THE SEESAW AS TRANSPORT OF RELATIONS

Prerequisites. From Chapter 23: the lift Θ_λ, the numbers c_a with Problems 23.2.1 and 23.2.2, and Section 23.4, in particular Problem 23.4.2. From Chapter 35: the ranks and spaces X_j, Y_j of Section 35.2 with Problems 35.2.2 and 35.2.3, the forms and lifts of Section 35.4, and the period map (B) of Section 35.5. From Chapter 44: the Bessel subgroup and the toy of Section 44.3. Valuations at finite places of a number field.

The period map B_u of Chapter 35 lifts a form of level 0 along the chain $K_m \cdots K_1$ and pairs the result at the top, along $U(X') \subset U(X_m)$. Chapter 44 turns a Bessel period at the bottom, along $U(X_0) \subset U(Y_0)$, into an inner product with a Poincaré series. How is a pairing at the top moved to the bottom? By seesaw identities [Ku], one pair of levels at a time. In the Lean formalization [Lean] they are proved in a weaker form than identities with explicit constants: every linear relation among the values at one end holds among the values at the other. This is all the argument needs, because the constants that an identity would carry only have to be nonzero, and units at the right places.

45.1. Transport of relations

Let I be a set and $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I}$ families in complex vector spaces X and Y. We say that *the relations of x pass to y* if $\sum_i \alpha_i x_i = 0$ implies $\sum_i \alpha_i y_i = 0$ for every finitely supported family (α_i) of complex numbers.

Problem 45.1.1. The relations of x pass to y if and only if there is a linear map T from the span of the x_i to Y with $Tx_i = y_i$ for all i; such a T is unique. If the relations also pass back from y to x, then T is injective.

Problem 45.1.2. Let $X = Y = \mathbb{C}$, suppose the relations of x pass to y, and let $x_{i_0} \neq 0$. Then $y_i x_{i_0} = y_{i_0} x_i$ for every i. If moreover $y_{i_0} \neq 0$, then $y = cx$ with $c = y_{i_0}/x_{i_0} \neq 0$.

Problem 45.1.3. If the relations of x pass to y and those of y pass to z, then those of x pass to z. If the relations of x pass to y and $c_i \in \mathbb{C}$, then those of $(c_i x_i)$ pass to $(c_i y_i)$.

The second half of Problem 45.1.2 is how proportionality is recovered: one nonzero value at both ends fixes the constant. The constants that arise must also have valuation zero at certain places. Let S be a set of primes. Following the formalization, call a pair (x, y) of complex numbers an *S-unit pair* if $y \neq 0$ and $(x/y)^n = \beta$ for some integer $n \geq 1$ and some $\beta \in \mathbb{Q}^\times$ such that every prime p with $v_p(\beta) \neq 0$ lies in S.

Problem 45.1.4. Let (x, y) be an S-unit pair, L a number field with an embedding $\sigma\colon L \to \mathbb{C}$, $a \in L$ with $\sigma(a) = x/y$, and w a finite place of L above no prime of S. Then $v_w(a) = 0$. For $L = \mathbb{Q}(\mathrm{i})$, $a = 1 + \mathrm{i}$ and $S = \{2\}$, $n = 4$ works, and $v_w(a)$ is 1 at the place above 2.

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45.2. The tower

By Problems 35.2.2 and 35.2.3, for $j < m$ the complement of X_j in Y_j has rank $m-1-j$. A *Bessel datum of rank* ρ at level j is a decomposition of this complement into ρ hyperbolic planes and an anisotropic line, as in Section 44.1; a *Fourier–Jacobi datum of rank* ρ is a decomposition into ρ hyperbolic planes. A Fourier–Jacobi period integrates over the unipotent radical of the stabilizer of an isotropic flag of length ρ, against a character and a theta function of the Weil representation, instead of a character alone [GGP, §§12, 23].

Problem 45.2.1. The complement of X_j in Y_j has the rank of a Bessel datum of rank ρ if and only if $j+2\rho+2=m$, and the complement of X_{j+1} in Y_{j+1} has the rank of a Fourier–Jacobi datum of rank ρ if and only if $j+2\rho+2=m$. So Bessel periods occur at even levels and Fourier–Jacobi periods at odd levels.

Problem 45.2.2. Let j be even and $j+2\rho+2=m$ (here ρ is a rank, not an embedding). At every embedding, the signatures of Problem 35.2.3 are $(\rho,\rho+1)$ at level j and (ρ,ρ) at level $j+1$: those of ρ hyperbolic planes and a negative line, and of ρ hyperbolic planes.

The seesaw moves from a Bessel datum of rank $\sigma+1$ at level j, through the Fourier–Jacobi datum of rank $\sigma+1$ at level $j+1$, to the Bessel datum of rank σ at level $j+2$: a *double step*.

Problem 45.2.3. The double step $(j,\sigma+1)\mapsto(j+2,\sigma)$ preserves $j+2\rho+2=m$. Starting from $(0,m/2-1)$ it reaches $(m-2,0)$ after $m/2-1$ steps. For $m=6$ the Bessel data are at levels $0,2,4$, of ranks $2,1,0$, and the Fourier–Jacobi data at levels $1,3,5$, of ranks $2,1,0$.

At level $m-2$ the Bessel datum has rank 0: the period is a restriction along a line. The last two lifts lead to X_m, where the pairing is the restriction along $X'=z^\perp\subset X_m$, of codimension one. For the smallest ranks $(m,k,t_0)=(2,8,15)$ of Problem 35.2.1 there is no double step at all.

45.3. The definite seesaw

Problem 23.4.2 of Chapter 23 is the smallest seesaw. For $\varphi\in\mathcal{A}_{(a)}(1)$ and $\xi\in\mathcal{A}_{(j)}(1)$, with the period P and the vector θ_z defined there, it says

$$P(\Theta_{(a)}\varphi,\xi)=\big\langle\varphi,\,(\Theta_{(j)}\xi)\,\theta_z\big\rangle.$$

The left side lifts φ and pairs at the bottom; the right side lifts ξ and pairs at the top. Kudla's principle is that both sides integrate one theta kernel, of a decomposed space, in two orders [Ku].

Problem 45.3.1. For $a=8$, $j=4$ and $\varphi(I_1)=\xi(I_1)=1$, both sides of the seesaw equal $c_4^2/16=\vartheta_0^2(\varpi/2\pi)^8=0.0012814849\ldots$, with $c_4=4\vartheta_0(\varpi/2\pi)^4=0.1431913370\ldots$.

Problem 45.3.2. With the Poincaré series $P\xi$ of Section 44.3, $\langle\Theta_{(a)}\varphi,P\xi\rangle=\langle\varphi,(\Theta_{(j)}\xi)\theta_z\rangle$. Hence, for fixed ξ, the relations of the family $\varphi\mapsto\langle\varphi,(\Theta_{(j)}\xi)\theta_z\rangle$ pass to $\varphi\mapsto\langle\Theta_{(a)}\varphi,P\xi\rangle$, and back.

So in the smallest case a period at the top of a chain is the inner product of the lifted form with a Poincaré series. The rest of this chapter, with Chapter 44 and the next chapter, is this statement for the chain of the argument.

45.4. The seesaw in the argument

We use the following data of the formalization, for one word and a family of twists t by Hecke characters; the dependence on t is not shown. For each level $j \le m$ there are *column forms*, forms on $U(X_j)$, with the twist S_j and the lift K_j from level j to level $j+1$; and *row forms*, a space C_j of automorphic forms on $U(Y_j)$ inside a space of theta lifts, with maps $\iota_j \colon C_{j+1} \to C_j$ (theta lifts down), a space C_T of forms on $U(X')$, and $\iota_T \colon C_T \to C_{m-1}$. There are nonzero constants $c_I(j)$, $c_V(j)$, $c_K(j)$, c_P. For $j+2\rho+2=m$, a column form ψ of level j and $G \in C_j$, let $\mathrm{per}_j(\psi, G)$ be the Bessel period of rank ρ of G against theta functions of ψ; it is a function of auxiliary variables, and "$=0$" below means identically. For a column form ψ of level m and $F \in C_T$, let $\mathrm{tail}(\psi, F)$ be the restriction pairing along $X' \subset X_m$.

Theorem 45.4.1 (double step)**.** *Let $j+2(\sigma+1)+2=m$. Let ψ_i be column forms of level j, possibly for different twists, $G_i \in C_{j+2}$, and $\alpha_i \in \mathbb{C}$, for i in a finite set. If*

$$\sum_i \alpha_i \, c_V(j) c_K(j) c_K(j+1) \, \mathrm{per}_{j+2}\big(K_{j+1}K_jS_j\psi_i,\, G_i\big) = 0,$$

then $\sum_i \alpha_i \, c_I(j) c_I(j+1) \, \mathrm{per}_j(\psi_i, \iota_j\iota_{j+1}G_i) = 0$.

Status: proved in the formalization from four laws. (i) Unfolding of a double step: a Bessel period of rank $\sigma+1$ of a function lifted twice from level $j+2$ is an iterated integral, over a quotient of the Fourier–Jacobi subgroup of rank $\sigma+1$ at level $j+1$ and then over a quotient of the Bessel subgroup of rank σ at level $j+2$, of a Bessel period of rank σ; it is derived from the descent of Weil data. (ii) The value of that integral for the column forms of the construction, which collapses the finite-adelic integrals to finite sums over cells and moves base points. (iii) Two consecutive lifts of column forms compose with the constants c_K. (iv) Rapid decay of cusp forms. We have read the statements of (i) and (ii) and the proof of (ii) down to named lemmas, not further.

Theorem 45.4.2 (last step)**.** *If $\sum_i \alpha_i \, c_V c_K c_K' c_P \, \mathrm{tail}\big(K_{m-1}K_{m-2}S_{m-2}\psi_i, F_i\big) = 0$, with $c_V = c_V(m-2)$, $c_K = c_K(m-2)$, $c_K' = c_K(m-1)$, then $\sum_i \alpha_i \, c_I(m-2) c_I(m-1) \, \mathrm{per}_{m-2}(\psi_i, \iota_{m-2}\iota_T F_i) = 0$.*

Status: proved in the formalization in the same way, from the law for the last two lifts and the evaluation of the final restriction pairing; not followed further.

Put

$$\mathrm{up}_0\,\psi := \psi, \quad \mathrm{up}_{l+1}\,\psi := K_{2l+1}K_{2l}S_{2l}\,\mathrm{up}_l\,\psi, \quad W_l := \prod_{i<l} c_V(2i)c_K(2i)c_K(2i+1),$$

$c_{\mathrm{top}}(l) := W_l c_P$, $c_{\mathrm{bot}} := \prod_{j<m} c_I(j)$, and $\mathrm{dn}_j := \iota_0\iota_1\cdots\iota_{j-1} \colon C_j \to C_0$.

Problem 45.4.1. Assume Theorem 45.4.1, and let $j+2(\sigma+1)+2=m$. If $\mathrm{per}_{j+2}(K_{j+1}K_jS_j\psi, G)=0$, then $\mathrm{per}_j(\psi, \iota_j\iota_{j+1}G)=0$. Hence a nonzero Bessel period at level j of ψ against $\iota_j\iota_{j+1}G$ forces a nonzero Bessel period at level $j+2$.

Problem 45.4.2. Assume Theorem 45.4.1, and let $2l+2\rho+2=m$. If $\sum_i \alpha_i W_l \, \mathrm{per}_{2l}(\mathrm{up}_l\,\psi_i, G_i)=0$, then $\sum_i \alpha_i \big(\prod_{j<2l} c_I(j)\big) \, \mathrm{per}_0(\psi_i, \mathrm{dn}_{2l}\, G_i) = 0$.

Problem 45.4.3. Let $m=2n+2$, and assume Theorems 45.4.1 and 45.4.2. If $\sum_i \alpha_i c_{\mathrm{top}}(n+1)\,\mathrm{tail}(\mathrm{up}_{n+1}\,\psi_i, F_i)=0$, then $\sum_i \alpha_i c_{\mathrm{bot}} \, \mathrm{per}_0(\psi_i, \mathrm{dn}_{m-1}\,\iota_T F_i) = 0$, where per_0 is the Bessel period of rank n at level 0.

Problem 45.4.3, which rests on Problem 45.4.2, is the following statement.

Theorem 45.4.3 (the seesaw)**.** *The relations of the family* $i \mapsto c_{\mathrm{top}}(n+1)\,\mathrm{tail}(\mathrm{up}_{n+1}\,\psi_i, F_i)$ *pass to the family* $i \mapsto c_{\mathrm{bot}}\,\mathrm{per}_0(\psi_i, \mathrm{dn}_{m-1}\,\iota_T F_i)$.

Problem 45.4.4. Suppose that instead of Theorem 45.4.1 one had identities $\mathrm{per}_j(\psi, \iota_j\iota_{j+1}G) = \kappa_j\,\mathrm{per}_{j+2}(K_{j+1}K_jS_j\psi, G)$ with $\kappa_j \neq 0$ independent of ψ, G and the twist. Then the relations of $i \mapsto \mathrm{per}_{j+2}(K_{j+1}K_jS_j\psi_i, G_i)$ and $i \mapsto \mathrm{per}_j(\psi_i, \iota_j\iota_{j+1}G_i)$ pass to each other.

Theorem 45.4.1 gives one direction only, but it allows the twists to vary within one relation, with the constants c_V, c_K, c_I of each twist; this is what the comparison between twists in Problem 45.5.1 uses.

45.5. From the top to the bottom

The theorem compares two abstract periods. The formalization ties them to the objects of Chapter 35 by five conditions, for each word u. Write $\mathrm{col}_0\varphi$ for the column form of level 0 and twist t attached to a form $\varphi \in M_u$, the space of forms of level 0 of u ramified at T_u (Section 35.5), $\mathrm{col}_m := K_{m-1}\cdots K_0$ for the chain (the chain $K_m\cdots K_1$ of Chapter 35, with the indexing of Section 45.4), $B_{u,t}$ for the period map (B) at the twist t, and $\mathrm{Bper}(\varphi, T)$ for the Bessel period at level 0 of the column form of φ against a row form T.

- (L1) $\mathrm{per}_0(\mathrm{col}_0\varphi, G) = \mathrm{Bper}(\varphi, G)$ at every twist t; the right side does not depend on t.
- (L2) To each row datum F of $B_{u,t}$ corresponds $\tilde{F} \in C_T$ with $\mathrm{tail}(\mathrm{col}_m\mathrm{col}_0\varphi, \tilde{F}) = B_{u,t}(\varphi, F)$ for every φ.
- (L3) There are a linear bijection A of M_u with $\|A\varphi\| = \nu\|\varphi\|$ for some $\nu > 0$, and $s \neq 0$, such that $\mathrm{tail}(\mathrm{up}_{n+1}\,\mathrm{col}_0\varphi, \tilde{F}) = s\,\mathrm{tail}(\mathrm{col}_m\mathrm{col}_0A\varphi, \tilde{F})$ for every $\tilde{F} \in C_T$ and every twist.
- (L4) The element $\tilde{F}$ of (L2) can be chosen with $\mathrm{dn}_{m-1}\,\iota_T\tilde{F}$ equal to a row form $\mathrm{row}_t(F)$ at the base; and every row form at the base is $\mathrm{dn}_{m-1}\,\iota_T\tilde{F}_0$ for some $\tilde{F}_0 \in C_T$ of the base twist t_0.
- (L5) For each twist t there is a set S_t of primes such that $\big(c_{\mathrm{top}}(n+1)/c_{\mathrm{bot}}\big)_t$ and $\big(c_{\mathrm{top}}(n+1)/c_{\mathrm{bot}}\big)_{t_0}$ form an S_t-unit pair.

Status: (L1)–(L5) and the laws behind Theorems 45.4.1 and 45.4.2 are fields of one package of the formalization, which is supplied when the family is constructed; we have not traced the proofs that the constructed family satisfies them.

Problem 45.5.1. Assume Theorem 45.4.3 and (L1)–(L4), and put $c_t := c_{\mathrm{top}}(n+1)_t s/c_{\mathrm{bot}}(t)$, with the constants of the twist t. For every row form T at the base there is a function p on M_u such that: if $\alpha p(\varphi_0) + \alpha' c_t B_{u,t}(\varphi, F) = 0$, then $\alpha\,\mathrm{Bper}(A^{-1}\varphi_0, T) + \alpha'\,\mathrm{Bper}(A^{-1}\varphi, \mathrm{row}_t(F)) = 0$.

Problem 45.5.2. $c_t \neq 0$. Under (L5), if $c_t/c_{t_0} = \sigma(a)$ with $a \in L$, then $v_w(a) = 0$ at every finite place w of L above no prime of S_t.

Since Bper is linear in its first argument, relations among the values at $A^{-1}\varphi$ are relations among the values at $U\varphi$, with $U := \nu A^{-1}$, an isometry of M_u. So Problems 45.5.1 and 45.5.2 prove the following.

Theorem 45.5.1 (relations at the bottom)**.** *For every word u there are a unitary operator U of M_u and nonzero constants c_t, with c_t/c_{t_0} of valuation zero at the places above no prime of S_t, such that for every row form T there is p with the following property: every relation $\alpha p(\varphi_0) + \alpha' c_t B_{u,t}(\varphi, F) = 0$ implies $\alpha \operatorname{Bper}(U\varphi_0, T) + \alpha' \operatorname{Bper}(U\varphi, \operatorname{row}_t(F)) = 0$.*

Status: proved in the formalization from Theorem 45.4.3 and (L1)–(L5). The corresponding statement in [Adr] is a seesaw relation for an abstract period map, [Adr, Hyp. 14.10], assumed there; the formalization does not state it.

In the Lean formalization. In [Lean] the tower is `GlobalTheory.SeeSaw.Comm` and `Tup`, with `per`, `perF`, `perT`, `tail`, `up`, `dn`, `cTop`, `cBot`; the Bessel and Fourier–Jacobi data are `BesLev` and `JacLev`, with `BesLev.per` and `JacLev.per`. Theorems 45.4.1, 45.4.2, 45.4.3 are `StepX`, `LastX`, `MainX`; Problem 45.4.2 is `up_ind`, Problem 45.4.3 is `mainX_of_stepX_lastX`, and Theorem 45.4.3 is assembled by `mainX_of_lawsX`. The laws are `LawsX`; (i)–(iv) are `DoubleStep` (from `besselPeriod_double_step_unfolding_of_descent`), `ValX` (proved by `GlobalTheory.ValFin.valX`), `TwoX` and `RapidVal`. The S-unit pairs are `UnitRatio`, and Problem 45.1.4 is `vw_eq_zero_of_unitRatio`. (L1)–(L5) are `LinkVanishes`, `LinkTailCondition`, `LinkCommutes`, `LinkDescends` with `LinkSurjective`, and `FamilyRatioCondition`, fields of `FamilyAllConditions`. Theorem 45.5.1 is `JointCondition`, proved by `joint_of_famAllQ_alt` and `rel_of_links_alt`.

Hints

§45.1.

45.1.1: Define $T(\sum_i \alpha_i x_i) := \sum_i \alpha_i y_i$.
45.1.2: Use the relation $x_{i_0} \cdot x_i - x_i \cdot x_{i_0} = 0$.
45.1.4: $a^n = \beta$, and $v_w(\beta)$ is a multiple of $v_p(\beta)$. $(1+\mathrm{i})^4 = -4$.

§45.2.

45.2.1: Compare $2\rho + 1$ and 2ρ with $m - 1 - j$ and $m - 2 - j$.
45.2.2: Put $m - j = 2\rho + 2$ in Problem 35.2.3.

§45.3.

45.3.1: Problems 23.2.1 and 23.2.2; $\vartheta_0 = 1.1803405990\ldots$ and $\varpi = 2.6220575542\ldots$.
45.3.2: Problem 44.2.1 and Problem 23.4.2.

§45.4.

45.4.1: A family with one term.
45.4.2: Induction on l, applying Theorem 45.4.1 with $j = 2l - 2$ to the coefficients $\alpha_i W_{l-1}$.
45.4.3: Theorem 45.4.2, where $m - 2 = 2n$, then Problem 45.4.2 with $l = n$; note $\mathrm{dn}_{2n}\, \iota_{2n} = \mathrm{dn}_{2n+1}$.

§45.5.

45.5.1: By (L4) choose $\tilde{F}_0$ of the base twist with $\mathrm{dn}_{m-1}\,\iota_T\tilde{F}_0 = T$, and define $p(\varphi_0)$ through the top pairing of $\mathrm{up}_{n+1}\,\mathrm{col}_0 A^{-1}\varphi_0$ with $\tilde{F}_0$, at the base twist. Apply Theorem 45.4.3 to a family of two terms, one for each twist, and use (L1)–(L4).

45.5.2: s does not depend on the twist; Problem 45.1.4.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

[GGP] W. T. Gan, B. H. Gross and D. Prasad, *Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups*, Astérisque **346** (2012), 1–109.

[Ku] S. S. Kudla, *Seesaw dual reductive pairs*, in: Automorphic Forms of Several Variables (Katata, 1983), Progr. Math. **46**, Birkhäuser, Boston, 1984, 244–268.

[Lean] The Lean community, *riemann-hypothesis*, a Lean formalization of Alpöge's argument, `https://github.com/leanprover-community/riemann-hypothesis`.