SLOPES OF LATTICES OF FORMS: THE GOOD PART AND DESCENT

Prerequisites. From Chapter 6, *The slope method*: normed spaces, degrees and slopes, $\mu^\sharp$ (Section 6.3), Problems 6.3.2 and 6.3.3, and Theorem 6.4.1. From Chapter 26, *Covolumes of lattices of sections*: the lattices E_D of binary forms with the Fubini–Study L^2 norm (Section 26.1) and Problem 26.1.7. The covolume estimate we are after is the first display of Theorem 47.2.2 of Chapter 47.

Theorem 47.2.2 bounds the covolume of a lattice M_j attached to the jth lift of the chain. In the Lean formalization [Lean] the bound is proved, and the proof splits into two unequal parts. The first part, treated here, is linear algebra: it reduces the covolume of M_j to a lower bound for the slope of a lattice of holomorphic automorphic forms. The second part, which proves that lower bound with arithmetic geometry on covers of a Shimura variety, is the subject of Chapter 51, and the period in the main term is the subject of Chapter 50. How much can one lose in passing from the lattice of all integral forms of a type to the part on which a finite group acts by a given character?

49.1. Normalized slopes and covolumes

Throughout, K is a number field. A *lattice* in a normed space V over K (Section 6.3) is a finitely generated $\mathcal{O}_K$-submodule $L \subseteq V$; we give $L \otimes K$ the restricted archimedean norms and the unit lattices $L \otimes \mathcal{O}_v$, and write $\deg L$, $\operatorname{rk} L$ for its degree and dimension. All archimedean norms in this chapter are Hermitian. For $L \neq 0$ put

$$\mu^\sharp(L) := \frac{2 \deg L}{[K : \mathbb{Q}] \operatorname{rk} L}, \qquad \operatorname{lcv}(L) := -\operatorname{rk}(L)\, \mu^\sharp(L),$$

and let $\mu^\sharp_{\max}(L)$ be the largest $\mu^\sharp$ of a nonzero sublattice of L. We call lcv the normalized logarithmic covolume. This $\mu^\sharp$ is the one of Chapter 6, and it does not change under extension of K (Problem 26.1.7).

Problem 49.1.1. For a fractional ideal $\mathfrak{a}$ of K with the standard norms, $\operatorname{lcv}(\mathfrak{a}) = 2 \log N\mathfrak{a}/[K : \mathbb{Q}]$. Over $\mathbb{Q}$, $\operatorname{lcv}(L) = 2 \log \operatorname{covol}(L)$ for every Euclidean lattice L.

Problem 49.1.2. Multiplying the norm at each archimedean place v by a constant $c_v > 0$ adds $-\frac{2}{[K:\mathbb{Q}]} \sum_v n_v \log c_v$ to $\mu^\sharp(L)$. In particular, if $|\log c_v| \leq \epsilon$ for every v, then $\mu^\sharp(L)$ moves by at most 2ϵ; and if $c_v \leq 1$ for every v, then $\mu^\sharp(L)$ does not decrease.

Problem 49.1.3. For an integer $g \geq 1$, $\mu^\sharp(g^{-1}L) = \mu^\sharp(L) + 2 \log g$, and $\mu^\sharp_{\max}(g^{-1}L) = \mu^\sharp_{\max}(L) + 2 \log g$.

Problem 49.1.4. Let $b_1, \dots, b_D$ be a K-basis of $L \otimes K$, let $M \subseteq K^D$ be the lattice of coordinate vectors of the elements of L, and give K^D the norms $\|c\|_v := s \left\| \sum_i c_i b_i \right\|_v$ with $0 < s \leq 1$. Then $\operatorname{lcv}(M)/D \leq -\mu^\sharp(L)$, with equality if $s = 1$.

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In the formalization M_j is such a coordinate lattice, and the problem is the first step of the proof: the covolume of M_j is bounded by minus the slope of a lattice L of forms.

49.2. The good part of a lattice

Let G be a finite group acting K-linearly on V by isometries at every archimedean place, and let $\chi\colon G \to K^\times$ be a character. Put $e := |G|^{-1}\sum_{h\in G}\chi(h)^{-1}h$.

Problem 49.2.1. e is idempotent, and at every archimedean place it is the orthogonal projection onto its image, so that $\|ex+(y-ey)\|^2 = \|ex\|^2 + \|y-ey\|^2$. If L is G-stable, then $|G|\,e(L) \subseteq L$.

From now on e is any idempotent endomorphism of V that is an orthogonal projection at every archimedean place, L is a lattice, and $g \geq 1$ is an integer with $g\,e(L) \subseteq L$. Put

$$L_{\mathrm{good}} := L \cap eV, \qquad L_{\mathrm{bad}} := L \cap \ker e.$$

Problem 49.2.2. $L\otimes K$ is stable under e, the lattices L_{good} and L_{bad} are saturated in L, and $\operatorname{rk} L_{\mathrm{good}} + \operatorname{rk} L_{\mathrm{bad}} = \operatorname{rk} L$.

Problem 49.2.3. The map $x \mapsto x - ex$ induces an isometry from L/L_{good}, with the quotient norms, onto the lattice $(1-e)L$. Hence $\deg L = \deg L_{\mathrm{good}} + \deg (1-e)L$.

Problem 49.2.4. $(1-e)L \subseteq g^{-1}L_{\mathrm{bad}}$. Hence, if $(1-e)L \neq 0$ and $L \subseteq L_{\mathrm{all}}$ for a lattice L_{all}, then $\mu^\sharp((1-e)L) \leq \mu^\sharp_{\max}(L_{\mathrm{all}}) + 2\log g$.

Problem 49.2.5. Suppose $L_{\mathrm{good}} \neq 0$ and $L \subseteq L_{\mathrm{all}}$, and let $0 \leq \delta \leq \frac{1}{12}$ and Q be real numbers with

$$\operatorname{rk} L_{\mathrm{bad}} \leq 3\delta \operatorname{rk} L, \qquad \mu^\sharp_{\max}(L_{\mathrm{all}}) \leq Q, \qquad -Q \leq \mu^\sharp(L), \qquad 2\log g \leq Q.$$

Then $\mu^\sharp(L_{\mathrm{good}}) \geq \mu^\sharp(L) - 12\delta Q$.

Problems 49.2.2–49.2.5 prove the following.

Theorem 49.2.1 (good part)**.** *Let $L \subseteq L_{\mathrm{all}}$ be lattices, e an orthogonal idempotent with $g\,e(L) \subseteq L$, and δ, Q as in Problem 49.2.5. If the part of L killed by e has rank at most $3\delta \operatorname{rk} L$, then removing it lowers the slope by at most $12\delta Q$.*

In the application G is a finite group of the construction, δ is a negative exponential of a power of x, and Q is a power of x times the error unit; so the loss $12\delta Q$ is negligible. The next two problems are the smallest cases.

Problem 49.2.6. Let $L = \mathbb{Z}^2$ with the standard norm, $G = \{1, \sigma\}$ acting by $\sigma(x_1, x_2) = (x_2, x_1)$, and $\chi = 1$, so $g = 2$. Then $\mu^\sharp(L) = 0$, $\mu^\sharp(L_{\mathrm{good}}) = -\log 2$, $\mu^\sharp(L_{\mathrm{bad}}) = -\log 2$ and $\mu^\sharp((1-e)L) = \log 2$; the inclusion of Problem 49.2.4 is an equality.

Problem 49.2.7. Let σ act on E_D by exchanging x and y, and $\chi = 1$. For $D = 4$,

$$\deg L_{\mathrm{good}} = \tfrac{1}{2}\log 750, \qquad \deg (1-e)E_4 = \log 20, \qquad \deg E_4 = \tfrac{1}{2}\log 300000.$$

For $D = 10$ one finds $\deg L_{\mathrm{good}} = 16.3474\ldots$ and $\deg (1-e)E_{10} = 15.8495\ldots$, whose sum is $\deg E_{10} = 32.1969\ldots$

49.3. Descent along a family of maps

The lower bound of Chapter 51 is proved on a cover of the Shimura variety and for a shifted weight. To bring it back, one maps a lattice A into a lattice B by several maps at once. How much slope survives? Let A and B be nonzero lattices in normed spaces V and W, let $\varphi_1, \dots, \varphi_n \colon V \to W$ be K-linear with $\varphi_h(A) \subseteq B$, and put $\Psi := (\varphi_1, \dots, \varphi_n) \colon V \to W^n$, where W^n carries the orthogonal sum of the norms.

Problem 49.3.1. Suppose Ψ is injective and $\sum_h \|\varphi_h f\|_v^2 \le e^{2P}\|f\|_v^2$ for every archimedean v and every f. Then $\mu^\sharp(\Psi(A)) \ge \mu^\sharp(A) - 2P$.

Problem 49.3.2. If a lattice B' contains elements $y_1, \dots, y_r$ that span $B' \otimes K$ and have norm at most e^{Λ} at every archimedean place, then $\mu^\sharp(B') \ge -2\Lambda$. The same holds for every nonzero quotient of B' with the quotient norms.

Problem 49.3.3. In addition to the hypotheses of Problem 49.3.1, suppose that B has a K-spanning set of elements of norm at most e^{Λ} at every archimedean place, with $\Lambda \ge 0$, that $n \operatorname{rk} B \le (1+\epsilon)\operatorname{rk} A$ with $\epsilon \ge 0$, and that $t \le \mu^\sharp(A) - 2P$. Then

$$\mu^\sharp(B) \ge t - \epsilon\big(|t| + 2\Lambda\big).$$

Problem 49.3.4. For $n = 1$ and $\epsilon = 0$, Problem 49.3.3 is the inequality $\mu^\sharp(B) \ge \mu^\sharp(A) - 2P$ for an injective map between lattices of the same rank, which is Theorem 6.4.1 in this normalization.

Problem 49.3.5. Multiplication by x^c maps E_D into E_{D+c} with operator norm at most 1, and E_{D+c} is spanned by monomials of norm at most 1. So Problem 49.3.3 applies with $n = 1$, $P = \Lambda = 0$ and $\epsilon = c/(D+1)$. For $D = 10$ and $c = 1$ it gives $\mu^\sharp(E_{11}) \ge \frac{10}{11}\mu^\sharp(E_{10}) = 5.3218\ldots$, while $\mu^\sharp(E_{11}) = 6.3925\ldots$

Lemma 49.3.1 (descent)**.** *Problems 49.3.1–49.3.3 prove: if n maps $A \to B$ are jointly injective with $\sum_h \|\varphi_h f\|^2 \le e^{2P}\|f\|^2$, if B is spanned by elements of norm at most e^{Λ}, and if $n \operatorname{rk} B \le (1+\epsilon)\operatorname{rk} A$, then every $t \le \mu^\sharp(A) - 2P$ satisfies $\mu^\sharp(B) \ge t - \epsilon(|t| + 2\Lambda)$.*

49.4. The reduction

We now describe the data of the formalization at one point of the family and one lift j, $1 \le j \le m$. We keep the notation of Chapter 47: the scale x, $y = x^{\varepsilon}$, $b = e^{y}$, $B_0 = fmby$, the places $\rho \in \Phi$, the weights $w_j(\rho)$, the signature $(p_j, q_j) = (k + j - 1, m - j)$ of the jth lift, and $\log \Omega$ as in Theorem 47.1.1. Put

$$u := B_0 \log\log x, \qquad T_j := \sum_{\rho \in \Phi} (p_j - q_j)\, w_j(\rho) \log \Omega.$$

The field K is totally complex. The data are:

- a lattice M_j of rank D, the lattice of the jth lift (this D is not the degree of E_D in the earlier sections);
- a normed space V of automorphic forms, lattices $L_{\mathrm{type}} \subseteq L_{\mathrm{all}}$ (the integral forms of the auxiliary type, and all integral forms), and an idempotent e as in Section 49.2, with $g\, e(L_{\mathrm{type}}) \subseteq L_{\mathrm{type}}$;
- the lattice L_{core} of integral holomorphic cusp forms of the weight of the jth lift on the unitary Shimura variety of signature (p_j, q_j), with the Petersson norms, defined over a subfield of K; write $\mu_{\mathrm{core}} := \mu^\sharp(L_{\mathrm{core}})$.

They satisfy, for constants A_1, C_P and $\delta = e^{-x^c}$ with $c > 0$:

(R1) M_j is the coordinate lattice of $L_{\mathrm{good}} := L_{\mathrm{type}} \cap eV$ in a basis, with metric s times that of V, $0 < s \le 1$;
(R2) $\mathrm{rk}(L_{\mathrm{type}} \cap \ker e) \le 3\delta \,\mathrm{rk}\, L_{\mathrm{type}}$, $\mu^\sharp_{\max}(L_{\mathrm{all}}) \le x^{A_1} b$, and $2 \log g \le b$;
(R3) if $K_0 \subseteq K$ is the field of definition of L_{core}, the map $L_{\mathrm{core}} \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_K \to V$ is injective with image L_{type}, and at every complex embedding σ of K the norm of V is c_σ times the Petersson norm at $\sigma|_{K_0}$, with $|\log c_\sigma| \le C_P u$;
(R4) $1 \le b \le u$ and $|T_j| \le x^{A_1} b$.

Status: in the formalization (R1)–(R4) are hypotheses of the assembly, each proved separately for the constructed family. We have read the statements of these proofs, not the proofs, except for (R1). We have not traced how M_j, V and the type correspond to the spaces of the theta correspondence in Chapter 47.

Problem 49.4.1. Under (R3), $\mu^\sharp(L_{\mathrm{type}}) \ge \mu_{\mathrm{core}} - 2C_P u$.

Problem 49.4.2. For $c > 0$ and real A_1, A_2, C, $12e^{-x^c}(x^{A_1} + x^{A_2} + C) \to 0$ as $x \to \infty$. For $c = \frac{1}{7}$ and $A_1 = A_2 = C = 1$, the inequality $12e^{-x^{1/7}}(2x + 1) \le 1$ holds for $x \ge 1$ exactly when $x \ge 7.9867\ldots \cdot 10^9$.

Combining Problems 49.1.4, 49.2.5 with $Q := (2x^{A_1} + C)u$ and $C := 2C_P + C_L + 1$, 49.4.1 and 49.4.2 gives the reduction.

Theorem 49.4.1 (reduction)**.** *Assume (R1)–(R4), $D \ge 1$ and $\mu_{\mathrm{core}} \ge -T_j - C_L u$ with $C_L \ge 0$. Then for x large in terms of c, A_1, C_P, C_L,*

$$\frac{\mathrm{lcv}(M_j)}{D} \le T_j + (2C_P + C_L + 1)\, u.$$

The hypothesis $\mu_{\mathrm{core}} \ge -T_j - C_L u$, with C_L independent of the point and the lift, is the lower bound proved in Chapter 51, as Theorem 51.4.1. With it, Theorem 49.4.1 is the covolume estimate in the first display of Theorem 47.2.2: for every admissible family there are C and x_0 with $\mathrm{lcv}(M_j)/D \le \sum_\rho (p_j - q_j) w_j(\rho) \log \Omega + C B_0 \log\log x$ for $x \ge x_0$, at every point and every lift. *Status*: proved in the formalization, from the inputs listed at the end of Chapter 51.

Where is the bound used? In the formalization the lattices $M_1, \ldots, M_m$ sit on the side of the construction at the auxiliary level, and a geometry-of-numbers inequality turns the sum of their covolume bounds into an upper bound for the slope of the dual of the target lattice at that level, with local index terms; a comparison between levels then gives the bound for $\mu(R^\vee)$ in Theorem 47.2.2. *Status*: we have read the statements of these two steps, not their proofs. So the arithmetic geometry is applied neither to the initial lattices of the construction nor to the target itself, but to the lattices of the lifts, which control the target.

In the Lean formalization. In [Lean] the covolume estimate is `CovolumeBoundOfX`, the inequality `(p.S false).L3 (p.N C CH)`, with lcv the function `Formalism.DivisionValues.lcv` and T_j, u the numbers `l3m`, `l3u` of `ZscOfNum`. It is proved by `topCovolSlim2_use_alt_alt` through `topCovolG_alt_alt`, whose last step is `covolX_of_lift_bounds_and_coreLe` with the inequality `lcv_le` (Theorem 49.4.1). The inputs are `RecAt` from `lcv_div_le_neg_slope_sq` (Problem 49.1.4); `slope_good_ge` (Problem 49.2.5) with `CutAt`, `BadAt`, `LamAt` ((R2)); `CoreAt` from `coreAt_of` with `GramAt`, `MeasAt`, `RatAt` ((R3), Problem 49.4.1); `SizeAt` ((R4)); `exists_loss_le` (Problem 49.4.2); and `LowAt`

(Theorem 51.4.1). Lemma 49.3.1 is `DescentFamily.descent`. The use of the bound is `IndefSide.boundsMu` followed by `Passage`.

Hints

§49.1.

49.1.1: Problem 6.3.6 and Problem 6.3.1.
49.1.2: Problem 6.3.2; a complex place has $n_v = 2$.
49.1.3: Problem 6.3.6 with $\mathfrak{a} = g^{-1}\mathcal{O}_K$.
49.1.4: The coordinate map is an isometry for $s = 1$; then Problem 49.1.2.

§49.2.

49.2.1: Reindex the double sum for e^2; for the projection, e commutes with the adjoint action since the h are isometries and $|\chi(h)| = 1$.
49.2.2: $x = ex + (x - ex)$, and a multiple of each part lies in L.
49.2.3: The quotient norm of x modulo eV is $\|x - ex\|$ by Problem 49.2.1; then Problem 6.3.3.
49.2.4: $g(x - ex) \in L$, and e kills it. Then Problem 49.1.3.
49.2.5: Problems 49.2.3 and 49.2.4, and $\operatorname{rk} L_{\mathrm{good}} \geq \frac{3}{4} \operatorname{rk} L$.
49.2.6: $e(x_1, x_2) = \frac{1}{2}(x_1 + x_2, x_1 + x_2)$.
49.2.7: The monomials are orthogonal (Problem 6.3.4); use the bases $x^a y^{D-a} \pm x^{D-a} y^a$.

§49.3.

49.3.1: Theorem 6.4.1 for $\Psi\colon A \to \Psi(A)$; the finite places contribute at most 0.
49.3.2: Choose a K-basis among the y_i; Hadamard's inequality for its wedge product.
49.3.3: Apply Problem 6.3.3 to the saturation of $\Psi(A)$ in B^n, and Problem 49.3.2 to the quotient.
49.3.5: $|z|^{2c} \leq (1 + |z|^2)^c$; for the values, Problem 26.1.1.

§49.4.

49.4.1: Problem 26.1.7, then Problem 49.1.2.
49.4.2: Take logarithms; $\log(12(2x + 1)) - x^{1/7}$ increases up to $x^{1/7} \approx 7$ and then decreases.

References

[Lean] The Lean community, *riemann-hypothesis*, a Lean formalization of Alpöge's argument, `https://github.com/leanprover-community/riemann-hypothesis`.