THE CM PERIOD AND THE FLAT LINE

Prerequisites. From Chapter 20, *The Kronecker limit formula and Chowla–Selberg*: the reduced forms Q, the points τ_Q with $y_Q = \operatorname{Im}\tau_Q$, Problems 20.3.3 and 20.3.5, all in Section 20.3. From Chapter 12: Problems 12.1.9 and 12.3.7, and $\Delta(\mathbb{Z}+\mathbb{Z}\tau) = (2\pi)^{12}\eta(\tau)^{24}$. From Chapter 28, *Covolumes and periods*: the Faltings height h_{F} in Deligne's normalization, Problems 28.1.6 and 28.1.9, and Problem 28.3.1. From Chapter 47: Theorem 47.1.1 and its notation, and Problem 47.1.2. From Chapter 49: $\mu^\sharp$ and the main term T_j of Section 49.4. The theory of complex multiplication for elliptic curves is used through three classical facts, recalled in Section 50.1.

The covolume estimate of Theorem 47.2.2 has the main term $T_j = \sum_\rho (p_j - q_j) w_j(\rho) \log\Omega$. Which period is Ω, and how does it reach a lattice of automorphic forms on a unitary Shimura variety? In the Lean formalization [Lean] the answer involves only elliptic curves: Ω is the Chowla–Selberg period of an elliptic curve with complex multiplication by $\mathcal{O}_K$, $K = \mathbb{Q}(\sqrt{-d})$, and it enters through the degree of one metrized line, the *flat line*, at points of the Shimura variety built from two such curves.

50.1. The constant

Let $d > 4$ be prime with $d \equiv 3 \pmod 4$, $K = \mathbb{Q}(\sqrt{-d})$ of class number h, χ its character, and $c_\chi = L'(1,\chi)/L(1,\chi)$. For a lattice $L \subset \mathbb{C}$ put $\mathcal{A}(L) := A(L)|\Delta(L)|^{1/6}$, which is invariant under homothety (Problem 28.1.6). Let

$$\log\Omega_K := \gamma + \log\frac{2\pi}{\sqrt{d}} - c_\chi .$$

For a CM field $E \supseteq K$ as in Theorem 47.1.1, the constant there is $\log\Omega = \log\Omega_K - \log(\mathrm{rd}_E/\sqrt{d})$; for $E = K$ the two agree.

Problem 50.1.1. $\mathcal{A}(\mathbb{Z}+\mathbb{Z}\tau) = 4\pi^2 \operatorname{Im}(\tau)\,|\eta(\tau)|^4$.

Problem 50.1.2. $\log\Omega_K = \dfrac{1}{h}\displaystyle\sum_Q \log\frac{\mathcal{A}(\mathbb{Z}+\mathbb{Z}\tau_Q)}{\pi} = \frac{1}{h}\sum_Q \log\bigl(4\pi y_Q|\eta(\tau_Q)|^4\bigr)$.

Problem 50.1.3. $\log\Omega_K = 1.426502\ldots,\ 1.300363\ldots,\ 1.027786\ldots$ for $d = 7, 11, 19$. For $d = 23$ the three classes give $\log(4\pi y_Q|\eta(\tau_Q)|^4) = 0.894533\ldots$ for $Q = [1,1,6]$ and $1.456932\ldots$ for $Q = [2,\pm1,3]$, with mean $\log\Omega_K = 1.269466\ldots$

The formula of Problem 50.1.2 also holds for $d = 4$, where it gives $1.476335\ldots$ (Problem 47.1.2).

Problem 50.1.4. If $E \supseteq K$ is a number field, $\mathrm{rd}_E := |\operatorname{disc} E|^{1/[E:\mathbb{Q}]}$ (for CM fields this is the root discriminant of Chapter 47) and $\mathfrak{d}_{E/K}$ the relative discriminant, then $\mathrm{rd}_E = \sqrt{d}\,N(\mathfrak{d}_{E/K})^{1/[E:\mathbb{Q}]}$. Hence $\log\Omega \le \log\Omega_K$, with equality exactly when E/K is unramified at every finite prime.

Date: November 26, 2026.

The formalization does not average over ideal classes but over Galois conjugates of one lattice. We use three classical facts. First, the reduced form $Q = [a, b, c]$ corresponds to the ideal $a\mathbb{Z} + \frac{-b+\sqrt{-d}}{2}\mathbb{Z}$ of $\mathcal{O}_K$, which is homothetic to $\mathbb{Z} + \mathbb{Z}\tau_Q$, and this gives a bijection from reduced forms to ideal classes [Co, Thm. 7.7]. Second, two lattices with the same j are homothetic [Si1, Ch. VI]. Third, the values $j(\mathfrak{a})$, for $\mathfrak{a}$ running over representatives of the h ideal classes of K, are distinct algebraic integers, and they are the roots of a monic irreducible polynomial over $\mathbb{Z}$ of degree h [Si2, Ch. II].

Problem 50.1.5. For $d = 23$, the three numbers $j(\tau_Q)$ are $-3493225.6999\ldots$ and $737.8499\ldots \pm 1764.0189\ldots\mathrm{i}$, and they are the roots of

$$X^3 + 3491750X^2 - 5151296875X + 12771880859375.$$

Problem 50.1.6. Let L be a lattice homothetic to an ideal of $\mathcal{O}_K$ with $\Delta(L) = 1$, and let $M \subset \mathbb{C}$ be a number field containing $g_2(L)$ and $g_3(L)$. For each embedding $\sigma\colon M \to \mathbb{C}$ let L^σ be the lattice with invariants $\sigma g_2(L)$, $\sigma g_3(L)$; these are the period lattices of the curve $E_0\colon y^2 = 4x^3 - g_2(L)x - g_3(L)$ over M with the differential dx/y. Then

$$\frac{1}{[M:\mathbb{Q}]}\sum_\sigma \log\frac{A(L^\sigma)}{\pi} = \log\Omega_K.$$

Problem 50.1.7. For every lattice L, the complex conjugate lattice $\bar{L}$ has $A(\bar{L}) = A(L)$ and $|\Delta(\bar{L})| = |\Delta(L)|$. If L is homothetic to an ideal $\mathfrak{a}$, then $\bar{L}$ is homothetic to $\bar{\mathfrak{a}}$, has $\Delta(\bar{L}) = 1$ and invariants $\overline{g_2(L)}$, $\overline{g_3(L)}$. So Problem 50.1.6 applies to $\bar{L}$ and the field $\bar{M}$, and the conjugate curve $\bar{E}_0$ has the same mean $\log\Omega_K$.

Status: Problem 50.1.6 is proved in the formalization from Chowla–Selberg in the form of Problem 50.1.2, itself obtained there from Kronecker's limit formula.

Problem 50.1.8. If E_0 is an elliptic curve over a number field with complex multiplication by $\mathcal{O}_K$, then $h_{\mathrm{F}}(E_0) = -\frac{1}{2}(\log\Omega_K + \log\pi)$. For $d = 7$ this is $-1.285615\ldots$. For $K = \mathbb{Q}(\mathrm{i})$, where $h = 1$ and the formula of Problem 50.1.2 also holds, the same argument gives $-1.310532\ldots$, in agreement with Problem 28.1.9.

50.2. The main term

The jth lift lives on a unitary group of signature $(p_j, q_j) = (k + j - 1, m - j)$, $1 \le j \le m$, so $p_j + q_j = k + m - 1$ does not depend on j. In the formalization $k \ge m + 3$.

Problem 50.2.1. $p_j - q_j = k + 2j - 1 - m$, and $\sum_{j=1}^m (p_j - q_j) = mk$. If $k \ge m+3$, then $p_j - q_j \ge 4$ for every j, and $q_m = 0$.

So only the last lift lives on a definite group, and every main term T_j has a positive coefficient in front of each weight. The slope moves linearly in the weight when a lattice of sections is tensored with a line.

Problem 50.2.2. Let M be a nonzero lattice and ℓ a lattice of rank one, in normed spaces over a number field, and give $M \otimes \ell$ the tensor product norms and unit lattices. Then $\mu^\sharp(M \otimes \ell) = \mu^\sharp(M) + \mu^\sharp(\ell)$. Hence $\mu^\sharp(M \otimes \ell^{\otimes w}) = \mu^\sharp(M) + w\,\mu^\sharp(\ell)$ for $w \ge 0$.

Problem 50.2.3. For the lemniscatic lattice $\varpi\mathbb{Z}[\mathrm{i}]$, $\mathcal{A}(\varpi\mathbb{Z}[\mathrm{i}])/\pi = 2\varpi^2/\pi$. So for $K = \mathbb{Q}(\mathrm{i})$ the constant Ω of Theorem 47.1.1 is the number Ω of Problem 12.3.7, and the shift $|\lambda| \log A(c\,\mathcal{O})$ of Problem 28.3.1 equals $|\lambda|(\log\Omega + \log\frac{\pi}{2})$.

Thus on a definite group ($q = 0$) the period enters the slope of the lattice of forms linearly in the weight, with coefficient $\log\Omega$ up to a constant per unit of weight. In the formalization the same happens in signature (p, q), with coefficient $(p-q)\log\Omega$ per unit of weight, through the flat line of the next section.

50.3. The flat line

Let X be the unitary Shimura variety of signature (p, q) attached to $E \supseteq K$, and let E_0 and E_0' be elliptic curves with complex multiplication by $\mathcal{O}_K$ through the two embeddings of K, with $\Delta = 1$. In the formalization certain points of X are built from E_0 and E_0', with p and q copies respectively, and carry a line of invariant differentials of a fixed type; a section of this line that is constant along such points is a *flat generator*, and its metrized line the *flat line*. We do not reproduce the definitions of the formalization, which fix the type and the metric through a Hermitian form on the $\mathcal{O}_E$-lattice of the point.

Theorem 50.3.1 (flat degree)**.** *The flat line at a point of type (p, q), over a field of definition F, has degree divided by $[F : \mathbb{Q}]$ (half the normalization of Theorem 47.1.1) equal to*

$$\frac{p-q}{2}\log\Omega + q\left(\log d - \log \mathrm{rd}_E\right).$$

Status: proved in the formalization. Its degree is a sum of local terms. At the archimedean places it is the logarithm of a Gram determinant of CM vectors, expressed through the norms of the invariant differentials of E_0 and E_0', whose mean over conjugates is $\log\Omega_K$ (Problems 50.1.6 and 50.1.7). At the finite places it is a transition determinant of absolute value 1 together with the different of $\mathcal{O}_E$. We have read this proof but not the definitions of the flat generator and of its metric, so we state only the result.

Problem 50.3.1. For $E = K$, the flat degree is $\frac{p-q}{2}\log\Omega_K + \frac{q}{2}\log d$.

Problem 50.3.2. Let F_j be the flat degree in signature (p_j, q_j). Then $F_{j+1} - F_j = \log\Omega_K - \frac{1}{2}\log d$ for $1 \le j < m$, whatever E is. For $d = 7$ this difference is $0.453546\ldots$

Problem 50.3.3. Write $F_j^\circ := \frac{p_j - q_j}{2}\log\Omega$ for the period part of F_j. Then $T_j = 2\sum_\rho w_j(\rho) F_j^\circ$, and $\sum_j T_j = fmkw\log\Omega$ when $w_j(\rho) = w$ for all j and ρ, where f is the number of places $\rho \in \Phi$.

By Problem 50.2.2, tensoring with w copies of a line whose $\mu^\sharp$ is twice its normalized degree F_j° moves $\mu^\sharp$ by $2wF_j^\circ$; Problem 50.3.3 says that T_j is exactly this shift for the weights $w_j(\rho)$. The formalization uses this as follows. An integer α_ρ, the *anchor*, is attached to each place, with

$$\left|T_j - \sum_\rho (p_j - q_j)\,\alpha_\rho \log\Omega\right| \le Cu,$$

in the notation of Section 49.4. Multiplication by a flat section of the corresponding tensor power of the flat line then compares two weights on a cover.

Problem 50.3.4. If $|\alpha_\rho - w_j(\rho)| \le \eta$ for every ρ, then $\left|T_j - \sum_\rho (p_j - q_j)\alpha_\rho \log\Omega\right| \le f(k+m-1)\,\eta\,|\log\Omega|$.

Theorem 50.3.2 (flat comparison)**.** *On a good cover of the Shimura variety of the jth lift, $j < m$, the lattices of integral cusp forms of the weights λ_- and λ_M of Chapter 51 satisfy*

$$\mu^\sharp(\lambda_-) \ge \mu^\sharp(\lambda_M) - T_j - C_F u,$$

with C_F independent of the point and the lift.

Status: proved in the formalization from Theorem 50.3.1, the anchor bound and the statements on the cover listed at the end of Chapter 51. The anchor bound is proved there from the definitions of the anchors, which we have not traced; we do not know how close α_ρ is to $w_j(\rho)$.
Where else the constant occurs. The same closed formula $\gamma + \log(2\pi/\operatorname{rd}_E) - c_\chi$ is used in the formalization for the main term T_j, for the flat degree, and in the bounds for the theta kernels of the chain, where Problem 50.1.6 is applied. So the cancellation of the period between the covolumes and the heights of the period maps is built into the definitions. Only elliptic curves with complex multiplication by $\mathcal{O}_K$ enter this step; no period relation for abelian varieties with complex multiplication by a larger field is used.

In the Lean formalization. In [Lean] $\log\Omega$ is `NumDatum.logOm` in the main term `l3m` and `GlobalTheory.CM.logOm` in the flat degree; `logOmVal_eq` identifies the value. Problem 50.1.2 is `chowla_selberg_eta` and `chowla_selberg_lattice`; Problem 50.1.6 is `mean_log_period` and `MeanPeriod` (proved as `MeanPeriodClosed.meanPeriod`, used in `factors_leS`). The curves E_0, E_0' come from `exists_curve0_hasPeriod`. Theorem 50.3.1 is `GlobalTheory.CompanionTheory.Ar.Fl.flatDegree`, assembled by `deg_eq_of_local_two_curves`; it reaches the covolume bound through `flatDegreeSt_of_resK_gaga` and `flatDeg_alt_alt`. The anchor bound is `AnchorAt`, proved by `anchor_holds_alt`, and Theorem 50.3.2 is `FlatAt`, proved by `flat_alt_alt`.

Hints

§50.1.

50.1.1: $A(\mathbb{Z}+\mathbb{Z}\tau) = \operatorname{Im}\tau$.

50.1.2: Problem 20.3.5 and Problem 50.1.1.

50.1.3: Problem 50.1.2, or Problem 20.3.3 for c_χ.

50.1.4: The discriminant tower formula $|\operatorname{disc} E| = |\operatorname{disc} K|^{[E:K]} N(\mathfrak{d}_{E/K})$.

50.1.5: The τ_Q of the three reduced forms of Problem 50.1.3 are as in Section 20.3; compute $j = E_4^3/\Delta$ numerically, and round the symmetric functions, which are integers.

50.1.6: $\mathcal{A}(L^\sigma) = A(L^\sigma)$ depends only on $j(L^\sigma) = \sigma j(L)$; count the embeddings of M above each embedding of $\mathbb{Q}(j(L))$.

50.1.8: The second formula for h_{F} in Chapter 28, which does not depend on the model; $\mathcal{A}$ is homothety invariant (Problem 28.1.6), so Problem 50.1.6 applies after rescaling.

§50.2.

50.2.2: A generator of $\det(M \otimes \ell)$ is $\omega \otimes e^{\otimes r}$ for generators ω of $\det M$ and e of ℓ.

50.2.3: Problem 12.1.9 gives $\Delta(\varpi\mathbb{Z}[\mathrm{i}]) = 64$.

§50.3.

50.3.2: Problem 50.2.1.

50.3.3: Problem 50.2.1.

References

[Lean] The Lean community, *riemann-hypothesis*, a Lean formalization of Alpöge's argument, `https://github.com/leanprover-community/riemann-hypothesis`.

[Co] D. A. Cox, *Primes of the Form $x^2 + ny^2$*, 2nd ed., Wiley, 2013.

[Si1] J. H. Silverman, *The Arithmetic of Elliptic Curves*, 2nd ed., Grad. Texts in Math. 106, Springer, 2009.

[Si2] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Grad. Texts in Math. 151, Springer, 1994.