COUNTING FORMS AND THE CRITICAL LEVEL

Prerequisites. The mass of a genus, Weyl's dimension formula and the count identity of Chapter 21; the dimension budget, the level indices and the interval for the auxiliary prime in §47.3. We use the order of $\mathrm{GL}_n(\mathbb{F}_q)$, the Euler product of the Dedekind zeta function of a number field, and $\log(1+t) \le t$.

This chapter develops the input (A6) of Chapter 47 from the Lean formalization [Lean], where the published expositions stop. There the dimension budget is not an inequality: the rank of the source lattice and the budget are both given exactly by the mass formula, and their ratio is a product over the embeddings of a *critical level* divided by the auxiliary prime. The prime is then pinned to the geometric mean of the critical levels, and the lower bound (A6) becomes a lower bound for each critical level, which is a Vandermonde estimate.

48.1. Two dimensions

How many forms does each side of the final pairing carry? Let F be a totally real number field of degree f, and $E \supset F$ a CM field unramified over F at every finite prime, so that $|d_E| = d_F^2$ for the discriminants. Let χ be the quadratic character of F attached to E, write $\mathrm{rd}_E = |d_E|^{1/(2f)}$ and $v = \mathrm{rd}_E/2$ as in Chapter 47, and let Φ be a CM type, a set of f complex embeddings ρ of E. For $M \ge 1$ let $\mathrm{mass}(M)$ be the mass of the genus of the standard lattice $\mathcal{O}_E^M$ with $h(x,y) = \sum_i \bar{x}_i y_i$, the sum of $1/|\mathrm{Aut}|$ over its classes.

Theorem 48.1.1 (Mass ratio)**.** *For $M \ge 3$,*

$$\frac{\mathrm{mass}(M)}{\mathrm{mass}(M-1)} = \big((M-1)!\,(v/\pi)^M\big)^f (2v)^{-f/2}\, L(M), \qquad L(M) = \prod_{\mathfrak{P}} e_{\mathfrak{P}}(M),$$

the product over the primes $\mathfrak{P}$ of E, with $e_{\mathfrak{P}}(M) = (1-(-1)^M N\mathfrak{P}^{-M/2})^{-1}$ if $\bar{\mathfrak{P}} = \mathfrak{P}$ and $e_{\mathfrak{P}}(M) = (1-N\mathfrak{P}^{-M})^{-1/2}$ otherwise.

Status: this follows from the exact mass formula of Gan, Hanke and Yu [GHY] and the functional equation. In [Lean] it is proved from a statement on representations of one by hermitian lattices; we have not followed that proof. Simplest incarnation: the Gaussian ratio $M_r/M_{r-1} = (r-1)!\,\pi^{-r}\kappa_r$ of (2), where $\mathbb{Q}(\mathrm{i})$ is ramified at two and so lies outside the present hypotheses.

Problem 48.1.1. For $M \ge 2$, $L(M) = \zeta_F(M)$ for even M, and $L(M) = L_F(M,\chi)$ for odd M.

Problem 48.1.2. For $M \ge 2$, $|\log L(M)| \le \log \zeta_F(M) \le f \log \zeta(M)$.

Fix even $m = 2s+2 \ge 2$, $k \ge m+3$ and $r = k+m-1$, and two primes $g_0 \ne g_1$ split completely in E. The level indices of §47.3 are $I_\Pi = (I_\Pi(g_0)I_\Pi(g_1))^f$ and $I_\Sigma = (I_\Sigma(g_0)I_\Sigma(g_1))^f$ with $I_\Pi(q) = \prod_{i=1}^k (q^i-1)$ and $I_\Sigma(q) = q^k \prod_{i=1}^r (q^i-1)$.

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Problem 48.1.3. $|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n(n-1)/2}\prod_{i=1}^{n}(q^i-1)$. With $N = s(2k+2s+1)$ one has $k^2+N+km+m(m-1)/2 = r^2$, and hence $I_\Sigma(q) = |\mathrm{GL}_r(\mathbb{F}_q)|/q^{N+k(k-1)/2}$.

Problem 48.1.4. $I_\Sigma(q)/I_\Pi(q) = q^{km+m(m-1)/2}\prod_{j=k+1}^{r}(1-q^{-j})$. For $k=5$ and $m=2$ this is $q^5(q^6-1)$.

Recall from (4) the Vandermonde product $V_n(z) = \prod_{i<i'}(z_i - z_{i'})$ and $D_n(z) = V_n(z)/\prod_{i=0}^{n-1} i!$. For each $\rho \in \Phi$ let $z_\rho^{\mathrm{row}} \in \mathbb{R}^r$ be the row parameter and, for each word u, $z_{u\rho}^{\mathrm{col}} \in \mathbb{R}^k$ the column parameter, both strictly decreasing. Let T_u be the auxiliary prime of the word, let d_u be the rank of the source lattice N_u, and D^* the dimension budget.

Theorem 48.1.2 (The two counts)**.**

$$d_u = \mathrm{mass}(k)\, I_\Pi \prod_{\rho\in\Phi} D_k(z_{u\rho}^{\mathrm{col}})\,(T_u^k-1)^f, \qquad D^* = \mathrm{mass}(r)\, I_\Sigma \prod_{\rho\in\Phi} D_r(z_\rho^{\mathrm{row}}).$$

Status: the first is the class number formula for the column group at level T_u together with Weyl's dimension formula; the second is the volume of the row group times Harish-Chandra's formal degrees [HC], which are given by Weyl's formula. Both are proved in [Lean] from these inputs; we have read the statements, not the proofs. In [Adr, Hyp. 15.23] the budget is only bounded below, with a factor $(2/3)^f$.

For each word u and $\rho \in \Phi$ define the *critical level* $T_{u\rho}^{\mathrm{cr}} > 0$ by

$$(T_{u\rho}^{\mathrm{cr}})^k = (v/\pi)^{(m-1)k+m(m-1)/2}\ \mathrm{rd}_E^{-(m-1)/2}\,(g_0g_1)^{km+m(m-1)/2}\,\frac{V_r(z_\rho^{\mathrm{row}})}{V_k(z_{u\rho}^{\mathrm{col}})},$$

and put

$$E_a = \prod_{M=k+1}^{r} L(M)\cdot \prod_{i=0,1}\prod_{j=k+1}^{r}(1-g_i^{-j})^f\cdot(1-T_u^{-k})^{-f}.$$

Problem 48.1.5. $\sum_{M=k+1}^{r} M = (m-1)k + m(m-1)/2$, and

$$\frac{\mathrm{mass}(r)}{\mathrm{mass}(k)} = \Big((v/\pi)^{(m-1)k+m(m-1)/2}\ \mathrm{rd}_E^{-(m-1)/2}\prod_{i=k}^{r-1} i!\Big)^f \prod_{M=k+1}^{r} L(M).$$

Problem 48.1.6. $\prod_{i=k}^{r-1} i!\cdot D_r(z)/D_k(z') = V_r(z)/V_k(z')$ for all $z \in \mathbb{R}^r$ and $z' \in \mathbb{R}^k$ with $V_k(z') \neq 0$.

Combining Theorem 48.1.2 with Problems 48.1.4, 48.1.5 and 48.1.6 gives the following.

Theorem 48.1.3 (The count ratio)**.** *For every word u,*

$$\frac{D^*}{d_u} = E_a \prod_{\rho\in\Phi}\Big(\frac{T_{u\rho}^{\mathrm{cr}}}{T_u}\Big)^k.$$

No factorial survives, as in the Gaussian count identity (Theorem 21.4.1). What remains of the masses is the power of v/π and the L-values in E_a.

48.2. The prime and the critical levels

How close is T_u to the critical levels? The factor E_a is exponentially close to one in k.

Problem 48.2.1. For $M \geq 3$, $\log \zeta(M) \leq 2^{1-M}$.

Problem 48.2.2. If $g_0, g_1 \geq 4$ and $T_u \geq 2$, then $|\log E_a| \leq 8f/2^k$.

The prime T_u is chosen so that d_u matches the budget up to the frame factor $\Gamma_0 \geq 1$ of §47.3, the same for all words: for a constant $C_0 \geq 1$,

$$C_0^{-1}\Gamma_0^{-2}D^* \leq d_u \leq C_0\Gamma_0^{-2}D^*. \tag{1}$$

The formalization takes $C_0 = 3$. The prime itself is part of its data, subject to the interval $Z_u \leq T_u \leq (1 + 1/(fk))Z_u$ of Problem 47.3.3.

Problem 48.2.3. Assume (1), $8f/2^k \leq \log 2$ and $\log \Gamma_0 \leq k/8$, together with the hypotheses of Problem 48.2.2. Then

$$(2C_0)^{-1/k} \leq \prod_{\rho \in \Phi} \frac{T_{u\rho}^{\mathrm{cr}}}{T_u} \leq (2C_0)^{1/k} e^{1/4}.$$

In the application, $\Gamma_0 = \exp(C_\Gamma m^5 f y)$ with $y = x^\varepsilon$, and the size condition $8C_\Gamma m^5 f y \leq k$ gives $\log \Gamma_0 \leq k/8$. Thus $\log T_u$ is the mean of $\log T_{u\rho}^{\mathrm{cr}}$ over Φ up to $O(1/f)$. The same comparison bounds the ratio of the primes of two words.

Problem 48.2.4. Let u, u' be words satisfying (1), with $|\log E_a| \leq 8f/2^k$ for both and $k \geq 5$. If $T_{u\rho}^{\mathrm{cr}} \leq C_1 T_{u'\rho}^{\mathrm{cr}}$ for every ρ, then $T_u \leq eC_0C_1T_{u'}$.

48.3. The critical level from below

Which part of $T_{u\rho}^{\mathrm{cr}}$ is large? Fix ρ and drop it from the notation. The row parameter is $z^{\mathrm{row}} = (\alpha_1, \dots, \alpha_{k-1}, -b_1, \dots, -b_m)$ with

$$\alpha_i = \alpha_{k-1} + 4\Delta(k-1-i), \qquad \alpha_{k-1} \geq 0, \quad \alpha_1 \leq 4k\Delta, \quad \Delta \geq 1,$$

and nodes $1 \leq b_1$, $b_{j+1} \geq b_j + 1$. A word is $\delta \in \{0,1\}^{k-1}$ with $\delta_{k-1} = 0$, and the column parameter is $z^{\mathrm{col}} = (\gamma_1, \dots, \gamma_k)$ with

$$\gamma_i = \alpha_{i-1} - \tfrac{1}{2} - \delta_{i-1}\Delta \quad (2 \leq i \leq k), \qquad \gamma_1 - \alpha_1 \geq 4k\Delta,$$

and we put $\eta = \gamma_1 - \alpha_1$. This is the case of a place ρ that is not a top. At a top the last entry is replaced by $\gamma_k = -b'$ with $b_1/2 \leq b' \leq b_1 - 3$, and we put $\eta = \gamma_1 b'/(k\Delta)$ instead.

Problem 48.3.1. $\gamma_1 > \alpha_1 > \gamma_2 > \alpha_2 > \cdots > \alpha_{k-1} > \gamma_k$, and z^{row} is strictly decreasing.

Problem 48.3.2. Let $B > 0$ with $B^k = (v/\pi)^{m(m-1)/2}\,\mathrm{rd}_E^{-(m-1)/2}(g_0g_1)^{m(m-1)/2}$. Then

$$T^{\mathrm{cr}} = (v/\pi)^{m-1}(g_0g_1)^m\, B\left(V_r(z^{\mathrm{row}})/V_k(z^{\mathrm{col}})\right)^{1/k},$$

and if $\mathrm{rd}_E \geq 1$ and $g_0g_1 \geq 2\pi$, then $1 \leq B \leq \exp\bigl(m^2 \log(g_0g_1\,\mathrm{rd}_E)/(2k)\bigr)$.

Problem 48.3.3.

$$V_r(z^{\mathrm{row}}) = V_{k-1}(\alpha) \prod_{i,j}(\alpha_i + b_j) \prod_{j<j'}(b_{j'} - b_j) \geq V_{k-1}(\alpha)\,(b_1 \cdots b_m)^{k-1}.$$

Problem 48.3.4. $V_k(z^{\mathrm{col}}) = \prod_{i=2}^{k}(\gamma_1 - \gamma_i) \cdot V_{k-1}(\gamma_2, \dots, \gamma_k)$, and

$$V_{k-1}(\gamma_2, \dots, \gamma_k) = R_\delta \, V_{k-1}(\alpha), \qquad R_\delta = \prod_{1 \le i < i' \le k-1} \Big(1 + \frac{\delta_{i'} - \delta_i}{4(i' - i)}\Big).$$

Problem 48.3.5. $R_\delta \le e^{(k-2)/4}$.

The formalization also proves $R_\delta \ge e^{-k/4}$; we do not need it.

Problem 48.3.6. At a non-top place, $\eta \le \gamma_1 - \gamma_i \le 2\eta$ for $2 \le i \le k$. Hence $V_k(z^{\mathrm{col}}) \le (2\eta)^{k-1} e^{k/4} V_{k-1}(\alpha)$.

Problem 48.3.7. $2 \cdot 0.38 \cdot e^{1/4} < 1$. Let $\beta = b_1 \cdots b_m$, let $b \ge 1$ with $b_j \le b^2$ for all j, and put $e_0 = \exp(-2m \log b/k)$. Then $e_0^k \beta \le 1$, and $(0.38\, e_0 \beta/\eta)^k \, V_k(z^{\mathrm{col}}) \le \frac{1}{2} V_{k-1}(\alpha) \beta^{k-1}$.

By Problems 48.3.2, 48.3.3 and 48.3.7, at a non-top place $(V_r(z^{\mathrm{row}})/V_k(z^{\mathrm{col}}))^{1/k} \ge 0.38\, e_0 \beta/\eta$, and $B \ge 1$. The top case runs the same way with a separate Vandermonde count for the last column entry.

Theorem 48.3.1 (Lower bound for the critical level)**.** *Let $k \ge 4$, $b \ge 1$, $b_j \le b^2$ for all j, $\mathrm{rd}_E \ge 1$ and $g_0 g_1 \ge 2\pi$. At a non-top place*

$$T^{\mathrm{cr}} \ge c_1 \, (g_0 g_1)^m (v/\pi)^{m-1} \, \frac{b_1 \cdots b_m}{\eta}, \qquad c_1 = 0.38 \exp(-2m \log b/k).$$

At a top, the same holds with c_1 divided by $1 + 4k\Delta/b'$, under $40mk\Delta \le b_1$ and $20m \log b \le k$. Status*: the non-top case is Problems 48.3.2–48.3.7; the top case is proved in* [Lean]*, and we have not checked its Vandermonde count by hand.*

Problem 48.3.8. If $40mk\Delta \le b_1$ and $20m \log b \le k$, then $c_1 \ge 1/5$ in both cases of Theorem 48.3.1.

The word enters T^{cr} only through $V_k(z^{\mathrm{col}})$. The formalization also proves that, for $40mk\Delta \le b_1$ and any two words δ, δ', $|\log(V_k(z^{\mathrm{col}}_{\delta'})/V_k(z^{\mathrm{col}}_{\delta}))| \le k/2 + 1$ [Lean].

Problem 48.3.9. Under that bound and $k \ge 4$, the critical levels of two words differ by a factor between $e^{-3/4}$ and $e^{3/4}$.

With Problem 48.2.4 this gives $T_u \le e^{7/4} C_0 T_{u'}$ for any two words, comparing the critical levels at each place.

48.4. The auxiliary prime

Combining Problem 48.2.3 with Theorem 48.3.1 and Problem 48.3.8 gives the following.

Theorem 48.4.1 (Lower bound for the prime)**.** *Under the hypotheses of Problems 48.2.3 and 48.3.8 and of Theorem 48.3.1 at every $\rho \in \Phi$, with nodes $b_{\rho j}$ and $\eta_{u\rho}$ at ρ,*

$$\log T_u \ge m \log(g_0 g_1) + (m-1) \log \frac{v}{\pi} + \frac{1}{f} \sum_{\rho \in \Phi} \Big(\sum_{j=1}^{m} \log b_{\rho j} - \log \eta_{u\rho} \Big) - \log 5 - \frac{1}{f} \Big(\frac{\log(2C_0)}{k} + \frac{1}{4} \Big).$$

Problem 48.4.1. Theorem 48.4.1 follows from Problem 48.2.3, Theorem 48.3.1 and Problem 48.3.8.

Problem 48.4.2. Let $y = \log b$, $g = \log(g_0 g_1)$, and suppose $\log(v/\pi) \le y$, $b_{\rho j} \le b^2$ and $\eta_{u\rho} \le b_{\rho 1} b^{C_u}$. Then Theorem 48.4.1 gives

$$\log T_u \ge \sum_{j=1}^{m} \frac{1}{f} \sum_{\rho \in \Phi} \log \frac{b_{\rho j} v}{\pi} + mg - (3 + C_u)y - \log 5 - \frac{1}{f}\Big(\frac{\log(2C_0)}{k} + \frac{1}{4}\Big).$$

This is Theorem 47.3.1, the form of (A6) with the nodes $b_{\rho j}$ in place of $w_{j\rho}$ and an error of order y. In [Adr, §15.5] the nodes satisfy $b_{\rho j} < w_{j\rho} \le b_{\rho j}(1+1/x)$, so the two forms differ by at most m/x. The formalization bounds the first column entry by $\gamma_1 \le b^{C_u}$, which gives $\eta_{u\rho} \le b_{\rho 1} b^{C_u}$ at a top; elsewhere it imposes $\eta_{u\rho} \le 8k\Delta$, which is at most $b_{\rho 1}$ when $40mk\Delta \le b_{\rho 1}$.

In the Lean formalization. The mass ratio is `SiegelMassRatioCM`, proved as `massRatio_all_unramified_cmFields`, with right side `massRatioRightSide` and $L(M)$ `lValEta`; the bound of Problem 48.1.2 is `log_l_value_field_bound`. Theorem 48.1.2 is `FamilyMidVolume`, proved by `admissible_ranks_of_mass_alt` through `Dstar_mass` and `admissibleRanks_of_mass_formula`, with formal degrees `DsigHC`. The critical level is `Tcrit`. Theorem 48.1.3 is `Sz.PropA`, used in `degree_ratio_product_formula`; Problems 48.2.2, 48.2.3 and 48.2.4 are `Sz.EaSize`, `Sz.CorA` and `level_ratio_bound`. Section 48.3 follows `BracketBound`, `VandLamLower`, `VandNonTop`, `VandTop`, `EdgeFactors` and `TcritLower` (proved as `critical_threshold_lower_bound`, used in `critical_level_lower_bound`), and the word comparison is `TcritWords`. The window (1) is part of `StandingNumericalData`; the interval for T_u is `levelPrime_window`. Downstream, the formalization uses $T_u \ge (g_0 g_1 v/\pi)^{m-2}$ (`TFrameAt`) and $b^{m/2} \le T_u \le b^{(4C_g+13)m}$ (`levelPrime_size`) [Lean].

Status: we have not traced where the formalization uses a lower bound for $\log T_u$ in the mean slope inequality, nor how it constructs a prime in the window; we located only the consequences named above. Theorem 48.4.1 is our combination of its lemmas.

Hints

§48.1.

48.1.1: Since E/F is unramified, $\bar{\mathfrak{P}} = \mathfrak{P}$ exactly when $\mathfrak{P}$ lies over an inert prime of F. Group the factors over each prime of F.

48.1.2: Compare each Euler factor with $(1 - N\mathfrak{p}^{-M})^{\pm 1}$, and the primes of F over p with $(1 - p^{-M})^{-f}$.

48.1.3: Count ordered bases. For the identity, expand r^2 with $r = k + 2s + 1$.

48.1.5: Multiply Theorem 48.1.1 over M; use $2v = \mathrm{rd}_E$.

§48.2.

48.2.1: Bound $\sum_{n \ge 3} n^{-M}$ by an integral.

48.2.2: Use $-\log(1-t) \le 2t$ for $0 \le t \le \frac{1}{2}$ and Problems 48.1.2 and 48.2.1.

48.2.3: Theorem 48.1.3 with $\frac{1}{2} \le E_a \le 2$.

48.2.4: Compare $T_u^{kf} \prod_\rho (T_{u\rho}^{\mathrm{cr}}/T_u)^k$ for u and u'; (1) gives $d_u \le C_0^2 d_{u'}$.

§48.3.

48.3.2: Substitute $\mathrm{rd}_E = 2v$ in B^k.

48.3.5: Group the pairs by $d = i' - i$.

48.3.6: γ_k is the smallest column entry. Then use Problems 48.3.4 and 48.3.5.

48.3.7: Insert Problem 48.3.6 and use $\eta \geq 1$.

48.3.8: At a top, $b' \geq b_1/2$; and $e^{-t} \geq 1 - t$.

§48.4.

48.4.1: Take logarithms in Problem 48.2.3 and average Theorem 48.3.1 over Φ.

References

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[Lean] The Lean community, *riemann-hypothesis*, a Lean formalization of Alpöge's argument, `https://github.com/leanprover-community/riemann-hypothesis`.