GAIN AND LOSS AT AN INERT PRIME

Prerequisites. Chapter 6, Problem 6.5.3: replacing the unit lattice L_v at a finite place by $\mathfrak{p}_v^{-k}L'_v$, for a full sublattice $L'_v \subseteq L_v$, raises $\deg V$ by $\log q_v\,(k\dim V - \mathrm{length}(L_v/L'_v))$. Chapter 25: the valuation ord of Problem 25.4.1 and Theorem 25.4.1. Chapter 37, Sections 37.3–37.5. Chapter 40 and Chapter 41, Theorem 41.4.1: the Hecke image N, the exponent τ, and the bound length $\le v(p)\,\mathrm{rk}\,M(\tau+2S\Sigma)$. The classes of inert primes and the Frobenius orbits of Section 53.4, and Problem 53.4.5: $\sum_{p\le N}(\log p)/p \le \log N + \log 4$. The notation $x, \varepsilon, a, b', X', X''$, k, m, the tops $\mathcal{T}$ and their weights a_t, as in Section 37.4.

At an inert prime p the chain of lifts gains a power of p, but only on the Hecke image N, and passing to N costs its colength. How are the two combined, and what does the formalization [Lean] actually assert at one prime? The answer has three parts: explicit numbers for the gain at each Frobenius orbit, a divisibility statement for the lifts on N, and a bookkeeping inequality between numbers. In the formalization the last two are proved on separate branches, and the bookkeeping never sees the lifts. The chapter ends with the form in which these statements reach the record at one scale.

42.1. The numbers at one orbit

Fix an inert prime $p \le X''$ of class I or II, and write $\mathcal{V}$ for the set of Frobenius orbits on the columns, Section 53.4. For an orbit v let $\mathcal{T}_v$ be the set of tops on it. A top is *even* if its distance along the orbit from a fixed base point of v, taken in $[0, n_p)$, is even, and *odd* otherwise. An odd power e is *excluded* at the top s if the e-th Frobenius image of the column of s is the column of another top, reached for the first time. Put $\gamma_p = p/(p^2-1)$.

The formalization defines, for each orbit v and each step $1 \le j \le m$, three real numbers. With $\alpha_s = a_s - 2c_G$ for a top s (where c_G is a small constant), with caps $c_j(s,e)$, given by the weight of the j-th lift at the cell of p^e in the column of s plus c_G, and with a small loss term $\theta_v \ge 0$, these are

$$g_j^0(v) = \sum_{s\in\mathcal{T}_v} \sum_{e \text{ odd, not excluded}} \Big(\frac{\alpha_s}{p^e} - c_j(s,e)\Big)_+ - \theta_v, \qquad g_j^U(v) = \gamma_p \sum_{s\in\mathcal{T}_v} \alpha_s - \sum_{s \text{ odd}} \sum_{e \text{ odd}} \min\Big(\frac{\alpha_s}{p^e}, c_j(s,e)\Big) - \theta_v,$$

and a *price* $\mathrm{Pr}_j(v) \ge 0$. The number g^0 is the gain of the j-th step without a Hecke operator at v, g^U its gain after the Hecke operator at its target, and Pr bounds the cost of commuting the Hecke operator through the composite, as in Theorem 37.3.2. *Status*: these are definitions in the formalization; that the lifts realize them is the input of Section 42.2. We assume $c_j(s,e) \ge 0$ and $\alpha_s \ge 0$, that is, $a_s \ge 2c_G$.

Problem 42.1.1. For a real $\alpha \ge 0$ and reals $c_e \ge 0$, indexed by the odd $e \ge 1$,

$$\gamma_p \alpha - \sum_{e \text{ odd}} \min\Big(\frac{\alpha}{p^e}, c_e\Big) = \sum_{e \text{ odd}} \Big(\frac{\alpha}{p^e} - c_e\Big)_+ .$$

For $p = 3$, $\alpha = 100$ and all $c_e = 2$, both sides equal $892/27$.

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Problem 42.1.2. If no power is excluded at any top of v, then $g_j^U(v) - g_j^0(v) = \sum_{s \text{ even}} \sum_{e \text{ odd}} \min(\alpha_s/p^e, c_j(s,e)) \geq 0$.

So the Hecke operator lets the even tops gain the full $\gamma_p \alpha_s$, while without it every top gains only the part of α_s/p^e above the cap.

Problem 42.1.3. For reals u and g, $\max(u, g_+) = g_+ + (u - g_+)_+$.

Only primes of class I are *served*. For such p, let κ_v be the defect at the orbit v: the explicit defect of Theorem 40.5.1 if $p < p_*$, and the least defect of Problem 40.3.2, for the depth t_v and denominator D_p of the formalization, if $p \geq p_*$; and $\Sigma(p) = m \sum_{v \text{ good}} \kappa_v$. If $p < X'$, its orbits are long enough, p avoids the excluded bands, and $\Sigma(p) \leq x^{-\varepsilon}$, the served set $\mathcal{V}_p$ is the set of good orbits; otherwise $\mathcal{V}_p = \varnothing$. Put

$$S_v = \Big\lceil \sum_{j=1}^{m} \big(g_j^U(v) - \Pr_j(v) - g_j^0(v)_+ \big)_+ \Big\rceil, \qquad \text{gain}_j(p) = \sum_{v \in \mathcal{V}_p} \max\big(g_j^U - \Pr_j,\, (g_j^0)_+\big)(v) + \sum_{v \notin \mathcal{V}_p} g_j^0(v)_+,$$

and $\text{gain}_j(p) = 0$ if p is of class I with Frobenius label $h_p = 0$. The exponents $\tau(p)$ and $\text{io}(p) = 2\Sigma(p) \max_{v \in \mathcal{V}_p} S_v$ are those of Chapter 41, with $\tau(p) = \text{io}(p) = 0$ when $\mathcal{V}_p = \varnothing$.

Problem 42.1.4. $\sum_j \text{gain}_j(p) \leq \sum_j \sum_{v \in \mathcal{V}} g_j^0(v)_+ + \sum_{v \in \mathcal{V}_p} S_v$. If $\mathcal{V}_p \neq \varnothing$, then $\text{io}(p) \leq 2x^{-\varepsilon} \max_{v \in \mathcal{V}_p} S_v$.

Problem 42.1.5. Let $\mathcal{V} = \mathcal{V}_p = \{v\}$, with $m = 2$, $g_j^0(v) = 0$, $g^U(v) = (3,2)$ and $\Pr(v) = (1,3)$. Then $\text{gain}_1(p) + \text{gain}_2(p) = 2$ and $S_v = 2$, the numbers of Problem 37.3.2.

42.2. The gain on the Hecke image

Let $w \mid p$ be a place, $\mathfrak{o}$ the completion at w, M the forms at p of Chapter 41, V^{ram} the chosen subspace, and K the composite of the m lifts, from M to the final lattice. Let ord be the valuation of Problem 25.4.1 on the final lattice, normalized by $\text{ord}(p) = 1$, and let low be a real number (the other local terms: $\sum_j (F_j - E_{u,j}) + v_p(D_\Gamma)$ in the notation of Chapter 53). Write y_v for the normalized Hecke operator at v. The input from the lifts is the following.

(G) For every $\mathcal{V}' \subseteq \mathcal{V}$ and every $F \in M$,

$$\text{ord}\Big(K \prod_{v \in \mathcal{V}'} y_v^m F\Big) \geq \sum_{j=1}^{m} \Big(\sum_{v \in \mathcal{V}'} \max\big(g_j^U - \Pr_j,\, (g_j^0)_+\big)(v) + \sum_{v \notin \mathcal{V}'} g_j^0(v)_+ \Big) + \text{low}.$$

Status: (G) is a hypothesis of the divisibility statement in the formalization, which is supplied by the construction of the lifts; we have not traced its proof. Theorem 37.3.2 is its consequence in the form of Theorem 42.2.1 below, and its case of one orbit is Theorem 25.4.1.

Let N be the module of Chapter 41 built from $\mathcal{V}_p$ and the S_v above, and $NP_w = N \cap p^{\lceil \tau(p) \rceil} M \cap V^{\text{ram}}$.

Problem 42.2.1. $\text{ord}(\sum_i c_i x_i) \geq \min_i \text{ord}(x_i)$ for $c_i \in \mathfrak{o}$, and $\text{ord}(p^s x) = s + \text{ord}(x)$.

Problem 42.2.2. Assume (G). For $U \subseteq \mathcal{V}_p$ and $F \in M$, the generator $\prod_{v \in \mathcal{V}_p \setminus U} y_v^m \, p^{\sum_{v \in U} S_v} F$ of N satisfies $\text{ord}(K \cdot) \geq \sum_j \text{gain}_j(p) + \text{low}$.

Problems 42.2.1 and 42.2.2 prove the following.

Theorem 42.2.1 (gain on the Hecke image)**.** *Assume (G). Then* $\operatorname{ord}(Kx) \geq \sum_j \operatorname{gain}_j(p) + \operatorname{low}$ *for every* $x \in NP_w$.

Status: proved in the formalization from (G). The corresponding statement in [Adr] is Theorem 37.3.2, for the served orbits.

Problem 42.2.3. If $\mathcal{V}_p = \varnothing$, then $N = M$, $NP_w = M \cap V^{\mathrm{ram}}$, and $\operatorname{gain}_j(p) = \sum_v g_j^0(v)_+$, unless p is of class I with $h_p = 0$, where $\operatorname{gain}_j(p) = 0$. Such a p is never served, since its orbits have length one.

Problem 42.2.4. If $K = 0$, the conclusion of Theorem 42.2.1 holds with any real number in place of $\sum_j \operatorname{gain}_j(p) + \operatorname{low}$.

42.3. Recording the gain and the loss

At a place $w \mid p$ with ramification index e_w and residue field of order q_w, the source lattice of the word u is

$$N_{u,w} = \mathfrak{p}_w^{-\lfloor e_w \operatorname{sg}_u(w) \rfloor} NP_w, \qquad \operatorname{sg}_u(w) = \sum_j \big(\operatorname{gain}_j(p) + F_j(w) - E_{u,j}(w)\big),$$

where F_j and $E_{u,j}$ are the other local terms of Chapter 53. Theorem 42.2.1 is what makes the pairing integral on N_u; the colength of NP_w is what it costs.

Problem 42.3.1. Let $M^{\mathrm{ram}} = M \cap V^{\mathrm{ram}}$ have rank d, and suppose $\operatorname{length}(M^{\mathrm{ram}}/NP_w) \leq e_w \operatorname{rk} M\,(\tau(p) + \operatorname{io}(p))$, as Theorem 41.4.1(b) gives with $M' = M^{\mathrm{ram}}$ and $v(p) = e_w$. Replacing M^{ram} by $N_{u,w}$ raises the degree by at least $\log q_w \big(d(e_w \operatorname{sg}_u(w) - 1) - e_w \operatorname{rk} M\,(\tau(p) + \operatorname{io}(p))\big)$.

Problem 42.3.2. Per unit of $e_w d \log q_w$, the change in Problem 42.3.1 is at least $\sum_j \operatorname{gain}_j(p) - (\operatorname{rk} M/d)(\tau(p) + \operatorname{io}(p))$, plus the other local terms, up to the rounding $1/e_w$. If V^{ram} is the whole space, so that $d = \operatorname{rk} M$, and $\mathcal{V}_p \neq \varnothing$, this is at least $\sum_j \operatorname{gain}_j(p) - \tau(p) - 2x^{-\varepsilon} \max_{v \in \mathcal{V}_p} S_v$ plus the other local terms, up to the rounding.

42.4. The inequality at one prime

For a top t and step j, the contribution of the odd powers of p to the prime sum of Section 37.5 is

$$V_j(t,p) = \sum_{i \text{ odd},\ i \leq \lfloor \log_p a_t \rfloor + 1} \Big(\frac{a_t}{p^i} - w_j(t,p,i)\Big)_+,$$

where $w_j(t,p,i)$ is the weight of the cell of p^i in the column of t. Let $\mathcal{T}_p$ be the set of tops on served orbits, or all tops if p is of class II or $p \geq X'$.

Theorem 42.4.1 (gain against the prime sum)**.** *There are a constant C and a function $\delta(x) \to 0$ such that, for all large x, every datum of every sufficiently large family in the class used by the formalization, and every inert $p \leq X''$ of class I or II,*

(i) $0 \leq \operatorname{gain}_j(p) \leq 4|\mathcal{T}|a/p$ *for every j;*

(ii) $\sum_j \sum_{t \in \mathcal{T}_p} \big(V_j(t,p) - \operatorname{lo}(p)\big) - m\,\operatorname{fl}(p) \leq \sum_j \operatorname{gain}_j(p) - \tau(p)$, *where* $\operatorname{fl}(p) = Cf(N_m^2 \log_p a + (\log_p a)^2)$ *and* $\operatorname{lo}(p)$ *is an explicit loss term: a multiple of* $\log_p(2a)$, *plus, at the primes of class I below X', terms that depend on the range of p, among them $b^{\delta(x)}$ for $p_* \leq p < X'$;*

(iii) *the source lattice satisfies the inclusion and the length bound of Theorem 41.4.1(b) at every $w \mid p$ with $p \neq T_u$.*

Status: proved in the formalization. Part (iii) is Theorem 41.4.1 with the defects of Theorems 40.5.1 and 40.5.2. Parts (i) and (ii) are inequalities between the explicit numbers of Section 42.1; their proof uses the size conditions of the family, the mean bound of Theorem 40.5.2, and the choice of the weights below. We have not followed the proof of (ii). The statement contains no lift: in the formalization it is proved with the chain of lifts replaced by the zero chain, for which the divisibility clause holds by Problem 42.2.4. The divisibility is Theorem 42.2.1, proved on a separate branch from (G), and the two meet only through the lattice N_u of Section 42.3. The corresponding statement in [Adr] is part of Theorem 37.5.1.

The weights are chosen so that few primes are rejected. For weights $(a_\bullet, b_\bullet)$ in the box of Theorem 40.5.2, let

$$\Phi(a_\bullet, b_\bullet) = \sum_{p<X' \text{ candidate}} \frac{\log p}{p}\Big(2 \cdot \mathbf{1}_{\Sigma(p)>x^{-\varepsilon}} + 4\Sigma(p)\Big).$$

Problem 42.4.1. The mean of Φ over the box is at most $(2x^{\varepsilon}+4)\sum_{p<X' \text{ candidate}} \frac{\log p}{p}\,\overline{\Sigma(p)}$, where $\overline{\Sigma(p)}$ is the mean of $\Sigma(p)$. Hence some weight in the box has Φ at most this bound.

Problem 42.4.2. Suppose that at every good orbit v of every candidate the mean of κ_v is at most $N_v x^{-12}$, where N_v is the number of tops on v. Then the bound of Problem 42.4.1 is at most $(2x^{\varepsilon}+4)(x^{\varepsilon/3}+\log 4)\, m|\mathcal{T}|x^{-12}$.

Status: the formalization proves the hypothesis of Problem 42.4.2 for its family, from Theorem 40.5.2 for $p \geq p_*$ and from the explicit defect for $p < p_*$, and its admissible weights satisfy $\Phi \leq x^{-9}$; we have not traced how the size conditions turn the bound of Problem 42.4.2 into this.

42.5. The record at one scale

For each word u the formalization records, at the scale x, the slopes of M_u and N_u, the rank, the height, and the sums $\sum_w 2\ell_w e_w(\sum_j \mathrm{gain}_j - \tau)$, $\sum_w 2\ell_w e_w \tau$ and $\sum_w \ell_w e_w \mathrm{io}$ over the places w, in the normalization of Chapter 53; these enter the accounting of Chapter 58. It produces such records by an argument about families, whose quantifiers we make explicit.

Let $\mathcal{K}$ be a set of *thresholds* with a binary operation $\vee$, and suppose that each family $\mathcal{F}$ at each ε has a predicate $\mathrm{Large}(\mathcal{F}, K)$ with $\mathrm{Large}(\mathcal{F}, K \vee K')$ equivalent to $\mathrm{Large}(\mathcal{F}, K)$ and $\mathrm{Large}(\mathcal{F}, K')$. A property P is *closed* on a class $\mathcal{C}$ of families if there is $\varepsilon_1 > 0$ such that for every $\varepsilon \leq \varepsilon_1$ there is K with $P(\mathcal{F})$ for every $\mathcal{F} \in \mathcal{C}$ at ε with $\mathrm{Large}(\mathcal{F}, K)$. The class $\mathcal{C}$ has *designs* if there is $\varepsilon_0 > 0$ such that for every $\varepsilon \leq \varepsilon_0$ and every K some $\mathcal{F} \in \mathcal{C}$ at ε has $\mathrm{Large}(\mathcal{F}, K)$.

Problem 42.5.1. If P and Q are closed on $\mathcal{C}$, so is $P \wedge Q$.

Problem 42.5.2. If $P_1, \ldots, P_9$ are closed on $\mathcal{C}$ and $\mathcal{C}$ has designs, then there is $\varepsilon_2 > 0$ such that for every $\varepsilon < \varepsilon_2$ some family in $\mathcal{C}$ at ε has all nine properties.

In the formalization the nine properties are the inputs of the accounting: the bound for the twisted places, the covering of the twisted places, the bound on irreducibility defects, the rank equality at small level, the divisibility and size statement

with its slope consequence, the rank condition, Theorem 42.4.1, the Gram conditions, and the passage to the chain record. The constants are $e_D = 2$ and $e_L = 0$. For each primitive character ψ the class consists of the families attached to ψ whose windows satisfy the condition of Chapter 58. The conclusion is: for every small ε there is *one* family, and x_0, such that at every $x \geq x_0$ in its window set a chain record exists. The record at x ties together, for each translate u_0 in the group of columns, one member of the family at the same x; these members share the frame, m, f, q, d and the top weights. *Status*: proved in the formalization. The corresponding step in [Adr] is [Adr, Hyp. 18.10], assumed there for every admissible datum; the formalization proves the weaker statement just described, which is what the window argument of Chapter 58 uses.

In the Lean formalization. In [Lean] the numbers of Section 42.1 are `g0`, `gU`, `Pr`, `Sv`, `gainP`, `tauP`, `ioP`, `Sig` and the served set `VV`; even tops are `IsV`, excluded powers `Excl`. The input (G) is `UAt`, inside `WeilJacquetShalikaLaw.unitBoundAtTwistedPlaces`; Theorem 42.2.1 is `WeilJacquetShalikaLaw.Conclusion1`, on the module `NP` of `InNP`. The lattice of Section 42.3 is given by `Iu` and `NprimeDef`, and the slope it records is `LinkMu`. Theorem 42.4.1 is `LinkG`, with V_j the `vonMangoldtClassSum` and lo, fl the terms `loG`, `flG`; it is proved by `linkG_without_kernels` with the zero chain `zeroCh`, from `AuxBlock8.Pf.stmtF`. The function Φ is the left side of `WReq`, and the mean hypothesis of Problem 42.4.2 is `MeanKappaBound`. Section 42.5: closed properties are `Closed`, designs are `DesignExists`, Problem 42.5.2 is `topDesWin`, and its conclusion is `FamilyClassConnected` with the record `ChainTied`.

Hints

§42.1.

42.1.1: Work term by term.
42.1.2: Problem 42.1.1 for the odd tops.
42.1.4: Problem 42.1.3, summed over j.

§42.2.

42.2.2: Input (G), and Problem 42.1.3 at the orbits of U.

§42.3.

42.3.1: Problem 6.5.3 with $k = \lfloor e_w \mathrm{sg}_u(w) \rfloor$ and $L'_w = NP_w$.
42.3.2: Problem 42.1.4.

§42.4.

42.4.1: Markov's inequality for the indicator.
42.4.2: Problem 53.4.5 with $N = \lfloor X' \rfloor$, and $\sum_v N_v \leq |\mathcal{T}|$.

§42.5.

42.5.1: The operation $\vee$.
42.5.2: Problem 42.5.1.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

[Lean] The Lean community, *riemann-hypothesis*, a Lean formalization of Alpöge's argument, `https://github.com/leanprover-community/riemann-hypothesis`.