THE BESSEL IDENTITY

Prerequisites. From Chapter 35: the lattices and the period maps B_u of Section 35.5, and the coefficient field L with its fixed embedding $\sigma\colon L \to \mathbb{C}$. From Chapter 43: Bessel realizations (1), Theorems 43.2.1, 43.2.2 and 43.3.1, and Problems 43.4.2 and 43.4.3. From Chapter 44: the Poincaré series P_u, the spaces C_u, the toy of Section 44.3 and Theorem 44.4.1. From Chapter 45: Problems 45.1.2, 45.1.3, 45.1.4 and 45.3.2, and Theorem 45.5.1. From Chapter 23: the period of Section 23.4. Inner products on complex Hilbert spaces are conjugate-linear in the first variable, as in Chapter 43.

Condition (d) of Theorem 43.3.1 asks that the period maps have a Bessel realization: each B_u is, up to unitary changes of variable and a scalar, an inner product with a Poincaré series. Chapter 45 moved the pairing at the top of the chain to a Bessel period at the bottom, and Chapter 44 wrote Bessel periods as inner products with Poincaré series. This chapter assembles these into the identity that the Lean formalization [Lean] proves, and says what it gives.

46.1. The statement

For one record, let $\mathcal{H}$ be the space of automorphic forms of Chapter 44, of dimension D. Let $\mathcal{E}_0 := (R \otimes L) \otimes_{L,\sigma} \mathbb{C}$ be the complexification of the row space of Chapter 35, of dimension $D_{\mathrm{r}} \le D$, with the inner product given by the norm of R at σ, and let $s\colon \mathcal{E}_0 \to \mathcal{H}$ be a conjugate-linear isometry, the row embedding. For each word u let $M_u \subset C_u$ be the complexification of the forms of level 0 of u ramified at T_u, with the inner product of C_u, and $P_u\colon C_u \to \mathcal{H}$ the Bessel–Poincaré map. The period map gives a $\mathbb{C}$-bilinear pairing $B_u\colon M_u \times \mathcal{E}_0 \to \mathbb{C}$; on vectors coming from the L-structures it is defined, as in (B) of Section 35.5, by the pairing at the end of the chain, and B_u^{alg} denotes the corresponding L-valued pairing of the lattices.

Theorem 46.1.1 (Bessel identity)**.** *For every record of the family at a large scale:*

(i) $B_u(\varphi, F) = \sigma\big(B_u^{\mathrm{alg}}(\varphi, F)\big)$ *for φ and F in the L-structures;*

(ii) *there are a unitary operator V of $\mathcal{E}_0$ and, for each word u, a number $a_u \neq 0$ and a unitary operator U_u of M_u with*

$$B_u(\varphi, F) = a_u \,\langle sVF,\ P_u U_u \varphi\rangle_{\mathcal{H}} \qquad (\varphi \in M_u,\ F \in \mathcal{E}_0).$$

Status: proved in the formalization, for the family it constructs; the proof is the subject of this chapter, and we say at each step how far we have followed it. In [Adr] the statement is Theorem 43.4.3 of Chapter 43, derived there from hypotheses assumed there, among them a seesaw relation for an abstract period map, which the formalization does not state.

Problem 46.1.1. The right side of (ii) is $\mathbb{C}$-linear in φ and in F. If s were $\mathbb{C}$-linear and (ii) held with B_u bilinear, then $B_u = 0$.

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Problem 46.1.2. By Theorem 46.1.1(i), B_u is the $\mathbb{C}$-bilinear extension of $\sigma \circ B_u^{\mathrm{alg}}$, that is, the map $B_{u,v}$ of Chapter 43 at the place v of σ; so Theorem 46.1.1(i) and (ii) give a Bessel realization (1) of the maps B_u, with $Y_v = \mathcal{E}_0$, $V_{u,v} = M_u$, $\iota_0 = s$, $U = V$, $\alpha_u = a_u$, ι_u the inclusion $M_u \subset C_u$, and the same U_u.

In the toy of Section 44.3, let $\mathcal{E}_0 := \mathcal{A}_{(a)}(2)$, with $s(F)$ the form whose coefficients in the basis e^ν are the complex conjugates of those of F, let $M_j = C_j := \mathcal{A}_{(j)}(1)$, and put $B_j(\xi, F) := B(sF, \xi)$, with the period B of Section 44.2 read with the inner product of this chapter, so that $B(\psi, \xi)$ is conjugate-linear in ψ; it is the complex conjugate of the period $P(\psi, \xi)$ of Chapter 23 when the latter is read as linear in ψ.

Problem 46.1.3. s is a conjugate-linear isometry of $\mathcal{A}_{(a)}(2)$, B_j is bilinear, and $B_j(\xi, F) = \langle sF, P\xi \rangle$: Theorem 46.1.1(ii) holds with $a_j = 1$ and $U_j = V = 1$. For $a = 8$: $B_0(\xi_0, e_1^8 + e_2^8) = 1/4$ and $B_4(\xi_4, e_1^4 e_2^4) = 1/280$.

Problem 45.3.2 of Chapter 45 adds the chain: for $F = \Theta_{(a)}\varphi$ with φ real, the period at the top equals $B_j(\xi, F)$.

46.2. From the kernel identity to the Bessel identity

In the formalization the row space is reached from below: there are a finite-dimensional inner product space V_0 of forms and a linear map $\iota\colon V_0 \to \mathcal{E}_0$ with $\iota(V_0) = \mathcal{E}_0$, and $B_u(\varphi, \iota q) = \mathrm{per}_u(\varphi, q)$ for $q \in (\ker \iota)^\perp$, where per_u is the pairing at the end of the chain of u at the base twist. The *row dictionary* is an isometry j of $\mathcal{E}_0$ into L^2 of the automorphic quotient with $s(F) = \overline{j(F)}$. The *column dictionary* is a unitary operator W_u of M_u: the column of the definite side attached to $\psi \in M_u$ is the element $W_u\psi$ of $M_u \subset C_u$. In Sections 46.3 and 46.4, $P_u\psi$ for $\psi \in M_u$ means $P_u W_u \psi$.

Theorem 46.2.1 (kernel identity)**.** *There are a unitary operator V of $\mathcal{E}_0$ and, for each word u, a number $a_u \neq 0$ and a unitary operator U_u of M_u such that*

$$\mathrm{per}_u(\varphi, q) = a_u \left\langle \overline{j(V\iota q)},\ P_u W_u U_u \varphi \right\rangle$$

for all $\varphi \in M_u$ and $q \in V_0$.

Status: proved in the formalization from Theorem 46.3.1 below by the argument of Section 46.3.

Problem 46.2.1. Let $\iota\colon V_0 \to \mathcal{E}_0$ be a linear map of finite-dimensional inner product spaces with $\iota(V_0) = \mathcal{E}_0$. For every $x \in \mathcal{E}_0$ there is $q \in (\ker \iota)^\perp$ with $\iota q = x$.

Problem 46.2.2. Theorem 46.2.1, the two dictionaries and $B_u(\varphi, \iota q) = \mathrm{per}_u(\varphi, q)$ imply Theorem 46.1.1(ii), with U_u replaced by $W_u U_u$.

46.3. Bilinear forms on bases

The step behind Theorem 46.2.1 compares two bilinear forms on coordinate vectors. Fix an L-basis $m_1, \ldots, m_d$ of the L-structure of M_u, and write $\varphi \in L^d$ also for $\sum_l \varphi_l m_l \in M_u$. The pairing per_u is $\mathbb{C}$-bilinear on $M_u \times V_0$, and V_0 has an L-structure with a basis $r_1, \ldots, r_n$, which is a $\mathbb{C}$-basis of V_0; for $F \in L^n$ put $\mathrm{row}(F) := \sum_k F_k r_k \in V_0$. For each twist t there are row coordinates $F \in L^{n_t}$ (with $n_{t_0} = n$), a $\mathbb{C}$-linear map Θ_t from $\mathbb{C}^{n_t}$ to $\mathcal{E}_0$, and an L-bilinear period $\mathrm{Pt}_{u,t}\colon L^d \times L^{n_t} \to L$ of the definite side; the period map at the twist t on coordinates is

$B_{u,t}(\varphi,F)=\sigma\,\mathrm{Pt}_{u,t}(\varphi,F)$, and at the base twist $\mathrm{per}_u(\varphi,\mathrm{row}(F))=\sigma\,\mathrm{Pt}_{u,t_0}(\varphi,F)$ (the *chain tie*). Finally $\iota(\mathrm{row}(F))=\lambda^{-1}\Theta_{t_0}(F)$ for a number $\lambda\neq 0$ (the *row tie*).

Theorem 46.3.1 (multiplicity identity)**.** *There are a unitary operator V of $\mathcal{E}_0$ and, for each word u, numbers $a_u\neq 0$, a unitary operator U_u of M_u, and numbers $\zeta(u,t)\neq 0$ with $\zeta(u,t_0)=1$, such that*

$$\zeta(u,t)\,\sigma\big(\mathrm{Pt}_{u,t}(\varphi,F)\big)=a_u\,\big\langle sV\Theta_t(F),\ P_uU_u\varphi\big\rangle$$

for all twists t and all coordinate vectors $\varphi\in L^d$, $F\in L^{n_t}$. Moreover, if $\zeta(u,t)=\sigma(a)$ with $a\in L$, then $v_w(a)=0$ at every finite place w of L above no prime of the set S_t of Chapter 45.

Problem 46.3.1. Let $\beta\colon L^d\times L^n\to L$ be L-bilinear and $\Phi\colon\mathbb{C}^d\times\mathbb{C}^n\to\mathbb{C}$ be $\mathbb{C}$-bilinear, with $\Phi(e_i,e_k)=\sigma\beta(e_i,e_k)$ on the standard basis vectors. Then $\Phi(\sigma x,\sigma y)=\sigma\beta(x,y)$ for all $x\in L^d$, $y\in L^n$, and Φ is determined by β.

Problem 46.3.2. Assume Theorem 46.3.1, the chain tie and the row tie. Then the two $\mathbb{C}$-bilinear forms $(\varphi,q)\mapsto\mathrm{per}_u(\varphi,q)$ and $(\varphi,q)\mapsto a_u\lambda\langle sV\iota q,P_uU_u\varphi\rangle$ on $M_u\times V_0$ agree on (m_l,r_k), hence everywhere; this is Theorem 46.2.1 with $a_u\lambda$ in place of a_u.

In the formalization the embedding in the chain tie may be the complex conjugate of σ, and then the scalars are conjugated accordingly; we suppress this.

46.4. The multiplicity identity

Each $e\in\mathcal{E}_0$ has a theta lift R_e, an automorphic form on the indefinite group at the top, whose restriction to the base is a row form T_e in the sense of Chapter 45; every row form at the base is some T_e, and $\mathrm{row}_t(F)$ of Chapter 45 is $T_{\Theta_t(F)}$. Recall that P_ux means P_uW_ux. Theorem 46.3.1 comes from five inputs.

(E) There are a vector v of the archimedean discrete series and an isometric bijection J from the space $\{\mathsf{R}_e:e\in\mathcal{E}_0\}$ onto $\mathcal{H}$, such that $J(\mathsf{R}_e)$ is almost everywhere the complex conjugate of the evaluation of R_e at v. *Status*: proved in the formalization from a proposition stating a classification of discrete series of unitary groups [HC]; we have not traced how that proposition is supplied.

(Pa) There are a unitary operator V of $\mathcal{E}_0$, induced by translation by a central element, and $\kappa\neq 0$ such that $\langle sVe,P_ux\rangle=\kappa\,b(e,x)$ for $e\in\mathcal{E}_0$ and $x\in M_u$, where $b(e,x):=\langle J(\mathsf{R}_e),P_ux\rangle_{\mathcal{H}}$ is the integral of the evaluated theta lift of e against the Poincaré series of x, without complex conjugation. *Status*: proved in the formalization; we have read only its statement.

(J) Theorem 45.5.1 of Chapter 45, with its unitary operator U of M_u, its constants c_t and, for each row form T, its function p.

(Bo) There is a unitary operator R_u of M_u such that for all $e,e'\in\mathcal{E}_0$, $x,x'\in M_u$ and $\alpha,\alpha'\in\mathbb{C}$: if $\alpha\,\mathrm{Bper}(x,T_e)+\alpha'\,\mathrm{Bper}(x',T_{e'})=0$, then $\alpha b(e,R_ux)+\alpha' b(e',R_ux')=0$. *Status*: proved in the formalization; by our reading it is the unfolding of Chapter 44, but we have not read its proof.

(NZ) For every word u there is $\varphi\in M_u$ with $P_u\varphi\neq 0$. *Status*: an assumption of the step of the formalization that proves Theorem 46.3.1; we have not traced where it is proved.

Problem 46.4.1. Assume the second inequality of Theorem 44.4.1 with $e_0 \le \frac{1}{2}$, and $\dim M_u > \frac{1}{2}\dim C_u$. Then (NZ) holds.

Problem 46.4.2. (E) and (NZ) give, for every word u, vectors $e^\circ \in \mathcal{E}_0$ and $x^\circ \in M_u$ with $b(e^\circ, x^\circ) \neq 0$.

Problem 46.4.3. Assume (J), (Bo) and the conclusion of Problem 46.4.2, and put $b(t,\varphi,F) := b(\Theta_t(F), R_uU\varphi)$. There are numbers $p_0 \neq 0$ and $b_0 \neq 0$, depending on u, such that $c_t\, B_{u,t}(\varphi,F)\, b_0 = p_0\, b(t,\varphi,F)$ for all t, φ, F.

Problem 46.4.4. Assume Problem 46.4.3 and (Pa). Then the identity of Theorem 46.3.1 holds with $\zeta(u,t) = c_t/c_{t_0}$, $a_u = p_0/(\kappa b_0 c_{t_0})$ and $U_u := R_uU$.

Problem 46.4.5. In Problem 46.4.4, $\zeta(u,t_0) = 1$ and $\zeta(u,t) \neq 0$; and if $\zeta(u,t) = \sigma(a)$ with $a \in L$, then $v_w(a) = 0$ at every finite place w of L above no prime of the set S_t of Chapter 45.

Problems 46.4.2–46.4.5 prove Theorem 46.3.1 from (E), (Pa), (J), (Bo) and (NZ); Problems 46.3.1 and 46.3.2 give Theorem 46.2.1; and Problems 46.2.1 and 46.2.2 give Theorem 46.1.1(ii). Part (i) is proved in the formalization from the same multiplicity identity and the chain tie, by an argument we have not followed.

46.5. What the identity gives

Fix orthonormal bases (b_k) of $\mathcal{E}_0$, (e_l) of M_u and (f_i) of $\mathcal{H}$. In the bases of the L-structures the matrix of B_u differs from the one below by invertible matrices, the Y_i and Y' of Theorem 43.2.1.

Problem 46.5.1. Let $\mathsf{S}_{ki} := \langle sVb_k, f_i\rangle$ and $\Pi_{il} := \langle f_i, P_uU_ue_l\rangle$. Then $B_u(e_l,b_k) = a_u(\mathsf{S}\Pi)_{kl}$, and S is a $D_{\mathrm{r}} \times D$ matrix with orthonormal rows.

Complete S to a unitary $D \times D$ matrix $\hat{\mathsf{S}}$, and put $A_u := \hat{\mathsf{S}}\Pi$, whose first D_{r} rows are a_u^{-1} times the matrix of B_u.

Problem 46.5.2. With $\gamma_u := c_0(u)^{-1}$ and K_{uu} as in Chapter 44, $\gamma_uA_u^*A_u - 1 = U_u^*(\iota_u^*K_{uu}\iota_u - 1)U_u$, where $\iota_u\colon M_u \to C_u$ is the inclusion; and $Q := \sum_u \gamma_uA_uA_u^*$ is unitarily equivalent to $\sum_u \gamma_uP_u\iota_u\iota_u^*P_u^*$. So the hypotheses of Theorem 43.2.2 for the matrices $\hat{\mathsf{S}}P_u$ and $V_u := \iota_uU_u$ are those for P_u and ι_uU_u.

Problem 46.5.3. Assume the second inequality of Theorem 44.4.1. Then the rank of B_u is at least $\dim M_u - e_0\dim C_u - (D - D_{\mathrm{r}})$.

With Theorem 44.4.1 Problem 46.5.2 gives conditions (v) and (vi) of Theorem 43.2.1, through Theorem 43.2.2, and Problem 46.1.2 gives (d) of Theorem 43.3.1.

In the Lean formalization. In [Lean], Theorem 46.1.1 is `BesselId`, a clause of `LinkDIV_at`; part (i) is `D7.BuRational`, proved through `buRat_rrFull_closed_alt`, and part (ii) is `adm13`, with $\langle sF, P_u\varphi\rangle$ the definition `D7.besS`. The period at the end of the chain is `perCol` (`PtilCol` of `ColumnChain.col`), tied to B_u by `Rows.BuDefQ`. Problems 46.2.1 and 46.2.2 are `exists_Q_iota` and `besselId_of_joins`, reached from `adm13_of_kernel_join` and `adm13_of_colJoin`; Theorem 46.2.1 is `KerJoinS`, and Problem 46.3.2 is `kernelJoin_of_multiplicity`. Theorem 46.3.1 is `MIstar2` (with `MIid`, `MIUnit2`, `AuNZ`), proved by `miStar2_at_realisation_alt` and `miStar2_of_anchored`. The

inputs are `exists_conjVec` (E), `pairForm_alt` (Pa), `JointCondition` (J), `botRel_of_pin_of_mem_center` (Bo) and `PuNZ` (NZ); Problem 46.4.3 is `anchor_pairs_alt` with `anchor_word_alt`. Problems 46.5.1 and 46.5.2 correspond to `blockReal_at_of` and `matK_at_of`.

Hints

§46.1.

46.1.1: Replace F by $\mathrm{i}F$.
46.1.3: Problem 44.2.1; Problem 44.3.2 and $\|e^{\nu}\|^2 = \nu!/a!$.

§46.2.

46.2.1: Subtract from any preimage its projection to $\ker \iota$.
46.2.2: Write the second argument as ιq with $q \in (\ker \iota)^{\perp}$, by Problem 46.2.1.

§46.3.

46.3.1: Expand in the bases.
46.3.2: Problem 46.3.1 at $t = t_0$, where $\zeta = 1$.

§46.4.

46.4.1: Problem 43.4.3.
46.4.2: J is onto.
46.4.3: Take $T := T_{e^{\circ}}$ in (J).
46.4.5: Theorem 45.5.1.

§46.5.

46.5.1: Expand $P_u U_u e_l$ in the basis (f_i); s is a conjugate-linear isometry.
46.5.2: $\hat{\mathsf{S}}^{*}\hat{\mathsf{S}} = 1$.
46.5.3: Problems 46.5.1 and 43.4.3; S has a kernel of dimension $D - D_{\mathrm{r}}$.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

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[Lean] The Lean community, *riemann-hypothesis*, a Lean formalization of Alpöge's argument, `https://github.com/leanprover-community/riemann-hypothesis`.