BESSEL–POINCARÉ SERIES AND THEIR GRAM OPERATORS

Prerequisites. From Chapter 11: Poincaré series on $\mathrm{SL}_2(\mathbb{Z})$, the unfolding of Problem 11.1.1, Problem 11.1.3, and the dimension as formal degree times covolume (Section 11.4). From Chapter 21: class sets, the groups Γ_L, forms and their norm (Section 21.3), Problems 21.1.2, 21.1.4, 21.3.2 and 21.3.5, and the mass $M_2 = 1/32$. From Chapter 23: the period of Section 23.4. From Chapter 35: the decomposition (S4) of Y_0 and Problem 35.2.4. From Chapter 43: the Gram operators and Theorem 43.4.1. The Bessel periods of Gan, Gross and Prasad [GGP, §12], at the level of the definition.

In Chapter 11 a Fourier coefficient of a cusp form is its inner product with a Poincaré series, and Petersson's formula expands the inner products of two Poincaré series. Chapter 43 needs the same two facts for a unitary group, with Fourier coefficients replaced by Bessel periods. What is the Poincaré series, and why is its Gram operator a sum over double cosets? We give the definitions in the form the Lean formalization [Lean] uses, work them out on a definite group where every sum is finite, and then state what the formalization proves about them. As in Chapter 35, hermitian forms h are linear in the first variable.

44.1. The Bessel subgroup

Let E/F be a CM extension, ψ_F a nontrivial character of $F\backslash\mathbb{A}_F$, and let V be a hermitian space over E of rank r with a decomposition

$$V = Ee_1 \oplus \cdots \oplus Ee_w \oplus X \oplus Ez \oplus Ef_1 \oplus \cdots \oplus Ef_w,$$

where $h(e_a, e_b) = h(f_a, f_b) = 0$, $h(e_a, f_b) = \delta_{ab}$, the space X of rank k is orthogonal to the e_a, f_a and z, and $h(z,z) = -T$ for a prime T. So $r = 2w + k + 1$. This is the decomposition (S4) of Y_0 in Chapter 35, with $w = m/2 - 1$, the hyperbolic planes $Ee_a + Ef_a$, and X the image of X_0.

Let N be the group of $n \in U(V)$ such that $(n-1)e_a$ lies in the span of $e_1, \ldots, e_{a-1}$ for every a, and $(n-1)y$ lies in the span of $e_1, \ldots, e_w$ for every y orthogonal to all e_a. It is the unipotent radical of the stabilizer of the isotropic flag $Ee_1 \subset Ee_1 + Ee_2 \subset \cdots$. Embed $U(X)$ in $U(V)$ as the pointwise stabilizer of the e_a, the f_a and z. Put $e_{w+1} := z$ and, for $n \in N$,

$$\lambda(n) := \sum_{a=1}^{w} h(ne_{a+1}, f_a), \qquad \psi_N(n) := \psi_F\big(\mathrm{Tr}_{E/F}\,\lambda(n)\big).$$

The *Bessel subgroup* is $R := N \rtimes U(X)$. For automorphic forms f on $U(V)$ and ϕ on $U(X)$, the Bessel period integrates f against $\overline{\psi_N}$ over $N(F)\backslash N(\mathbb{A}_F)$, and the result against $\bar{\phi}$ over $U(X)(F)\backslash U(X)(\mathbb{A}_F)$ [GGP, §12]. For $w = 0$ the group N is trivial, and the Bessel period is the restriction period along $U(X) \subset U(X \perp Ez)$.

Problem 44.1.1. For $n \in N$, $\lambda(n)$ is the sum over $a \le w$ of the coefficient of e_a in ne_{a+1}. The map $\lambda\colon N \to E$ is a homomorphism, and $\lambda(gng^{-1}) = \lambda(n)$ for $g \in U(X)$. Hence ψ_N is a character of $N(\mathbb{A}_F)$, trivial on $N(F)$ and fixed by $U(X)$.

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Problem 44.1.2. $U(X)$ normalizes N, and $U(X) \cap N = \{1\}$.

Problem 44.1.3. As an algebraic group over F, N has dimension $w(r+k)$. For $(w,k) = (1,0)$ it is the three-dimensional unipotent radical of a Borel subgroup of $U(V)$, V of rank 3.

Different words of the construction use different lines. The record contains, for each word u, a number $c_u \in E^\times$ with $c_u \bar{c}_u T_u = T_{u_0}$, where T_u is the terminal prime of u (Section 35.5), and the line of u is spanned by $z_u := c_u^{-1} z$, with $z = z_{u_0}$.

Problem 44.1.4. $h(z_u, z_u) = -T_u$. Replacing z by z_u changes neither N nor $U(X)$; it multiplies the term $a = w$ of λ by c_u^{-1} and leaves the others unchanged.

44.2. Poincaré series

The Poincaré series of the argument is built like that of Chapter 11, with Γ_∞ replaced by R and $e(mz)$ by a Bessel function. Let $G := U(V)$. The formalization fixes a discrete series representation π of $G(F \otimes \mathbb{R})$, an open compact subgroup $K \subset G(\mathbb{A}_{F,f})$ with a character χ, and the space $\mathcal{H}$ of automorphic forms on G of type π at infinity on which K acts by χ. For a word u it fixes a representation π_u of $U(X)(F \otimes \mathbb{R})$ (the weight of the word), and a function $W_u \colon G(F \otimes \mathbb{R}) \to \pi_u$ with

$$W_u(nhg) = \psi_N(n)\, \pi_u(h) W_u(g) \qquad (n \in N(F \otimes \mathbb{R}),\ h \in U(X)(F \otimes \mathbb{R})),$$

which is reproduced by convolution with a matrix coefficient of π and has norm one in L^2 of a fundamental domain for R. It is the archimedean Bessel function. Let C_u be the space of $U(X)(F \otimes \mathbb{R})$-equivariant maps T from π_u to functions on $U(X)(F) \backslash U(X)(\mathbb{A}_F)$ with a fixed level and character: algebraic forms on the definite group $U(X)$ of weight π_u, the column forms of the word. For $g \in G(\mathbb{A}_F)$ whose finite part is $n\, h\, x_u\, \kappa$ with $n \in N(\mathbb{A}_{F,f})$, $h \in U(X)(\mathbb{A}_{F,f})$, $\kappa \in K$ and a fixed translate x_u, put $\Phi_{\mathsf{T}}(g) := \psi_N(n) \chi(\kappa)\, \mathsf{T}(W_u(g_\infty))(h)$, and $\Phi_{\mathsf{T}}(g) := 0$ for the other g. The *Bessel–Poincaré series* is

$$P_u \mathsf{T}(g) := \sum_{\delta \in R(F) \backslash G(F)} \Phi_{\mathsf{T}}(\delta g).$$

Status: these are the definitions of the formalization. It proves that the series converges in L^2, lies in $\mathcal{H}$ and is linear in T, under a hypothesis that $r \geq 1801$ and that consecutive weights b_j of the chain, which enter the Harish-Chandra parameter of π, differ by at least $16r^{19}$, from a decay statement for W_u that rests on uniqueness of archimedean Bessel models with a gap hypothesis; compare [JSZ]. We have not followed the proof of the decay statement. Formally, unfolding the sum against $f \in \mathcal{H}$ gives the Bessel period of f against T, as in Problem 11.1.1.

On a definite group with one class every quotient is finite, and the series is a finite sum. Let Γ be a finite group acting unitarily on a space V_π, and let the forms on G be the vectors V_π^Γ, with $\|\psi\|^2 := |\Gamma|^{-1} \|\psi\|_{V_\pi}^2$, as in Chapter 21 for one class. Let $\Gamma_X \subset \Gamma$ be a subgroup, W a unitary representation of Γ_X, and $\iota \colon W \to V_\pi$ a Γ_X-equivariant linear map with $\|\iota\xi\|^2 = c_0 \|\xi\|^2$ for a constant $c_0 > 0$. The forms on X are W^{Γ_X}, with $\|\xi\|^2 := |\Gamma_X|^{-1} \|\xi\|_W^2$. Put

$$B(\psi, \xi) := |\Gamma_X|^{-1} \langle \psi, \iota\xi \rangle, \qquad P\xi := \sum_{\gamma \in \Gamma / \Gamma_X} \gamma\, \iota\xi.$$

Problem 44.2.1. $P\xi$ is well defined and lies in V_π^Γ, and $\langle \psi, P\xi \rangle = B(\psi, \xi)$ for every $\psi \in V_\pi^\Gamma$.

Problem 44.2.2. The adjoint of P for the two norms is $P^*\psi = \iota^*\psi$. In particular $P \neq 0$ if and only if $B(\psi, \xi) \neq 0$ for some ψ and ξ.

Problem 44.2.3. Let (W', ι') be a second pair as above, with Poincaré map P'. For $\xi \in W^{\Gamma_X}$ and $\xi' \in W'^{\Gamma_X}$,

$$\langle P\xi, P'\xi' \rangle = \sum_{\delta \in \Gamma_X \backslash \Gamma / \Gamma_X} \frac{\langle \delta\, \iota\xi, \iota'\xi' \rangle}{|\Gamma_X \cap \delta \Gamma_X \delta^{-1}|},$$

and if $(W', \iota') = (W, \iota)$ the term of $\delta = 1$ is $c_0 \langle \xi, \xi' \rangle$.

So $K := c_0^{-1} P^* P$ is the identity plus a sum over the nontrivial double cosets. This is the finite form of Petersson's formula, Theorem 11.2.1, where the double cosets $\Gamma_\infty \backslash \Gamma / \Gamma_\infty$ give the Kloosterman terms.

44.3. The smallest case

Take $\mathcal{O} = \mathbb{Z}[\mathrm{i}]$, $\Gamma := \Gamma_{I_2}$, of order 32 by Problem 21.1.2, and $\mathrm{Cl}_2 = \{I_2\}$ by Problem 21.1.4. Let $z := e_2$ and $X := \mathcal{O}e_1$; here $h(z, z) = 1$, since everything is definite. The stabilizer of z is $\Gamma_X = \{\mathrm{diag}(u, 1) : u \in \mu_4\}$. Let $V_\pi := \mathrm{Sym}^a(\mathbb{C}^2)$, $W := \mathrm{Sym}^j(\mathbb{C}e_1)$ and $\iota(\xi) := \xi\, e_2^{a-j}$. Then B is the period $P(\psi, \xi)$ of Section 23.4, and Problem 44.2.1 says that it is the inner product with $P\xi$. For $4 \mid j$ let $\xi_j \in W^{\Gamma_X}$ be the form with value e_1^j.

Problem 44.3.1. $\Gamma_X \backslash \Gamma / \Gamma_X$ has five elements: the classes of $\mathrm{diag}(1, u)$, $u \in \mu_4$, with four elements each, and the class of the sixteen antidiagonal matrices.

Problem 44.3.2. Let $4 \mid j$ and $0 \leq j \leq a$. Then $P\xi_j = 4\big(e_1^j e_2^{a-j} + e_1^{a-j} e_2^j\big)$ if $4 \mid a$, and $P\xi_j = 0$ otherwise. For $a = 8$: $P\xi_0 = 4(e_1^8 + e_2^8)$ and $P\xi_4 = 8e_1^4 e_2^4$.

Problem 44.3.3. Here $c_0 = \binom{a}{j}^{-1}$. Let $4 \mid a$ and $4 \mid j, j'$, and put $K_{jj'} := \big(c_0(j) c_0(j')\big)^{-1/2} \langle P\xi_j, P\xi_{j'} \rangle / (\|\xi_j\| \|\xi_{j'}\|)$. Then

$$K_{jj'} = 4\big(\delta_{jj'} + \delta_{j', a-j}\big).$$

In the expansion of Problem 44.2.3 the identity contributes 1 to K_{jj}, the three classes $\mathrm{diag}(1, u)$ with $u \neq 1$ contribute 3, and the antidiagonal class contributes $4\delta_{j', a-j}$.

Problem 44.3.4. Let $4 \mid a$. The forms $P\xi_j$ with $4 \mid j$ and $2j \leq a$ are an orthogonal basis of $\mathcal{A}_{(a)}(2)$, their number is $\lfloor a/8 \rfloor + 1$, and $P\xi_{a-j} = P\xi_j$.

As in Problem 11.1.3, distinct Poincaré series can coincide.

Problem 44.3.5. Let $4 \mid a$ and $D^* := 4M_2 \dim \mathrm{Sym}^a(\mathbb{C}^2)$. Then $D^* = (a+1)/8$, and $0 < \dim \mathcal{A}_{(a)}(2) - D^* \leq 7/8$.

So in the toy the words j with $2j < a$ give exactly orthogonal Poincaré series which span all but at most one dimension of the space, and $\dim \mathcal{A}_{(a)}(2)$ is D^* up to a bounded error, as $\dim S_k$ is $(k-1)/12$ up to a bounded error in Problem 11.4.2. But the diagonal is 4, not 1: the terms of the elements $\mathrm{diag}(1, u)$, which preserve the line Ez without fixing z, are as large as the identity term. The statement of the next section needs every term other than the identity term to be small, and the Bessel data are built so that the space of forms of each word is a small part of $\mathcal{H}$. We have not followed how the formalization disposes of the terms of such elements.

44.4. The Gram operators in the argument

Return to Section 44.2. Let $d_u := \dim C_u$, and let $c_0(u)$ be the volume of the level subgroup, times the volume of a fundamental domain for N at infinity, times the volume of $U(X)(F \otimes \mathbb{R})$, divided by $\dim \pi_u$. The Gram operators are

$$K_{uu'} := \big(c_0(u)c_0(u')\big)^{-1/2} P_u^* P_{u'} \colon C_{u'} \to C_u,$$

as in Chapter 43. Let $D := \dim \mathcal{H}$, and let D^* be the formal degree of π times the volume of $G(F)\backslash G(\mathbb{A}_F)$ times the value at 1 of the level idempotent. Fix the scale x, $y := x^\varepsilon$ and $b := e^y$, and put $\Gamma_0 := \exp(C_\Gamma m^5 f y)$ for a constant $C_\Gamma > 0$ of the family.

Theorem 44.4.1 (near orthogonality)**.** *For every family of records of the formalization satisfying its size conditions there is x_0 such that for every record at a scale $x \geq x_0$ there are $\eta_0, \eta_1, e_0 \geq 0$ with $\eta_0, \eta_1, \sqrt{e_0} \leq \exp(-x^{\varepsilon/8})$ with the following property. If $1/3 \leq d_u \Gamma_0^2 / D^* \leq 3$ for every word u, then*

$$\|K_{uu'}\|_{\mathrm{HS}}^2 \leq (1+\eta_0)\frac{d_u d_{u'}}{D^*} \quad (u \neq u'), \qquad \|K_{uu} - 1\|_{\mathrm{HS}}^2 \leq e_0 d_u, \qquad D \leq (1+\eta_1) D^*.$$

Status: proved in the formalization. The proof writes $K_{uu'}$ as an absolutely convergent sum over a set of terms containing an identity term; by our reading these are the double cosets $R(F)\backslash G(F)/R(F)$ of Problem 44.2.3, but we have not read their definition. It then proves four bounds: the terms of a class of degenerate terms sum to zero when $u \neq u'$; their sum is small when $u = u'$; the other terms of an off-diagonal block have squared norm at most $(1 + b^{-6}) d_u d_{u'} / D^*$; and the other terms of a diagonal block differ from the identity by little. The off-diagonal bound is derived from the Siegel–Weil formula in its convergent range and from a lower bound for Poincaré amplitudes; we have not followed the proofs of these four bounds. The statement is Theorem 43.4.1 of Chapter 43; in [Adr] it is derived from hypotheses assumed there.

The third inequality comes from a count of forms.

Theorem 44.4.2 (the count)**.** *There are $C, c, c_b > 0$ such that, if every gap of the Harish-Chandra parameter of π is at least $C r^{19} f^5 \log(b \operatorname{rd}_E)$, $\log f \leq \sqrt{r}$ and $\log\log b \leq c_b r f \log r$, then $|D - D^*| \leq \exp(-c r f \log r)\, D^*$. Here rd_E is the root discriminant of E.*

Status: proved in the formalization from the Siegel–Weil formula for unitary groups in its convergent range, through a statement on volumes; we have not followed the proof. It is a limit multiplicity statement with a rate, in the tradition of de George and Wallach [dGW]. Section 11.4 is its simplest case, where the identity term is the formal degree times the covolume, and Problem 44.3.5 is the definite case of rank two.

Problem 44.4.1. For $x \geq 1$, $b^{-6} \leq \exp(-x^{\varepsilon/8})$. Under the hypotheses of Theorem 44.4.2, if moreover $c r f \log r \geq x^{\varepsilon/8}$, Theorem 44.4.2 gives the third inequality of Theorem 44.4.1 with $\eta_1 = \exp(-c r f \log r)$.

Problem 44.4.2. Suppose $K_{uu'} = K^0_{uu'} + K''_{uu'}$ with $K''_{uu'} = 0$ for $u \neq u'$ and $\|K^0_{uu'}\|_{\mathrm{HS}}^2 \leq (1 + b^{-6}) d_u d_{u'} / D^*$. Then the first inequality of Theorem 44.4.1 holds with $\eta_0 = b^{-6}$, for every $x \geq 1$.

Problem 44.4.3. In the toy of Section 44.3, with $4 \mid a$, the words j with $4 \mid j$ and $2j < a$, and $D^* = (a+1)/8$: the third inequality of Theorem 44.4.1 holds with $\eta_1 = 7/(a+1)$, the left side of the first is 0, and the second fails for every $e_0 < 9$.

In Chapter 43 the three inequalities are (e) of Theorem 43.3.1. The identification of the period maps with these Poincaré series, which is (d) there, is the subject of the following chapters.

In the Lean formalization. In [Lean] the Bessel datum is `HidaSetting`: `eX`, `fY`, `zu` with `zu_norm`, the sets `Ninf`, `Nfin`, the character `psiInf`, `psiFin` (whose argument is divided by the product of the two level primes), the Bessel function `Wu` with `IsWu`, and the level character `chi2`. The numbers c_u are `D7.cOf` with `cOf_eq`, and `D7.zmOf` is $c_u^{-1} z$. The seed, the series and its L^2 class are `Ph`, `Pser`, `PuFun` in `PoincareInHH`, with `InHH` and the normalizing volume `c0X`. The Gram operator is `Kmat`, and Theorem 44.4.1 is `LinkS_at` and `LinkS`, proved by `link_alt_alt_bis` through `theoremS_transport_link_zero` and `theoremS_link_of_evaluation`. The four bounds are `ClKppOff` (from `KppOffC_holds`), `ClKppDiag` (from `kernel_diagonal_positive`), `ClOff` (from `TheoremSSelected`, proved by `GlobalTheory.CarrierGap.offDiag`) and `ClDiag` (from `diagonal_positive_final`). Theorem 44.4.2 is `CountAllPointsClass`, proved by `count_constants_of_siegelWeil`; the decay is `WuDecayOfGap`, proved by `wuDecayOfGap_holds'`, under `GapHyp`.

Hints

§44.1.

44.1.1: Write $ne_{a+1} = e_{a+1} + \sum_{b \le a} c_b(n) e_b$, and compute the coefficients of $nn' e_{a+1}$; an element of $U(X)$ fixes every e_b and z.

44.1.2: An element of $U(X)$ preserves the span of the e_a and its orthogonal complement.

44.1.3: First the unipotent radical of the stabilizer of $Ee_1 + \cdots + Ee_w$: $2w(r-2w)$ dimensions from maps between $V/(\dots)^\perp$, the middle and the flag, and w^2 from a hermitian matrix. Then the unipotent radical of a Borel subgroup of $\mathrm{GL}_w(E)$.

44.1.4: h is linear in the first variable.

§44.2.

44.2.1: $\iota\xi$ is fixed by Γ_X, and $\langle \psi, \gamma v \rangle = \langle \gamma^{-1}\psi, v \rangle$.

44.2.2: Problem 44.2.1.

44.2.3: Apply Problem 44.2.1 with $\psi = P\xi$, and group the cosets $\gamma\Gamma_X$ by double cosets.

§44.3.

44.3.1: Multiply $\mathrm{diag}(u_1, u_2)$ and the antidiagonal matrices by $\mathrm{diag}(u, 1)$ on both sides.

44.3.2: Representatives of Γ/Γ_X: $\mathrm{diag}(1, u)$ and $\begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}$; sum the powers of u.

44.3.3: Problem 21.3.2 for c_0; then Problems 44.2.3 and 44.3.1.

44.3.4: Problem 21.3.5.

44.3.5: Write $a = 8q$ or $a = 8q + 4$.

§44.4.

44.4.1: $6x^{\varepsilon} \geq x^{\varepsilon/8}$.

44.4.3: Problems 44.3.3 and 44.3.5; here $d_j = 1$.

References

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