FRAMES AT ODD POSITIONS AND THE UNIT SCALARS

Prerequisites. Chapter 37, Section 37.3, with conditions (a)–(c) of Theorem 37.3.3, which are Adriazola's hypotheses (N3), (N1) and (N2) [Adr, Hyps. 11.42, 11.22 and 11.38], and Chapter 38, which proves (a) for the kernels of the Lean formalization. We use the data at an inert prime of [Adr, §11.1]: the CM field $E \supset K$ with CM type Φ, a prime $p \geq 5$ inert in K and unramified in E, W, σ, $\mathcal{K}_0$ and ι as in Chapter 38, and ord_p with $\mathrm{ord}_p(p) = 1$. Linear algebra over a discrete valuation ring is all that is needed.

Hypothesis (N2), condition (c) of Theorem 37.3.3, asks that at the odd positions of an orbit the passage from bounds on kernel entries to inclusions of lattices works as it does at the even positions, with a determinant factor of the opposite sign. Hypothesis (N1), condition (b), asks that the scalar normalizing each step kernel change only by units when the lattice at one orbit changes. Why are these not automatic, and why does the Lean formalization [Lean] not state (N2) at all? The answer to the second question is a choice of bases: at an odd position the Lean uses the dual of the basis at the preceding even position.

39.1. Positions and Frobenius

An *orbit* is a place v of the maximal real subfield E^+ above p that splits in E. Fix $w \mid v$ as in [Adr, §11.1], put $Q_v = E_w$, which is unramified of degree 2ν over $\mathbb{Q}_p$, let W_v be its ring of integers, and fix an embedding $\rho_v \colon Q_v \to \mathcal{K}_0$ extending $\iota|_K$. The *positions* of v are $s \in \mathbb{Z}/2\nu$, and to each corresponds $\tau_{v,s} \in \Phi$ with

$$\iota\tau_{v,s}(x) = \sigma^s \rho_v(x) \quad (s \text{ even}), \qquad \iota\tau_{v,s}(x) = \sigma^s \rho_v(\bar{x}) \quad (s \text{ odd})$$

for $x \in E$. Even positions see the completion at w, odd positions the completion at $\bar{w}$.

Problem 39.1.1. For every $s \in \mathbb{Z}/2\nu$ and $x \in E$, $\sigma\big(\iota\tau_{v,s}(x)\big) = \iota\tau_{v,s+1}(\bar{x})$. In particular $\sigma^2 \circ \iota\tau_{v,s} = \iota\tau_{v,s+2}$.

So Frobenius composed with complex conjugation moves each position to the next. The number of positions is even because Q_v contains the completion $\mathbb{Q}_{p^2}$ of K.

39.2. Dual frames

For a W_v-lattice $\Lambda \subset Q_v^N$ with basis matrix $g \in \mathrm{GL}_N(Q_v)$ put $\operatorname{len}\Lambda = -\operatorname{ord}_p \det g$, and let $\Lambda^* = \{y \in Q_v^N : y^{\mathsf{T}} x \in W_v \text{ for all } x \in \Lambda\}$.

Problem 39.2.1. $g^{-\mathsf{T}}$ is a basis matrix of Λ^*. Hence $\Lambda^{**} = \Lambda$ and $\operatorname{len}\Lambda^* = -\operatorname{len}\Lambda$.

Let $N \subset E^n$ be an $\mathcal{O}_E$-lattice that is self-dual for $h_n(x,y) = \sum_i x_i \bar{y}_i$. At the orbit, $N \otimes \mathbb{Z}_p$ has the two components $N_w \subset Q_v^n$ and $N_{\bar{w}}$, and self-duality says

Date: November 26, 2026.

that $N_{\bar w}$ is the image of N_w^* under complex conjugation. By the formula for $\iota\tau_{v,s}$, a basis matrix g of N_w therefore gives the frame

$$g_s = \sigma^s\rho_v(g) \ \ (s \text{ even}), \qquad g_s = \sigma^s\rho_v\big(g^{-\mathsf T}\big) \ \ (s \text{ odd})$$

of N at the position s.

Problem 39.2.2. $\sigma(g_s) = g_{s+1}^{-\mathsf T}$ and $\sigma^2(g_s) = g_{s+2}$ for every $s \in \mathbb Z/2\nu$. Moreover $\operatorname{ord}_p \det g_s = -\operatorname{len} N_w$ for even s and $\operatorname{ord}_p \det g_s = \operatorname{len} N_w$ for odd s.

The Lean states the same relation globally. It chooses, for each $\tau \in \Phi$, a matrix $P_\tau \in \mathrm{GL}_n(E)$ whose columns form a basis of N at the place induced by τ, subject to

$$\overline{P_{\tau_{v,s+1}}}^{\,\mathsf T} P_{\tau_{v,s}} = 1 \qquad \text{for all orbits } v \text{ and positions } s.$$

Problem 39.2.3. Let F be a field with an involution $x \mapsto \bar x$, and $h(x,y) = x^{\mathsf T}\bar y$ on F^n. For $P, Q \in \mathrm{GL}_n(F)$, $\bar Q^{\mathsf T} P = 1$ if and only if $h(Pe_i, Qe_j) = [i = j]$ for all i, j, that is, the columns of Q are the h-dual basis of the columns of P. For $F = E$ with complex conjugation, if $\bar Q^{\mathsf T} P = 1$, then $g_s = \iota\tau_{v,s}(P)$ and $g_{s+1} = \iota\tau_{v,s+1}(Q)$ satisfy $\sigma(g_s) = g_{s+1}^{-\mathsf T}$.

Problem 39.2.4. If $\gamma \in \mathrm{GL}_n(E)$ is unitary, $\bar\gamma^{\mathsf T}\gamma = 1$, then $\bar\gamma^{-\mathsf T} = \gamma$. Hence $(P_\tau) \mapsto (\gamma P_\tau)$ preserves the relation displayed above.

Problem 39.2.4 is why the unitary group acts on the frames at all positions by the same matrix, as in the covariance of the lifts [Adr, Lemma 11.11]: at an odd position the contragredient action $g \mapsto g^{-\mathsf T}$ of the model is the ordinary action on the dual frame.

39.3. The odd positions, (N2)

Recall from [Adr, Def. 11.3, Lemma 11.4] that the integral structure $H_s(\Lambda)$ at a position is the image of the minimal lattice under the frame, twisted by a power of its determinant, and that an edge (Λ'', Λ) of type i, meaning $p\Lambda \subset \Lambda'' \subset \Lambda$ with Λ/Λ'' of dimension i, has a basis g of Λ with gd a basis of Λ'', where $d = \operatorname{diag}(p, \dots, p, 1, \dots, 1)$ has i entries p.

Problem 39.3.1. In this situation Λ''^* has basis matrix $g^{-\mathsf T}d^{-1}$, and $p\Lambda''^* \subset \Lambda^* \subset \Lambda''^*$ with Λ''^*/Λ^* of dimension i.

A *canonical bideterminant* in an $n \times n$ matrix X of indeterminates is a product $B = \prod_c \Delta_{S_c}$, where Δ_S is the minor on the rows S and the first $\#S$ columns [Adr, Def. 11.1].

Problem 39.3.2. For $D = \operatorname{diag}(p^{a_1}, \dots, p^{a_n})$, $B(DX) = p^{\sum_c \sum_{r \in S_c} a_r} B(X)$. In particular $B(d^{\pm 1}X) = p^{\pm \sum_c \#(S_c \cap \{1,\dots,i\})} B(X)$, and $0 \le \#(S_c \cap \{1, \dots, i\}) \le \min(\#S_c, i)$.

At an even position the frame of the edge lattice is $\sigma^s\rho_v(gd)$. The change of frame divides the bideterminants by $p^{\sum_c \#(S_c \cap \{1,\dots,i_0\})}$, and the normalizing power $p^{\sum_{a \le i_0} \lambda(a)}$ of the Hecke operator leaves the exponent $\sum_c \big(\min(h'_c, i_0) - \#(S_c \cap \{1, \dots, i_0\})\big) \ge 0$ [Adr, Lemma 11.4, Thm. 11.35]. At an odd position the frame of the edge lattice is $\sigma^s\rho_v(g^{-\mathsf T}d^{-1})$ by Problem 39.3.1. The diagonal matrix enters with the opposite exponent, and by Problem 39.3.2 the bideterminants are multiplied by $p^{\sum_c \#(S_c \cap \{1,\dots,i_0\})}$ instead. This is the exponent $\#(S_c \cap \{1, \dots, i_0\})$ in part (a) of (N2) [Adr, Hyp. 11.38]. Part (b) concerns the determinant twists.

Problem 39.3.3. Let N_w and N'_w have lengths ℓ and ℓ', with frames g_s, g'_s as in Problem 39.2.2, and let w_s be integers. Then

$$\operatorname{ord}_p \prod_s \Big(\frac{\det g_s}{\det g'_s}\Big)^{w_s} = \mu\,(\ell' - \ell), \qquad \mu = \sum_{s \text{ even}} w_s - \sum_{s \text{ odd}} w_s.$$

At an odd position $\big(\det g_s/\det g'_s\big)^{w_s} = \sigma^s \rho_v\big((\det g/\det g')^{-w_s}\big)$.

With $w_s = w_{j+1}(v,s)$, the number μ is $\mu_v(j+1)$, and the product cancels the factor $p^{\mu_v(j+1)(\operatorname{len} L_v - \operatorname{len} L'_v)}$ in the step map κ_j [Adr, Def. 11.9]. The last assertion of Problem 39.3.3 is part (b) of (N2).

Status: Adriazola states (N2) as a hypothesis and says that the bookkeeping of the determinant powers at the odd positions remains to be written [Adr, Hyp. 11.38, Remark 11.39]. Problems 39.3.1–39.3.3 are that bookkeeping for the frames and the edges. The Lean formalization has no separate statement. Its frames at odd positions are the dual frames of Section 39.2, and the lattices of the chain are described at every position, even or odd, by the integrality of the pairings of the coordinates with one family of products of minors, so both parities are handled by one argument. We have not checked in detail how this description at the odd positions matches the dual model weights of [Adr, §11.1].

39.4. The unit scalars, (N1)

Each step kernel carries a scalar $s_j(c, L; c', L')$, built from a step character μ_{j+1} of the norm-one ideles, a frame factor and powers of a Hecke character χ of E [Adr, Def. 11.23]. (N1) asks that when the lattice changes at one orbit v, the scalar change by $a_v^{\Delta \operatorname{len}} a_v'^{\Delta \operatorname{len}'}$ with units a_v, a'_v. Let $\mathfrak{P}_w, \mathfrak{P}_{\bar w}$ be the primes of $\mathcal{O}_E$ at $w, \bar w$. The elements $\tau \in \Phi$ that induce w or $\bar w$ are exactly the $\tau_{v,s}$; every other $\tau \in \Phi$ induces a place above p other than $w, \bar w$ [Adr, proof of Lemma 11.25].

Problem 39.4.1. The idele π_v with component p at w, p^{-1} at $\bar w$ and 1 elsewhere satisfies $\pi_v \bar\pi_v = 1$, and its ideal is $\mathfrak{P}_w \mathfrak{P}_{\bar w}^{-1}$.

Problem 39.4.2. Let $k \geq 1$ and $\beta \in E$ with $\beta\bar\beta = 1$ and $\beta\mathcal{O}_E = (\mathfrak{P}_{\bar w}\mathfrak{P}_w^{-1})^k$. Then $\operatorname{ord}_p \iota\tau_{v,s}(\beta)$ is $-k$ for even s and k for odd s, and $\sum_{\tau\in\Phi} \operatorname{ord}_p \iota\tau(\beta) = 0$.

Such β exist for k twice the class number of E, and for them $\chi(\mathfrak{P}_{\bar w}\mathfrak{P}_w^{-1})^k = \prod_{\tau\in\Phi} \tau(\beta)^{-1}$; so Problem 39.4.2 shows that $\iota\chi(\mathfrak{P}_{\bar w}\mathfrak{P}_w^{-1})$ is a unit [Adr, Lemma 11.30].

Problem 39.4.3. Let $k \geq 1$ and $z \in E$ with $z\bar z = 1$ and $z\mathcal{O}_E = (\mathfrak{P}_w\mathfrak{P}_{\bar w}^{-1})^k$, let ε_τ ($\tau \in \Phi$) and c be integers, and put $w_s = \varepsilon_{\tau_{v,s}} - c$. Then $\sum_{\tau\in\Phi} \varepsilon_\tau \operatorname{ord}_p \iota\tau(z) = k\big(\sum_{s \text{ even}} w_s - \sum_{s \text{ odd}} w_s\big)$.

If $\mu_{j+1}(z) = \prod_\tau \tau(z)^{\varepsilon_\tau}$ on norm-one elements and μ_{j+1} is trivial on the local units at v, then $\mu_{j+1}(\pi_v^k z^{-1})$ is a root of unity, and Problem 39.4.3 gives $\operatorname{ord}_p \iota\mu_{j+1}(\pi_v) = \mu_v(j+1)$ [Adr, Lemma 11.25]. With the effect of changing a lattice on the determinant idele [Adr, Lemma 11.26], this proves (N1) for Adriazola's normalization, with $a_v = p^{-\mu_v(j+1)}\iota\mu_{j+1}(\pi_v)\,\iota\chi((\pi_v))^{a_j}$ and $a'_v = p^{\mu_v(j+1)}\iota\mu_{j+1}(\pi_v)^{-1}\iota\chi((\pi_v))^{-a_{j+1}}$ [Adr, Prop. 11.27].

Status: (N1) for the normalization of [Adr, Def. 11.23] is proved in [Adr, Prop. 11.27]. The Lean formalization writes each step map in local coordinates as a scalar times the pairing of the source coordinates with the kernel series in

adapted frames, and proves that the scalar is a p-adic unit. We have not followed that proof, and we have not compared the Lean's scalar with s_j term by term.

In the Lean formalization. The dual frames are `GlobalTheory.KerStd.StdBases`, whose second clause is the relation $\overline{P_{\tau_{v,s+1}}}^{\mathsf{T}} P_{\tau_{v,s}} = 1$. Problem 39.1.1 is `exists_frob_step`, and Problems 39.2.2 and 39.2.3 are `frob_moves_coords`. The description of the lattices by pairings, at all positions, is in the fields `lat_S` and `levelFree_lattice_condition` of the structure `KerEnt`; its field `kform` writes the step map as `un` times the pairing with the kernel. The unit property of `un` is `KerEnt.UnUnit`, proved by `GlobalTheory.T.kernelPairScalars_isUnit_trueColumns`. The theorem `GlobalTheory.PdvClose.at_of_true_columns` combines it with the kernel law of Chapter 38 [Lean].

Hints

Section 39.1. 39.1.1. Treat even and odd s separately; $\sigma^{2\nu}$ fixes $\rho_v(Q_v)$.
Section 39.2. 39.2.1. $y \in \Lambda^*$ if and only if $g^{\mathsf{T}} y \in W_v^N$. 39.2.2. The definition of g_s, and $\sigma^{2\nu}\rho_v = \rho_v$ as in Problem 39.1.1. 39.2.3. Transpose; then Problem 39.1.1.
Section 39.3. 39.3.1. Problem 39.2.1 for Λ''. 39.3.2. Multiplying a row by p^a multiplies every minor containing it by p^a. 39.3.3. Problem 39.2.2.
Section 39.4. 39.4.2. The formula for $\iota\tau_{v,s}$, and $\mathrm{ord}_w(\bar{\beta}) = \mathrm{ord}_{\bar{w}}(\beta)$; there are ν positions of each parity. 39.4.3. As in Problem 39.4.2.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

[Lean] The Lean community, *riemann-hypothesis*, a Lean formalization of Alpöge's argument, `https://github.com/leanprover-community/riemann-hypothesis`.