LOCAL EXPONENTS OF THE LIFTS

Prerequisites. We use the notation of Chapter 52, Section 52.1: the scale x, $y = x^\varepsilon$ with $0 < \varepsilon < 1/52$, $b = e^y$, $a = xb$, the CM type Φ of E with $f = 2n$ elements, the field L with its places w, weights ℓ_w and ramification indices e_w, the level primes $g_\iota = \pi_\iota \bar{\pi}_\iota$, the terminal prime $T_u = \varpi_u \bar{\varpi}_u$, the total weight $A = \sum_t a_t$ of the tops $t \in \mathcal{T}$, and $B_0 = fmby$. Recall $\sum_{w|p} \ell_w e_w = \log p$, and Problem 52.1.1: if $\pi \in \mathcal{O}_K$ and $\pi\bar{\pi} = g$ is prime, then $\sum_{w|g} \ell_w v_w(\tau\bar{\pi}) = \frac{1}{2} \log g$ for every automorphism τ of L. Section 53.4 uses Legendre's formula for the exponent of p in $n!$ and Chebyshev's bound $\sum_{p \le n} \log p \le n \log 4$ [AZ, Ch. 2].

Chapter 52 states the bound for the local terms, Theorem 52.1.1, in the form this chapter proves. In [Adr, Thm. 16.6] its inputs are size bounds for the numbers κ_j, δ_j, ε_u^T, ε_j^d and $\varepsilon_{j,p}^{\mathrm{in}}$ [Adr, (E1)–(E6), Hyps. 16.13–16.14]. In the Lean formalization [Lean] these numbers are given by formulas. What are the formulas, and why are the bounds true? Almost everything turns out to be elementary: a congruence at the terminal prime, a count of rows at the level primes, Minkowski's bound for finite groups, and, at the inert primes, a linear programme along Frobenius orbits summed with Mertens' estimate.

53.1. The exponent at each place

Besides the scale, the data have integer ranks $k \ge 2$, $m \ge 1$, $t \ge 0$ with $N_i = k + i(t+1)$, weights $w_j(\rho) \ge 0$ for $1 \le j \le m$ and $\rho \in \Phi$, a width Θ of the windows in which these weights lie, and a size parameter σ. Put $X' = \exp(x^{\varepsilon/3})$, $b' = a/X'$ and $X'' = xnX'$. In the formalization the elements of Φ restrict to the identity on K. The lifts gain only at primes of two classes, called class I and class II below, both consisting of primes inert in K.

The exponent $E_{u,j}(w)$ at a place w with residue characteristic p is defined as follows. If $p = T_u$, it is ε_u^T when $v_w(\varpi_u) > 0$ and 0 otherwise, with

$$\varepsilon_u^T = 2 \sum_{\rho \in \Phi} W_u(\rho) + fN_m + \tfrac{1}{4}|\mathcal{T}|(t+1)(m+1).$$

Here $W_u(\rho)$ is the largest of the numbers $|\lambda - \lfloor i(t+1)/2 \rfloor|$ over the entries λ at ρ of the highest weights of the chain u at the levels $i \le m$, the top entry at the top places excepted. If $p \ne T_u$, then $E_{u,j}(w) = v_p(D_\Gamma) + E'$, where

$E' = -\frac{1}{2}\Delta_j + e_w^{-1} \sum_\rho n_{u,j}(\rho) v_w(\tau_\rho \bar{\pi}_\iota) + \kappa_j$ with $\kappa_j = c_g f(N_m^2 + m^2\Theta)$ if $p = g_\iota$;
$E' = \varepsilon_j^d$ if $p = d$;
$E' = \varepsilon_{j,p}^{\mathrm{cl}} = 3fN_m^2 \log_p a + (p^2-1)^{-1} \sum_\rho w_j(\rho)$ if $p \le X''$ is of class I and its Frobenius label h_p of Section 53.4 vanishes;
$E' = \varepsilon_{j,p}^{\mathrm{orb}}$, defined in Section 53.4, for the other $p \le X''$ of classes I and II;
$E' = 0$ otherwise.

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Here Δ_j is the Fock degree, $n_{u,j}(\rho)$ the exponent of $\bar\pi_\iota$ at ρ in the j-th lift, $c_g \geq 10$ a constant, and ε_j^d an explicit expression in the weights at the two places of Φ above each column. This is the shape of [Adr, Def. 16.5], with its free numbers now given by formulas.

Sizes. The family of data satisfies size conditions. Those we use are:

(S1) $0 \leq w_j(\rho) \leq 3b$ and $W_u(\rho) \leq 5b$;
(S2) $N_m^2 \leq \Theta$, $4N_m \leq \Theta$, $1 \leq y \leq \Theta$, $f\Theta \leq b$, $X''\Theta \leq b$ and $X''N_m^2 \leq b$;
(S3) $2 \leq m \leq f$, $|\mathcal{T}| \leq f$, $(t+1)(m+1) \leq b$, $kb' \leq b$ and $k(b'+\Theta) \leq mb$;
(S4) $\log g_\iota \leq C_Y y$ and $\log d \leq y$;
(S5) $\log T_u \leq C_T m y$, and D_Γ divides Minkowski's number $M(2fN_m)$, where $M(r) = \prod_p p^{\mu_p(r)}$ with $\mu_p(r) = \sum_{i\geq 1} \lfloor r/(p^{i-1}(p-1)) \rfloor$.

Minkowski proved that the order of every finite subgroup of $\mathrm{GL}_r(\mathbb{Z})$ divides $M(r)$ [Se, §1]. *Status*: in the Lean formalization (S1)–(S4) follow from the size conditions imposed on the family, and (S5) is part of the definition of the class of families to which the main argument is applied; we have not traced how the constructed family is shown to satisfy them. Consequently $X'' \leq b \leq a$ and $fN_m \leq b$.

Problem 53.1.1. Let $\varpi \in \mathcal{O}_K$ with $\varpi \equiv 1 \pmod{4\mathcal{O}_K}$, and suppose $T = \varpi\bar\varpi$ is prime. Then $T \equiv 1 \pmod 4$, the trace $\varpi + \bar\varpi$ is $\equiv 2 \pmod 4$, and $T \nmid \varpi + \bar\varpi$. For $K = \mathbb{Q}(\mathrm{i})$ and $\varpi = 1 + 4\mathrm{i}$ one has $T = 17$ and trace 2.

Problem 53.1.2. In Problem 53.1.1, at each place $w \mid T$ of L exactly one of $v_w(\varpi)$, $v_w(\bar\varpi)$ is positive, and it equals e_w. For $\varpi = 1 + \mathrm{i}$, which is not $\equiv 1 \pmod 4$, both are positive at the place above 2.

In the construction $\varpi_u \equiv 1 \pmod{Q_S qd(g_0 g_1)^2}$ [Adr, Rem. 16.8], where Q_S is the part of the conductor of E prime to qd; it is divisible by 4, since in the formalization $2^{e+2} \mid Q_S$ for an exponent $e \geq 2$ of the 2-power tower. So Problems 53.1.1 and 53.1.2 apply to T_u.

Problem 53.1.3. Let S_u be the set of places $w \mid T_u$ with $v_w(\varpi_u) > 0$. Then $\sum_{w\in S_u} \ell_w e_w = \frac{1}{2}\log T_u$, and the places above T_u contribute exactly $\varepsilon_u^T \log T_u$ to $\sum_w 2\ell_w e_w E_{u,j}(w)$.

[Adr, Thm. 16.6] bounds this contribution by $2\varepsilon_u^T \log T_u$; the congruence halves it.

Problem 53.1.4. Under (S1), (S3) and $N_m \leq b$, one has $\varepsilon_u^T \leq 12fb$. With (S5), the places above T_u contribute at most $12C_T B_0$.

53.2. The level primes

By Problem 52.1.2, the places above g_ι contribute $(2v_{g_\iota}(D_\Gamma) - \Delta_j + 2\kappa_j + n_{u,j}) \log g_\iota$, where $n_{u,j} = \sum_\rho n_{u,j}(\rho)$. The main term $A \log g_\iota$ appears once we know that $n_{u,j} - \Delta_j$ is close to A.

The exponent $n_{u,j}(\rho)$ is a sum of N_{j-1} row exponents, read off from the highest weights of the j-th lift. There are $k-2$ middle rows, each with exponent within $\sigma + N_m$ of $w_j(\rho)$; $N_{j-1} - k$ block rows, each with exponent in $(0, 2\Theta]$; and two end rows. The two end rows together have exponent close to $w_j(\rho) + a(\rho) + w_j'(\rho)$. Here $a(\rho) = a_t$ and $w_j'(\rho) = 0$ if ρ is the top place of $t \in \mathcal{T}$, and $a(\rho) = 0$, $w_j'(\rho) = w_j(\rho)$ if ρ is not a top place. Each top t has exactly one top place, and distinct tops have

distinct top places. On the other side $\Delta_j = \sum_\rho \Delta_j(\rho)$, where $\Delta_j(\rho)$ is a sum of $k+m-1$ absolute values, $k-1$ of them close to $w_j(\rho)$ and the others small.

Lemma 53.2.1 (degree identity)**.** *For every u and $1 \le j \le m$,*

$$|n_{u,j} - \Delta_j - A| \le \delta_j, \qquad \delta_j = 2\sum_\rho w_j(\rho) + 40\big(f(k\sigma + mt\Theta + kN_m) + |\mathcal{T}|b'/x\big),$$

and $\delta_j \le 166fb$ under the size conditions.

Status: proved in the Lean formalization by the row-by-row comparison just described. The row exponents and the Fock degree are given by formulas in the construction of the lifts, which we have not re-derived. The corresponding step in [Adr] is the degree part of [Adr, Hyp. 16.14], assumed there.

Problem 53.2.1. Suppose that for every ρ

$$\big|n_{u,j}(\rho) - (k-1)w_j(\rho) - a(\rho) - w_j'(\rho)\big| \le e_1(\rho), \qquad \big|\Delta_j(\rho) - (k-1)w_j(\rho)\big| \le e_2(\rho).$$

Then $|n_{u,j} - \Delta_j - A| \le \sum_\rho \big(w_j(\rho) + e_1(\rho) + e_2(\rho)\big)$.

Problem 53.2.2. Assume (S2)–(S4) and Lemma 53.2.1. Then $\kappa_j \log g_\iota \le 2c_g C_Y B_0$, and $\delta_j \log g_\iota \le 166 C_Y B_0$.

Problem 53.2.3. For $\iota = 0, 1$,

$$\Big|\sum_{w \mid g_\iota} 2\ell_w e_w E_{u,j}(w) - A\log g_\iota\Big| \le (\delta_j + 2\kappa_j + 2v_{g_\iota}(D_\Gamma))\log g_\iota.$$

53.3. The other places

Problem 53.3.1. For every $r \ge 1$, $M(r)$ divides $(2r)!$, so $\log M(r) \le 2r\log(2r)$. For instance $M(1) = 2$ and $M(2) = 24$.

Problem 53.3.2. Assume (S3), (S5) and $fN_m \le b$. Then $\log D_\Gamma \le 12 fN_m \log(fN_m)$ and $2\log D_\Gamma \le 24 B_0$.

Lemma 53.3.1. *Under the size conditions, $|\varepsilon_j^d| \le |\mathcal{T}|b + fkb'$.*

Status: proved in the Lean formalization, from the windows for the weights at the two places above each column.

Problem 53.3.3. Under (S3), (S4) and Lemma 53.3.1, $2\log d\,|\varepsilon_j^d| \le 4B_0$.

Write P^{in} for the primes $p \le X''$ of classes I and II, and $\varepsilon_{j,p}^{\mathrm{in}}$ for $\varepsilon_{j,p}^{\mathrm{cl}}$ or $\varepsilon_{j,p}^{\mathrm{orb}}$, as in Section 53.1.

Problem 53.3.4. For every u and j,

$$\sum_{w \nmid g_0 g_1} 2\ell_w e_w |E_{u,j}(w)| \le \varepsilon_u^T \log T_u + 2\log D_\Gamma + 2\log d\,|\varepsilon_j^d| + \sum_{p \in P^{\mathrm{in}}} 2\log p\,|\varepsilon_{j,p}^{\mathrm{in}}|.$$

53.4. Orbit losses at inert primes

It remains to bound the last sum in Problem 53.3.4; this is [Adr, Hyp. 16.13] together with [Adr, Lemma 16.12]. The $f = 2n$ places of Φ are labelled by pairs $z = (\xi, h) \in \{0,1\} \times H$, where H is an abelian group of order n indexing the columns; we write $w_j(z)$ for the weight at the place labelled z. The tops are elements of H, and the top places are the labels $(0,t)$ with $t \in \mathcal{T}$. For p of class I or II there is an element $h_p \in H$, read off from the Frobenius at p, and Frobenius acts on labels by

$$\phi_p(\xi, h) = (\xi + \xi_p,\ h + h_p), \qquad \xi_p = 0 \text{ for class I}, \quad \xi_p = 1 \text{ for class II},$$

with ξ taken modulo 2. Let n_p be the order of ϕ_p.

Problem 53.4.1. For p of class I, n_p is the order of h_p; for p of class II, $n_p = 2\,\mathrm{ord}(2h_p) = \mathrm{lcm}(2, \mathrm{ord}\, h_p)$. Every orbit of ϕ_p on labels has exactly n_p elements.

The loss along an orbit is the defect of a linear programme. For an integer $l \geq 2$, an integer $d \geq 1$ and numbers $D_s \geq 0$ indexed by $s \in \mathbb{Z}/d$, let $V(l,d,D)$ be the supremum of $\sum_{s,i} x_{s,i}$ over the real $x_{s,i} \geq 0$, with $s \in \mathbb{Z}/d$ and i odd, $0 < i < d$, subject to

$$\sum_i l^i x_{s,i} + \sum_i x_{s-i,i} \leq D_s \qquad \text{for every } s \in \mathbb{Z}/d.$$

Put $F(l,d,D) = \frac{l}{l^2-1} \sum_s D_s - V(l,d,D)$. The factor $l/(l^2-1)$ is the gain per unit weight at an inert prime l, Chapter 18.

Problem 53.4.2. $V(l,d,D) \leq (l+1)^{-1} \sum_s D_s$, hence $0 \leq F(l,d,D) \leq \frac{l}{l^2-1} \sum_s D_s$.

Problem 53.4.3. If $d = 1$, then $F(l,1,D) = lD_0/(l^2-1)$. If $d = 2$ and $D_0 = D_1 = D$, then $V = 2D/(l+1)$ and $F(l,2,D) = 2D/(l^2-1)$.

For a label z put $F_{j,p}(z) = F(p, n_p, D)$ with $D_s = (k-1) w_j(\phi_p^s z)$. Let Z_p be the set of labels with $\xi = 0$ if p is of class I, and $Z_p = \emptyset$ if p is of class II; let Z_p' be its complement and $Z_p'^{\mathrm{top}}$ the top places in Z_p'. Let P^{odd} be the set of primes of class I whose $h_p \neq 0$ has odd order, and $\gamma_p = p/(p^2-1)$. Then

$$\varepsilon_{j,p}^{\mathrm{orb}} = \max\Big(0, \frac{1}{n_p} \sum_{z \in Z_p} F_{j,p}(z)\Big) + \gamma_p |Z_p'^{\mathrm{top}}| (b + \Theta + 1 + 3\sigma) + \frac{12}{p} \sum_{z \in Z_p'} w_j(z)$$

$$+ |Z_p'| N_{j-1} N_j \big(\log_p(4 n_p N_m a) + 1\big) + 5 |Z_p'| N_m^2 \log_p(2a) + 4 |Z_p'^{\mathrm{top}}| (\log_p 2a)^2 + \varepsilon_{j,p}^{\mathrm{odd}},$$

where $\varepsilon_{j,p}^{\mathrm{odd}} = (k-1) p^{-3} \sum_{z \in Z_p'} w_j(z) + 12 f N_m^2 \log_p a$ for $p \in P^{\mathrm{odd}}$, and $\varepsilon_{j,p}^{\mathrm{odd}} = 0$ otherwise. The first term is the orbit loss; the others are errors.

Lemma 53.4.1. *Under the size conditions, for every prime $p \leq b$ of class I or II,*

$$|\varepsilon_{j,p}^{\mathrm{orb}} - \varepsilon_{j,p}^{\mathrm{odd}}| \log p \leq 126 f m b \frac{\log p}{p} + 46 f \Theta y, \qquad |\varepsilon_{j,p}^{\mathrm{odd}}| \log p \leq 9 f b k \frac{\log p}{p^3} + 24 f \Theta y,$$

and $k \sum_{p \in P \cap P^{\mathrm{odd}}} (\log p)/p^3 \leq 2$ for every finite set P of primes.

Status: proved in the Lean formalization. The first bound treats the seven terms one at a time; its main term is Problem 53.4.4 below. The last bound rests on a congruence $p \equiv \pm 1 \pmod{2^e}$ for the primes of P^{odd}, with 2^e large compared to k; we have not followed its proof.

Problem 53.4.4. Suppose $w_j(z) \le 3(b' + \Theta)$ whenever z has $\xi = 0$, and assume (S3). Then for $p \ge 2$ of class I or II,

$$\max\Big(0, \frac{1}{n_p} \sum_{z \in Z_p} F_{j,p}(z)\Big) \le \frac{6fmb}{p}.$$

Problem 53.4.5. For every integer $N \ge 1$, $\sum_{p \le N} (\log p)/p \le \log N + \log 4$.

Problem 53.4.6. Assume (S2), (S3), Lemma 53.4.1 and $X'' \le b$. For every set P of primes $p \le X''$ of classes I and II, $\sum_{p \in P} 2 \log p \, |\varepsilon^{\mathrm{orb}}_{j,p}| \le 848 B_0$.

Problem 53.4.7. For $x \ge 1$, $\log a \le (1/\varepsilon + 1)y$.

Problem 53.4.8. Assume (S1) and (S2), and $X'' \le a$. For every set P of primes $p \le X''$, $\sum_{p \in P} 2 \log p \, \varepsilon^{\mathrm{cl}}_{j,p} \le 18(1/\varepsilon + 1) B_0$.

Problems 53.4.6 and 53.4.8 bound the last sum of Problem 53.3.4 by $(848 + 18/\varepsilon + 18)B_0$. The constant depends on ε, as in [Adr, Rem. 16.9].

53.5. The theorem

Problems 53.1.3–53.2.3, 53.3.2, 53.3.3, 53.3.4, 53.4.6 and 53.4.8 prove the following.

Theorem 53.5.1 (local terms)**.** *Let $0 < \varepsilon < 1/52$, and consider a family of data satisfying the size conditions, Lemmas 53.2.1, 53.3.1 and 53.4.1. There are C and x_0 such that for every datum at a scale $x \ge x_0$, every u and every $1 \le j \le m$,*

$$\Big|\sum_w 2\ell_w e_w E_{u,j}(w) - A \log(g_0 g_1)\Big| \le C B_0, \qquad \sum_{w \nmid g_0 g_1} 2\ell_w e_w |E_{u,j}(w)| \le C B_0.$$

Status: proved in the Lean formalization, for the family it constructs. The first inequality is the bound of [Adr, Thm. 16.6] for these exponents. The corresponding steps in [Adr] are [Adr, Hyps. 16.13 and 16.14], assumed there; here Lemmas 53.2.1, 53.3.1 and 53.4.1 take their place. The second inequality is not stated in [Adr]. Summing over j gives (A5), $E_u \le mAg + e_5$, with $e_5 = CmB_0$, as in Problem 52.1.3.

In the Lean formalization. In [Lean] the exponent is `Eval`, with the pieces `EatT`, `EatG`, `EatD`, `EatC`, `EatI`, `EatX`, the orbit loss `FOrb` and the programme `lpF`. The two inequalities of Theorem 53.5.1 are `LocalQuadraticParts.adm11`; the first is a clause of `LinkDIV_at`, supplied by `linkDIV_at_words`. They are assembled by `secondComplement_of_split` from `C2atG_of` (Problem 53.2.3) and `C2off_of` (Problem 53.3.4). The family version is `familySecondComplementInputs_of_tuSize_of_dGamSize`, whose hypotheses `TuSize` and `DGamSize` are (S5). Its inputs are: `typePrime_valuationSum_eq_logGap_all` (Problem 52.1.1); `not_dvd_trace`, `coprime_conj`, `sum_over_eq`, `ETpi_le` and `admissible_off_type_level_budget` (Problems 53.1.1–53.1.4); `admissibleDegree_count_at_parameter_const`, via `degree_fine`, `nuAbs_place_fine` and `letters_place` (Lemma 53.2.1); `typePrime_sizes_le_budget_single` (Problem 53.2.2); `dGamSize_of_dvd` and `log_minkM_le_two` (Problems 53.3.1 and 53.3.2); `abs_EatD_le` and `offDiagonalRamified_bound_const_four` (Lemma 53.3.1, Problem 53.3.3); `factLpF` (Problem 53.4.2), `first_term`, `eatI_bound`, `eatX_bound`, `sum_cube_le`, `prime_log_sum_bound` and `sum_eatI` (Lemma 53.4.1, Problems 53.4.4–53.4.6); `bound_one` and `sum_primes_le` (Problem 53.4.8). They are combined in `inertPrimes_loss_log_sum_admissible`.

In the formalization the orbit losses $F_{j,p}(z)$ also enter other local terms of the lifts, and a separate loss term enters the gain inequality at inert primes. We have not traced how these are related to the orbit losses of this chapter.

Hints

§53.1.

53.1.1: Write $\varpi = 1+4\beta$, and compare a nonzero multiple of T with $|\varpi+\bar{\varpi}| \le 2\sqrt{T}$.
53.1.2: The trace and T are coprime integers, and $v_w(\varpi) + v_w(\bar{\varpi}) = v_w(T)$.
53.1.3: Problem 52.1.1 for $\pi = \bar{\varpi}_u$, and Problem 53.1.2.
53.1.4: Bound the three terms of ε_u^T separately.

§53.2.

53.2.1: Sum over ρ; use $\sum_\rho a(\rho) = A$ and $0 \le w_j' \le w_j$.
53.2.2: (S2) and (S3).
53.2.3: Problem 52.1.2 and Lemma 53.2.1.

§53.3.

53.3.1: Compare $\mu_p(r)$ with Legendre's formula for $v_p((2r)!)$, term by term.
53.3.2: Problem 53.3.1 with $r = 2fN_m$; then $\log 4 \le 2\log(fN_m)$.
53.3.4: Group the places by residue characteristic; use Problem 53.1.3 at T_u, and $\sum_p v_p(D_\Gamma)\log p \le \log D_\Gamma$.

§53.4.

53.4.1: Compute ϕ_p^e.
53.4.2: Sum the constraints over s.
53.4.3: For $d = 2$ the only i is 1; use Problem 53.4.2 for the upper bound.
53.4.4: Problem 53.4.2 for each z, with $\gamma_p \le 2/p$; there are at most f labels.
53.4.5: Legendre's formula for $\log N!$.
53.4.6: Problem 53.4.5 with $N = \lfloor X'' \rfloor$, $|P| \le X''$, and $m \ge 2$.
53.4.7: $\log u \le u$.
53.4.8: Problem 53.4.7, and $\sum_{n\ge 2} n^{-2} \le 1$.

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