# LOWER SLOPE BOUNDS ON COVERS

Prerequisites. From Chapter 6: $\mu^\sharp$ and Problem 6.3.3. From Chapter 26, *Covolumes of lattices of sections*: the lattices $E_D$, Problems 26.1.1 and 26.1.2, Theorem 26.2.1 and formula (1). From Chapter 49: Lemma 49.3.1 and the notation of Section 49.4 (the lift $j$, $y = x^\varepsilon$, $b = e^y$, the error unit $u = B_0 \log\log x \geq 1$, the main term $T_j$, and $\mu_{\mathrm{core}}$). From Chapter 50: Theorem 50.3.2, and the notation $d$, $c_\chi$, $E$, $\mathrm{rd}_E$ of Section 50.1. Heights of points and arithmetic intersection numbers are recalled in Section 51.3 with references.

Chapter 49 reduced the covolume estimate to a lower bound $\mu_{\mathrm{core}} \geq -T_j - C_L u$ for the slope of a lattice of holomorphic cusp forms on the Shimura variety $S_j$ of the $j$th lift. How can one bound a slope from below with an error that is a fixed multiple of $u$, uniformly in the point and the lift? In the Lean formalization [Lean], for $j < m$, one passes to a cover of $S_j$, changes the weight, applies an effective arithmetic Hilbert–Samuel inequality on each component, and descends. The last lift, $j = m$, has $q_m = 0$ and is treated separately; we have not read that proof.

Throughout, $F$ is a number field, the field of definition of the cover; for a lattice $\Lambda$ of sections we write $\widehat{\deg}\Lambda := \deg \Lambda/[F:\mathbb{Q}]$, so that $\mu^\sharp(\Lambda) = 2\widehat{\deg}\Lambda/\operatorname{rk}\Lambda$.

## 51.1. Hilbert–Samuel on one component

Let $Z$ be an integral projective arithmetic variety over $\mathcal{O}_F$ whose generic fibre has dimension $\nu$, and $L$ a line bundle on $Z$ with a smooth Hermitian metric invariant under complex conjugation, ample on the generic fibre. Write $\delta_Z := \deg L_F$ and, following Zhang [Zha],

$$h_Z := \frac{\hat{c}_1(L)^{\nu+1}}{(\nu+1)\,[F:\mathbb{Q}]\,\delta_Z}.$$

For an integer $T \geq 1$ give $H^0(Z, L^{\otimes T})$ its $L^2$ norms; we use the inequality

$$\widehat{\deg} H^0(Z, L^{\otimes T}) \geq \binom{T-1}{\nu} \delta_Z \left( T h_Z - 3(\nu+1)|h_Z| - A\log(2+|h_Z|) - R \right), \quad (1)$$

where $A > 0$ and $R \geq 0$ are numbers with $A \log\max(A,1) + R \leq C_2 u$.

*Status*: in the formalization (1) is proved, for the line bundles and components that occur and for the tensor power $T$ chosen there, from published theorems: Kawamata–Viehweg and Nadel vanishing, principalisation, toroidal compactification, and the arithmetic Hilbert–Samuel theorem; the proof also uses Chow forms (Chapter 27). We have read its statement, not its proof, and we have not checked that the formalization's height is normalized as $h_Z$ above. Problem 51.1.2 shows that the leading terms agree with this normalization. Unlike Theorem 26.2.1, (1) is effective: it holds at one $T$, with all errors explicit.

**Problem 51.1.1.** For $T > \nu \geq 1$, $(T-\nu)^\nu/\nu! \leq \binom{T-1}{\nu} \leq T^\nu/\nu!$. For $T = 100$ and $\nu = 3$ the three numbers are $152112.1\ldots$, $156849$ and $166666.6\ldots$

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**Problem 51.1.2.** For fixed $Z$, $L$, $A$ and $R$, the right side of (1) is $\hat{c}_1(L)^{\nu+1}T^{\nu+1}/((\nu+1)!\,[F:\mathbb{Q}]) + O(T^\nu)$ as $T \to \infty$, the leading term of Theorem 26.2.1.

**Problem 51.1.3.** For $Z = \mathbb{P}^1_{\mathbb{Z}}$ and $\mathcal{O}(1)$ with the Fubini–Study metric, $\nu = 1$, $\delta_Z = 1$ and $h_Z = \frac{1}{4}$. With $A$ and $R$ replaced by 0, (1) reads $\deg E_T \geq (T-1)(\frac{T}{4} - \frac{3}{2})$. This holds for $1 \leq T \leq 400$, as one checks by machine, and for all large $T$ by Problem 26.1.2; for instance $\deg E_{10} = 32.1969\ldots \geq 9$ and $\deg E_{50} = 677.4825\ldots \geq 539$.

**Problem 51.1.4.** For $y \geq 0$ and $A > 0$, $2y - A\log(2+y) \geq -A\log\max(A,1) - 1$.

**Problem 51.1.5.** Let $c_s \geq 0$, $h \geq -c_s$ and $T \geq 3\nu + 5$. Then

$$Th - 3(\nu+1)|h| - A\log(2+|h|) \geq (T-3\nu-5)\,h - (6\nu+10)\,c_s - A\log\max(A,1) - 1.$$

## 51.2. From the components to the cover

Let $Y$ be a cover of $S_j$, $j < m$, of the kind described in Section 51.4. In the formalization a weight $\lambda_M$ is fixed which is $T$ times a weight $\lambda_1$, and the integral cusp forms on $Y$ of weight $\lambda_M$ are compared with the sections of $L^{\otimes T}$ on the components $Z$ of a model attached to $Y$ and $\lambda_1$. The number $\nu$ is attached to the signature; $T$ is chosen of size about $\nu^2$ times a large parameter, so that $T \geq 3\nu+5$. Write $\mu_Y(\lambda) := \mu^\sharp$ of the lattice of integral cusp forms of weight $\lambda$ on $Y$, and, for each component $Z$, $r_Z$ and $\widehat{\deg}_Z$ for the rank and the normalized degree of its lattice of sections. Let $B := \binom{T-1}{\nu}$. The formalization proves:

- (U1) $\mu_Y(\lambda_M) \geq 2\sum_Z \widehat{\deg}_Z \big/ \sum_Z r_Z$;
- (U2) $r_Z \geq B\delta_Z$ for every $Z$;
- (U3) there are $\delta' > 0$ and $0 \leq \varrho < 1$ with $\delta' \leq \delta_Z \leq \delta'/(1-\varrho)$ for every $Z$;
- (U4) there are $c_s, c_c, X \geq 0$ and real numbers $s_Z$ with $h_Z + c_s \geq 0$, $(1-\varrho)\max(0, s_Z + c_s) \leq h_Z + c_s + c_c$, and $\frac{1}{N}\sum_Z s_Z \geq -X$, where $N$ is the number of components;
- (U5) the inequality (1) on every $Z$;
- (U6) $TX \leq C_1 u$, $Tc_c \leq C_1 u$ and $(6\nu + 10 + 2\varrho T)\,c_s \leq C_1 u$.

**Problem 51.2.1.** If the lattice of forms on $Y$ is the orthogonal sum of the lattices of sections on the components, then (U1) is an equality.

**Problem 51.2.2.** Under (U3) and (U4),

$$\frac{\sum_Z \delta_Z h_Z}{\sum_Z \delta_Z} \geq -X - 2\varrho c_s - c_c.$$

**Problem 51.2.3.** Let $\varrho = \frac{1}{2}$, two components with $\delta = (1,2)$, $s = (2,-2)$, $c_s = 1$, $c_c = 0$, $X = 0$, and $h = (\frac{1}{2}, -1)$. Then (U3) and (U4) hold, the weighted mean of $h$ is $-\frac{1}{2}$, its plain mean is $-\frac{1}{4}$, and the bound of Problem 51.2.2 is $-1$.

**Problem 51.2.4.** Under (U1)–(U6),

$$\begin{aligned}\mu_Y(\lambda_M) &\geq -2\big(TX + Tc_c + (6\nu + 10 + 2\varrho T)c_s + A\log\max(A,1) + R + 1\big)\\ &\geq -2(3C_1 + C_2 + 1)\,u.\end{aligned}$$

Problems 51.1.5, 51.2.2 and 51.2.4 prove the following.

**Theorem 51.2.1** (uniform lower bound)**.** *Under (U1)–(U6),* $\mu_Y(\lambda_M) \geq -C_U u$ *with* $C_U = 2(3C_1 + C_2 + 1)$.

The formalization proves Theorem 51.2.1 with 4 in place of 1 in $C_U$, under a version of (U2) with an upper bound for $r_Z$ as well, which the argument above does not need. The content is in (U4) and (U6): the heights of the components must be bounded below on average by $-X$ with $TX \leq C_1 u$.

## 51.3. Heights of components

Where does (U4) come from? For an algebraic point $P$ of $Z_F$ over a finite extension $F'$, extended to $\mathcal{O}_{F'}$, let $h_L(P) := \deg(P^*L)/[F' : \mathbb{Q}]$. For a real $c$, let $L(-c)$ be $L$ with its norms multiplied by $e^c$; then $h_{L(-c)}(Z) = h_Z - c$ [Zha].

**Problem 51.3.1.** $h_{L(-c)}(P) = h_L(P) - c$ for every algebraic point $P$.

**Problem 51.3.2.** If $s \in H^0(Z, L^{\otimes k})$ is integral with $\|s\|_{\sup} \leq e^{-kc}$ at every archimedean place, and $P$ is not in the zero locus of $s$, then $h_L(P) \geq c$.

Zhang used Problem 51.3.2 and Minkowski's theorem in the proof of his inequalities for successive minima [Zha, proof of Thm. 5.2]. Together with Moriwaki's arithmetic Hilbert–Samuel formula for nef bundles [Mo] the same argument gives two comparisons, as explained in [Zh, §4.3]: if $h_L(P) \leq b$ on a Zariski dense set of points, then $h_Z \leq b$; and if $h_L(P) \geq a$ for all points, then $h_Z \geq a$.

**Problem 51.3.3.** On $\mathbb{P}^1_{\mathbb{Z}}$ with the Fubini–Study metric, $h((p : q)) = \frac{1}{2}\log(p^2 + q^2)$ for coprime integers $p, q$, and $h((1 : \zeta)) = \frac{1}{2}\log 2 = 0.3465\ldots$ for every root of unity $\zeta$. The second family is Zariski dense, which is consistent with $h_Z = \frac{1}{4}$.

Zariski dense points of small height also exist on Grassmannians: for an $a \times a'$ matrix $W$, the row space of $(I_a\ W)$ is a point of the Grassmannian of $a$-planes in $F^{a+a'}$, and $\frac{1}{2}\log\det(I_a + WW^*)$ is the archimedean part of its Plücker height for the Fubini–Study metric [Zh, §4.3].

**Problem 51.3.4.** If $W$ is an $a \times a'$ matrix whose entries are roots of unity, then $\frac{1}{2}\log\det(I_a + WW^*) \leq \frac{a}{2}\log(a' + 1)$. For $a = a' = 2$ and $W = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ both sides equal $\log 3$.

The lower bound for heights needs small sections when $L$ is only nef. Moriwaki proved that if $L$ is nef on every fibre and on the generic fibre, with a smooth semipositive metric invariant under conjugation, then $\widehat{\mathrm{vol}}(L) \geq \hat{c}_1(L)^{\nu+1}$, where $\widehat{\mathrm{vol}}$ is the arithmetic volume of Chapter 26 [Mo], after Yuan [Yu]. *Status*: this statement is proved in the formalization, from resolution of singularities, a Poincaré–Lelong formula and a density statement for curves; we have not read below these.

At the finite places, the heights of points on the Shimura variety involve the integral structure of the invariant differentials, which the formalization controls with the following theorem. Let $C$ be an algebraically closed complete extension of $\mathbb{Q}_p$, $G$ a $p$-divisible group over $\mathcal{O}_C$, and $\alpha_G \colon T_p(G) \otimes \mathcal{O}_C \to \omega_{G^\vee}$ its Hodge–Tate map: an element $x$ of $T_p(G) = \mathrm{Hom}(G^\vee, \mu_{p^\infty})$ is sent to the pullback $x^*(dT/T)$ of the invariant differential of $\mu_{p^\infty}$. Then $\kappa \cdot \omega_{G^\vee} \subseteq \mathrm{im}\,\alpha_G$ for every $\kappa$ with $v_p(\kappa) \geq 1/(p-1)$ (with $v_p(\kappa) \geq 2$ if $p = 2$) [Fa], and the kernel of $\alpha_G$ is the annihilator, under the Tate pairing, of the kernel of the logarithmic differential of $G^\vee$ [ALB]. *Status*: the second statement is assumed in the formalization as a published theorem, and the first is derived from it there.

**Problem 51.3.5.** For $G = \mathbb{Q}_p/\mathbb{Z}_p$, $\alpha_G$ maps $T_p(G) \otimes \mathcal{O}_C$ onto $\omega_{\mu_{p^\infty}} = \mathcal{O}_C\, dT/T$; for $G = \mu_{p^\infty}$, $\omega_{G^\vee} = 0$.

In the formalization, (U4) and (U6) follow from the following statement on covers. For every lift $j < m$ with $m^2 \le k$, and every good cover $Y$ of $S_j$ whose level has at most fifteen prime factors, the mean over the components of certain volume slopes $s_Z$ is at least $-X$, where $X$ is a constant times an error depending on the weight and on $\log \mathrm{rd}_E + |c_\chi|$. *Status*: proved in the formalization from Moriwaki's theorem, the arithmetic Hilbert–Samuel formula for nef bundles, an absolute Siegel lemma [Zh, §4.3], resolution, the extension of homomorphisms of semi-abelian schemes, the Hodge–Tate theorem above, Zarhin's trick, and a Riemann–Roch statement for nef bundles. We have not followed this proof, which goes through towers of covers and their volumes, and we do not state the error term.

## 51.4. Back to the core level

A *good cover* of $S_j$ in the formalization has full level $15^e g_0 g_1$, with an exponent $e$ fixed by the size parameters of the family and $g_0, g_1$ the level primes, and a field of definition of bounded size; such covers exist by class field theory. Besides $\lambda_M$ it carries weights $\lambda_-$ and $\lambda_+$, shifts of the weight of the $j$th lift, and a lattice $L_{\mathrm{deep}}$ of forms on $Y$ pulled back from the core level. With a size parameter $\Lambda_j > 0$, the formalization proves:

- (D1) $1 \le b \le u$, $|T_j| \le \Lambda_j$ and $b \le \Lambda_j$;
- (D2) $\mu^\sharp(L_{\mathrm{deep}}) - 2b \le \mu_{\mathrm{core}}$;
- (D3) there are $n$ maps from the forms of weight $\lambda_-$ to the forms of the weight of the $j$th lift on $Y$, jointly injective, mapping integral forms into $L_{\mathrm{deep}}$, with $\sum_h \|\varphi_h f\|^2 \le e^{2b}\|f\|^2$, and $n \,\mathrm{rk}\, L_{\mathrm{deep}}$ is at most the rank for $\lambda_+$;
- (D4) the rank for $\lambda_+$ is at most $(1 + b/\Lambda_j)$ times the rank for $\lambda_-$;
- (D5) $L_{\mathrm{deep}}$ is spanned by elements of norm at most $e^{\Lambda_j}$.

*Status*: we have read the statements of (D2)–(D5), not their proofs; in particular we do not know which maps $\varphi_h$ are used.

**Problem 51.4.1.** Assume (D1)–(D5), Theorem 50.3.2 and Theorem 51.2.1. Then Lemma 49.3.1 with $t = -T_j - (C_F + C_U)u - 2b$, $P = b$, $\Lambda = \Lambda_j$ and $\epsilon = b/\Lambda_j$ gives, if $C_F \ge 0$ (which we may assume),

$$\mu_{\mathrm{core}} \ge -T_j - \big(2(C_F + C_U) + 9\big)u.$$

**Theorem 51.4.1** (lower bound)**.** *For every admissible family there is $C_L$ such that, for $x$ large, at every point and for every lift $j$, $\mu_{\mathrm{core}} \ge -T_j - C_L u$.*

*Status*: proved in the formalization. For $j < m$ it is Problem 51.4.1, from the inputs listed in this chapter and in Theorem 50.3.2; for $j = m$ it is a separate argument that we have not read. With Theorem 49.4.1 it gives the covolume estimate of Theorem 47.2.2.

*In the Lean formalization.* In [Lean] Theorem 51.4.1 is `LowAt`, proved by `low_of_cover_hypotheses`; for $j < m$ the step of Problem 51.4.1 is `lowAt_of_cover`, and for $j = m$ the input is `covering_low_defect_alt`. Good covers are `Deep` with `Deep.Good`, constructed by `exists_good_sized`. (D1)–(D5) are `SizeAt`, `RingAt`, `MapsAt`, `RkAt`, `GminAt`. Theorem 51.2.1 is `UAt`, from `u_low_of_cover_hyps` and the numerical lemma `UniformLowerBoundOfSums.core`; (U1)–(U5) are `SumAt`,

`ShareAt`, `DegAt`, `HeightAt`, `HSAt`, and (1) is proved by `hilbertSamuel_of_component_estimates_alt`. (U4) comes from `height_lower_bound_general_alt` and the statement on covers `DeepStR`, proved by `deepStR_of_use`. Moriwaki's theorem is `MoriwakiBigStatement`, proved by `moriwakiBig_outright`; the Hodge–Tate theorem is `FarguesHodgeTate`, derived by `farguesHodgeTate_of_farguesExact`.

## Hints

**§51.1.**

**51.1.1:** $\binom{T-1}{\nu} = \prod_{i=1}^{\nu}(T-i)/\nu!$.
**51.1.2:** Problem 51.1.1 and the definition of $h_Z$.
**51.1.3:** Formula (1) for $N = 1$; compare with Problem 26.1.1.
**51.1.4:** Minimize over $y \geq 0$, separately for $A \leq 4$ and $A > 4$.
**51.1.5:** Problem 51.1.4 with $y = |h|$.

**§51.2.**

**51.2.1:** Problem 6.3.3.
**51.2.2:** Bound $\sum_Z \delta_Z(h_Z + c_s + c_c)$ below by $\delta'(1-\varrho)\sum_Z \max(0, s_Z + c_s)$, and $\sum_Z \delta_Z$ above by (U3).
**51.2.4:** Problem 51.1.5 on each component, then Problem 51.2.2; a negative numerator only gets smaller in absolute value when the denominator $\sum r_Z$ grows.

**§51.3.**

**51.3.1:** The degree of a metrized line on $\mathcal{O}_{F'}$ changes by $-[F' : \mathbb{Q}]c$.
**51.3.2:** Compute $\deg P^*L^{\otimes k}$ with the section $s(P)$.
**51.3.3:** Use the section $x$ or $y$ that does not vanish at the point.
**51.3.4:** Hadamard's inequality for $I_a + WW^*$.
**51.3.5:** $\alpha_G(x)$ is the pullback of $dT/T$ along $x\colon G^\vee \to \mu_{p^\infty}$.

**§51.4.**

**51.4.1:** Theorems 50.3.2 and 51.2.1 bound $\mu^\sharp$ for $\lambda_-$.

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