_q)$ and $\sqrt{-d} \in \mathbb{Q}(\zeta_d)$; compare the four factors of E_x.

57.2.2: Problem 54.1.5 with $N_0 = 4N_\psi$ and $P = qd$.

§57.3.

57.3.1: Induction on R.

57.3.2: Problem 57.3.1, with $u = \log x$.

57.3.3: $2C_0 u_{i+1} = \eta u_i$.

57.3.5: The cardinality of a union.

§57.4.

57.4.1: Theorem 54.4.2 with $\ell = 2$ and Problem 57.1.1; $\log Z \le C_0 J \log x$, and $2^{c_0} \le 16o$ by Problem 54.1.3.

57.4.2: Theorem 54.4.2, Problems 57.2.2 and 54.4.1; the window gives $e \le x^{\varepsilon/(2C_H)}$.

57.4.3: $4/5 > 3/4$.

§57.5.

57.5.1: $[E_x : \mathbb{Q}] \le N_x$.

57.5.2: Problem 57.1.2 for the size conditions on q and d; Theorem 57.4.1 for the layers.

57.5.3: Theorem 57.4.1 with Lemma 57.2.1, and Problems 57.5.1, 57.5.2; all exponents are below ε.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

[Lean] The Lean community, *riemann-hypothesis*, a Lean formalization of Alpöge's argument, `https://github.com/leanprover-community/riemann-hypothesis`.

[Zh] R. Zhang, *A preliminary note on Alpöge's proposed proof of the generalized Riemann hypothesis*, 22 November 2026, `https://hexagonmath.org/pdf/2611.00809v1`.