THE NORMALIZED KERNEL LAW

Prerequisites. Chapter 12 on the Weierstrass functions, the quasi-period map η and Legendre's relation; Chapter 13 on the operators U_x and theta functions of a lattice; Chapter 14 on formal groups and Lubin–Tate theory; Chapter 37, whose notation we keep: $K = \mathbb{Q}(\sqrt{-d})$ with $d \equiv 7 \bmod 16$ prime, $p \geq 5$ inert in K, $W = W(\overline{\mathbb{F}}_p)$ with Frobenius σ, $\mathcal{K}_0 = W[1/p]$, $\lg(X) = \sum_{k\geq 0} X^{p^{2k}}/p^k$, and the isomorphism $\iota\colon \mathbb{C} \to \mathbb{C}_p$. A series over $\mathbb{C}_p$ is *integral* if its coefficients have absolute value at most 1. Formal groups and reduction of elliptic curves are as in [AEC, Chs. IV and VII]. Since σ is Frobenius, we write σ_Λ for the Weierstrass sigma function.

Theorem 37.1.2 writes a theta kernel at an inert prime as $c\exp(-\sum_e \beta(x_e, y_e))\,U$ with U integral, and says nothing about the scalar $c \in \mathbb{C}_p^\times$. Condition (a) of Theorem 37.3.3, Adriazola's hypothesis (N3) [Adr, Hyp. 11.42], asks that $|c| \leq 1$ for the kernels of the chain. Which normalization of the theta function gives this, and how is it proved? In the Lean formalization [Lean] every kernel is built from one theta function of one elliptic curve with complex multiplication by $\mathcal{O}_K$, normalized to take the value 1 at the origin. The scalar is then controlled by a fourth tensor power, in which it appears as a coefficient.

38.1. One curve and its reduced theta function

Let $\Lambda \subset \mathbb{C}$ be a lattice with $\mathcal{O}_K\Lambda \subset \Lambda$, and let $A = |\mathrm{Im}(\bar{\omega}_1\omega_2)|$ be its covolume, for a basis ω_1, ω_2. For $\omega, \lambda \in \Lambda$ the number $\mathrm{Im}(\bar{\omega}\lambda)$ is an integer multiple of A. Let $\eta\colon \Lambda \to \mathbb{C}$ be the quasi-period map, $\zeta_\Lambda(z+\omega) = \zeta_\Lambda(z) + \eta(\omega)$.

Problem 38.1.1. There is a unique $s_2 = s_2(\Lambda) \in \mathbb{C}$ with $\eta(\omega) = s_2\omega + \frac{\pi}{A}\bar{\omega}$ for all $\omega \in \Lambda$. Moreover $s_2(c\Lambda) = c^{-2}s_2(\Lambda)$ for $c \in \mathbb{C}^\times$.

The *reduced theta function* is $\theta(z) = \sigma_\Lambda(z)e^{-s_2 z^2/2}$. It is odd, $\theta'(0) = 1$, its zeros are the points of Λ, and $\theta_{c\Lambda}(cz) = c\,\theta_\Lambda(z)$. Put $H(z,w) = z\bar{w}/A$ and $(U_xG)(z) = G(z+x)\,e^{-\pi H(z,x) - \frac{\pi}{2}H(x,x)}$, as in Chapter 13.

Problem 38.1.2. For $\omega \in \Lambda$, $\theta(z+\omega) = \psi(\omega)\,\theta(z)\exp\big(\frac{\pi}{A}\bar{\omega}(z+\frac{\omega}{2})\big)$, with ψ as in Problem 12.1.4. Equivalently $\psi(\omega)\,U_\omega\theta = \theta$ for $\omega \in \Lambda$.

Let $t \in \mathbb{C}\setminus\Lambda$ with $Nt \in \Lambda$ for some integer $N \geq 1$ prime to p. The *normalized translate* of θ by t is $B_t = U_t\theta/(U_t\theta)(0)$.

Problem 38.1.3. $(U_t\theta)(0) = \theta(t)e^{-\pi|t|^2/(2A)} \neq 0$, and $B_t(z) = \theta(z+t)\,e^{-\pi\bar{t}z/A}/\theta(t)$; in particular $B_t(0) = 1$. The quotient $k_t = B_t/\theta$ satisfies $k_t(z+\omega) = e^{2\pi\mathrm{i}\,\mathrm{Im}(\bar{\omega}t)/A}\,k_t(z)$ for $\omega \in \Lambda$, so k_t^N is Λ-periodic.

The model at p. By the theory of complex multiplication, $j(\Lambda)$ is an algebraic integer, and there is an algebraic u with $|\iota(u)| = 1$ such that $y^2 = 4x^3 - G_2x - G_3$, with $G_2 = u^4g_2(\Lambda)$ and $G_3 = u^6g_3(\Lambda)$, has good reduction over W [Adr, Prop. 8.4]. (Adriazola cites Serre–Tate for this; the Lean derives it from global Artin reciprocity.) Let $\mathcal{E}$ be this curve over W, $T = -2x/y$ its parameter at the

Date: November 26, 2026.

origin, $T_1 +_{\mathcal{E}} T_2$ its formal group law, and $\ell(T) = T + \cdots$ its logarithm. Then $z = u\,\ell(T)$ as power series [Adr, §8.2]. Put

$$\Upsilon(T) = -\log \frac{\theta(u\,\ell(T))}{u\,T}.$$

Problem 38.1.4. If $\wp_\Lambda(z) = z^{-2} + \sum_{k\geq 2} e_k z^{2k-2}$, then $\log(\theta(z)/z) = -\frac{s_2}{2}z^2 - \sum_{k\geq 2} \frac{e_k}{(2k-1)2k} z^{2k}$. Hence $\Upsilon(T) = -\log \frac{\ell(T)}{T} + \frac{u^2 s_2}{2}\ell(T)^2 + \sum_{k\geq 2} \frac{u^{2k} e_k}{(2k-1)2k}\ell(T)^{2k}$.

The numbers $u^2 s_2$ and $u^{2k} e_k$ lie in $\mathcal{K}_0$; for s_2 this comes from an expression of s_2 through values of $\wp_\Lambda$ at torsion points of order prime to p [Adr, Lemma 8.3(iv) and §8.2]. So Υ is a series over $\mathcal{K}_0$ without constant term. It is not integral. For series B, B' without constant term we write $B \simeq B'$ when $\exp(B - B')$ is integral.

Problem 38.1.5. The relation $\simeq$ is an equivalence relation. If $B_1 \simeq B_1'$ and $B_2 \simeq B_2'$, then $B_1 + B_2 \simeq B_1' + B_2'$, and $B_1 \circ \phi \simeq B_1' \circ \phi$ for every family ϕ of integral series without constant term.

Theorem 38.1.1 (One variable)**.** *For t as above, $e^{\Upsilon(T)} B_t(u\ell(T))$ is an integral series with constant term* 1*.*

Status: proved in [Adr, Prop. 8.8]. By Problem 38.1.3, k_t^N is a rational function on the curve with divisor $N[-t] - N[0]$; the point $-t$ does not reduce to the origin, so near the origin of the special fibre k_t^N is T^{-N} times a unit, up to a power of a uniformizer. The normalization $B_t(0) = 1$ removes that power, and an N-th root is taken by the binomial series, since N is prime to p.

38.2. Many variables on one curve

For $\alpha \in \mathbb{Z}_{p^2}$ let $[\alpha]$ be the endomorphism of the formal group of $\mathcal{E}$ with $\ell([\alpha]T) = \alpha\,\ell(T)$; it has coefficients in W by Lubin–Tate theory, as in Chapter 14. On $\mathbb{Z}_{p^2}$ the Frobenius σ is the nontrivial automorphism, and $\sigma(\iota(a)) = \iota(\bar{a})$ for $a \in \mathcal{O}_K$. Put $\delta\Upsilon(S,T) = \Upsilon(S +_{\mathcal{E}} T) - \Upsilon(S) - \Upsilon(T)$.

Theorem 38.2.1 (Biextension rules)**.** *For all $\alpha \in \mathbb{Z}_{p^2}$:*

(R1) $\delta\Upsilon(S +_{\mathcal{E}} S', T) \simeq \delta\Upsilon(S,T) + \delta\Upsilon(S',T)$*, and the same in the second variable;*
(R2) $\delta\Upsilon([\alpha]S, T) \simeq \delta\Upsilon(S, [\sigma\alpha]T)$*;*
(R3) $\Upsilon([\alpha]T) \simeq \alpha\,\sigma(\alpha)\,\Upsilon(T)$*.*

Status: proved in [Adr, Props. 8.9, 8.10 and Lemma 8.13]. Rule (R1) is the theorem of the cube: the quotient $\theta(z_1+z_2+z_3)\theta(z_1)\theta(z_2)\theta(z_3)/\theta(z_1+z_2)\theta(z_1+z_3)\theta(z_2+z_3)$ is a rational function on the cube of the curve whose divisor near the origin consists of kernels of homomorphisms with a unit coefficient (compare Problem 12.2.8). Rules (R2) and (R3) come in the same way from quotients of θ at az and $\bar{a}w$ for $a \in \mathcal{O}_K$, and extend to $\mathbb{Z}_{p^2}$ by continuity. Only complex multiplication by $\mathcal{O}_K$ enters.

Problem 38.2.1. For every $c \in \mathbb{Z}_{p^2}$, $\delta\Upsilon(T, [c]T) \simeq (c + \sigma(c))\,\Upsilon(T)$.

Problem 38.2.2. For integral series $A_1, \ldots, A_k$ without constant term, $\Upsilon(A_1 +_{\mathcal{E}} \cdots +_{\mathcal{E}} A_k) \simeq \sum_j \Upsilon(A_j) + \sum_{j<l} \delta\Upsilon(A_j, A_l)$.

Problem 38.2.3. Let $\gamma \in M_m(\mathbb{Z}_{p^2})$ satisfy $\sum_i \sigma(\gamma_{ij})\gamma_{il} = [j = l]$, and put $T_i' = [\gamma_{i1}]T_1 +_{\mathcal{E}} \cdots +_{\mathcal{E}} [\gamma_{im}]T_m$. Then $\sum_i \Upsilon(T_i') \simeq \sum_j \Upsilon(T_j)$.

A *hyperbolic pair* of size m is a pair of $m \times 2m$ matrices f, f' over $\mathbb{Z}_{p^2}$ with $\sum_i \sigma(f_{e,i}) f_{e',i} = 0$, $\sum_i \sigma(f'_{e,i}) f'_{e',i} = 0$ and $\sum_i \sigma(f_{e,i}) f'_{e',i} = [e = e']$.

Problem 38.2.4. For a hyperbolic pair of size m put $T_i = [f_{1,i}]X_1 +_{\mathcal{E}} \cdots +_{\mathcal{E}} [f_{m,i}]X_m +_{\mathcal{E}} [f'_{1,i}]Y_1 +_{\mathcal{E}} \cdots +_{\mathcal{E}} [f'_{m,i}]Y_m$ for $i \le 2m$. Then $\sum_{i \le 2m} \Upsilon(T_i) \simeq \sum_{e \le m} \delta\Upsilon(X_e, Y_e)$.

Let h be a hermitian form on a $\mathbb{Z}_{p^2}$-module, linear in the first variable, so $h(y,x) = \sigma(h(x,y))$. Since $\mathbb{Q}_{p^2}/\mathbb{Q}_p$ is unramified, the norm $\kappa \mapsto \kappa\sigma(\kappa)$ maps $\mathbb{Z}_{p^2}^{\times}$ onto $\mathbb{Z}_p^{\times}$.

Problem 38.2.5. Let $h(f,f) = h(f',f') = 0$, $h(f,f') = 1$, and $\kappa\sigma(\kappa) = -1$. Then $e_1 = f + \frac{1}{2}f'$ and $e_2 = \kappa(f - \frac{1}{2}f')$ satisfy $h(e_i, e_j) = [i = j]$ and span the same $\mathbb{Z}_{p^2}$-module as f, f'.

The coboundary in Lubin–Tate coordinates. The formal group of $\mathcal{E}$ and the group with logarithm lg are isomorphic over W: there are $\varepsilon \in W^{\times}$ and $\varkappa \in W[[T]]$ with $\varkappa'(0) \in W^{\times}$ and $\lg(\varkappa(T)) = \varepsilon\,\ell(T)$ [Adr, Prop. 8.11], [LT]. Put $\beta_0(x,y) = \delta\Upsilon(\varkappa^{-1}x, \varkappa^{-1}y)$. This is the series β of Chapter 37.

The Lean uses an averaged form of β_0. Let 2ν be the length of a Frobenius orbit, as in Theorem 37.2.1, with one variable X_s for each position $s \in \mathbb{Z}/2\nu$. The torus $\mu_{p^{2\nu}-1}$ acts by $X_s \mapsto \zeta^{p^s} X_s$ for even s and $X_s \mapsto \zeta^{-p^s} X_s$ for odd s, where $0 \le s < 2\nu$.

Problem 38.2.6. (i) For $0 \le s, t < 2\nu$, $p^s \equiv p^t \pmod{p^{2\nu}-1}$ only if $s = t$.

(ii) The average of a series over the torus keeps exactly the monomials $\prod_s X_s^{m_s}$ with $\sum_{s \text{ even}} m_s p^s - \sum_{s \text{ odd}} m_s p^s \equiv 0 \pmod{p^{2\nu}-1}$.

(iii) It removes every monomial $X_s X_t$ with s even and t odd, and keeps $X_s^p X_{s+1}$ for every $s \in \mathbb{Z}/2\nu$.

The monomials $X_s^p X_{s+1}$ are those of the class h_υ of Section 37.2. In the Lean, β is the torus average of $\sum_r \beta_0\big(x_r, y_r\big)$, where x_r, y_r are the Lubin–Tate coordinates obtained from the trace-dual bases ω_r, ω_r^* as in Theorem 37.2.1. It satisfies four conditions: (B1) its coefficients lie in $\mathcal{K}_0$; (B2) it is invariant under the torus; (B3) each of its monomials has positive degree in the even and in the odd variables and is not of bidegree $(1,1)$; (B4) the exponentials of its coboundaries in the even and in the odd variables, taken in logarithmic coordinates, are integral with constant term 1. *Status*: proved in the Lean formalization from (R1)–(R3) and Hazewinkel's functional-equation lemma. The Lean also proves that replacing the exponent by its torus average changes the kernel only by an integral factor; we have not followed that argument. Adriazola works with β_0 itself [Adr, Thm. 8.12].

38.3. The scalar

Write $f^{\otimes 4}$ for the product $f(x^{(1)})f(x^{(2)})f(x^{(3)})f(x^{(4)})$ of four copies of a series $f(x_1, \dots, x_n)$ in disjoint sets of variables.

Problem 38.3.1. If every coefficient of $f^{\otimes 4}$ has absolute value at most R, then every coefficient of f has absolute value at most $R^{1/4}$. In particular, if $f^{\otimes 4} = \kappa g$ with g integral, $g(0) = 1$ and $|\kappa| \le 1$, then f is integral and $|f(0)| = |\kappa|^{1/4}$.

Adriazola uses the same idea with two copies. For a theta function G of a hermitian lattice he writes $G \otimes G$, after a change of coordinates over K, as a

combination $\sum_t r_t B_t$ of products of normalized translates, and obtains a bound $R = \max_t |r_t|$ for the coefficients of the tensor square [Adr, Thm. 8.19]. Nothing controls the r_t, so the conclusion is cU with an unknown scalar c. The Lean fixes the normalization of the theta function so that the corresponding coefficients are integral.

By Landherr's theorem, positive definite hermitian spaces over K of equal dimension whose discriminants differ by a norm from K are isometric [Sch, Ch. 10, §6].

Problem 38.3.2. The discriminant of the sum of k copies of a hermitian space over K is the k-th power of its discriminant, and for even k it is a norm from K. Hence the sum of four copies of a positive definite hermitian space over K is isometric to the standard space $\sum_i x_i \bar{y}_i$ of the same dimension.

The kernels. Let $E \supset K$ be the CM field of Chapter 37, Φ its CM type, and $\mathfrak{a}$ the $\mathcal{O}_E$-ideal used there for the hermitian lattices. Take $\Lambda = \delta \mathcal{O}_K$ with $\delta^{12} = g_2(\mathcal{O}_K)^3 - 27 g_3(\mathcal{O}_K)^2$, so that $g_2^3 - 27 g_3^2 = 1$ for Λ. For even N, N' and self-dual $\mathcal{O}_E$-lattices $L \subset E^N$, $L' \subset E^{N'}$, let $Y(L, L') = \mathfrak{a}\,\mathrm{Hom}(L, L')$ with the K-valued form $\mathrm{Tr}_{E/K} \sum_{i,j} x_{ij} \bar{y}_{ij}$. Its theta functions live on $Y(L, L') \otimes_{\mathcal{O}_K} \mathbb{C}$, with the lattice $Y(L, L') \otimes_{\mathcal{O}_K} \Lambda$; only the curve $\mathbb{C}/\Lambda$ enters. The construction has three layers.

(a) A theta function $\theta_{\mathfrak{a}}$ on $\mathfrak{a} \otimes \mathbb{C} = \mathbb{C}^{\Phi}$ for the lattice $\mathfrak{a} \otimes \Lambda$. Fix a K-linear isometry γ from four copies of $\mathfrak{a}$ to the standard hermitian space of the same dimension (Problem 38.3.2). The function $\theta_{\mathfrak{a}}$ is defined by

$$\theta_{\mathfrak{a}}^{\otimes 4} = \kappa \cdot I_{\gamma}\big(B_{t_0}^{\otimes 4|\Phi|}\big) \circ \gamma,$$

up to a fourth root of unity. Here B_{t_0} is the normalized translate by a point t_0 of order two, I_{γ} is the average of the translates U_x over the cosets of the standard lattice in the image under γ of four copies of $\mathfrak{a} \otimes \Lambda$, weighted by a semicharacter and divided by the square root of the index, and κ is an algebraic number. For each inert p the same function satisfies $\theta_{\mathfrak{a}}^{\otimes 4} = \kappa_p \cdot I_{\gamma_p}(B_{t_0}^{\otimes 4|\Phi|}) \circ \gamma_p$ for an isometry γ_p whose index is prime to p, with $|\iota(\kappa_p)| \leq 1$.

(b) On standard lattices, the product of copies of $\theta_{\mathfrak{a}}$, one for each matrix entry.

(c) For general (L, L'), coset averages of (b), with index factors and a character factor.

The *kernels* at (L, L') are the combinations $\sum_k r_k U_{t_k} e_k$ with $|r_k| \leq 1$, t_k torsion of order prime to p, and e_k the function of (c) for a pair equal to (L, L') at p.

Theorem 38.3.1 (Kernel law in orthonormal coordinates)**.** *Let $b_1, \ldots, b_r$ be an orthonormal basis of $Y(L, L') \otimes_{\mathcal{O}_K} \mathbb{Z}_{p^2}$. For every kernel F at (L, L'),*

$$\exp\Big(\sum_{i=1}^{r} \Upsilon(T_i)\Big) \cdot F\Big(\sum_{i=1}^{r} b_i \otimes u\,\ell(T_i)\Big)$$

is integral. There is no scalar.

As in Chapter 37, F stands for its Taylor series moved to $\mathbb{C}_p$ by ι, and b_i is substituted through $\mathcal{O}_K \subset \mathbb{Z}_{p^2} \subset W$. *Status*: proved in the Lean formalization. The proof is the following chain; each step is proved there.

(1) *One copy.* For $\theta_{\mathfrak{a}}$ and its translates by torsion of order prime to p, the assertion holds in one orthonormal frame of $\mathfrak{a} \otimes \mathbb{Z}_{p^2}$. By (a), after a change

of variables by a σ-unitary matrix over W, the fourth tensor power becomes κ_p times a combination $\sum_k r_k \prod_w B_{t_{k,w}}$ with $|r_k| \le 1$. Theorem 38.1.1 handles each factor, Problem 38.2.3 (applied in four copies) handles the change of variables, and Problem 38.3.1 takes the fourth root. That the values at 0 entering the r_k are p-integral is proved in the Lean; we have not followed that proof.

(2) *Products.* The case of (b) follows, since integrality is preserved by products in disjoint variables.

(3) *Change of frame.* The assertion for one orthonormal frame implies it for every orthonormal frame (Problem 38.3.3).

(4) *Standard at p.* If (M, M') equals the standard pair at p, every $U_t e(M, M')$ is a p-integral combination of translates of (b). We have not followed this step, which is a computation with the coset averages (c).

(5) *Weak approximation and covariance.* Every pair is $(\gamma M, \gamma' M')$ with γ, γ' unitary over E and (M, M') standard at p, and the kernel of $(\gamma M, \gamma' M')$ is the kernel of (M, M') composed with (γ, γ').

(6) *Combinations.* The assertion is stable under combinations with integral coefficients and depends on (L, L') only at p.

Problem 38.3.3. Let $b'_j = \sum_i \gamma_{ij} b_i$ with both bases orthonormal. Then $\sum_i \sigma(\gamma_{ij})\gamma_{il} = [j = l]$, and $\sum_j b'_j \otimes u\ell(T_j) = \sum_i b_i \otimes u\ell(T'_i)$ with T'_i as in Problem 38.2.3. Deduce that Theorem 38.3.1 for the basis b implies it for b'.

Theorem 38.3.2 (Normalized kernel law, (N3))**.** *Let F be a kernel at (L, L'), and suppose $Y(L, L') \otimes \mathbb{Z}_{p^2} = P \oplus P^\perp$ with $P^\perp$ orthonormal and P spanned by $f_1, \dots, f_m, f'_1, \dots, f'_m$ with $h(f_e, f_{e'}) = h(f'_e, f'_{e'}) = 0$ and $h(f_e, f'_{e'}) = [e = e']$. Put $\Theta(x, y) = F\big(\sum_e f_e \otimes u\varepsilon^{-1} \lg(x_e) + \sum_e f'_e \otimes u\varepsilon^{-1} \lg(y_e)\big)$, in the coordinates of Theorem 37.1.2. Then*

$$\Theta(x, y) = \exp\Big(-\sum_{e=1}^{m} \beta_0(x_e, y_e)\Big)\, U(x, y), \qquad U \text{ integral.}$$

Problem 38.3.4. Theorem 38.3.1, Problems 38.2.5, 38.2.4 and 38.1.5, and the relation $\lg \circ \varkappa = \varepsilon\ell$ imply Theorem 38.3.2. Thus the scalar of Theorem 37.1.2 can be taken to be 1 for the kernels of the chain, and Theorem 37.2.1 holds with $c = 1$.

Theorem 38.3.2 is Theorem 37.1.2 with $c = 1$, for the kernels just described, which are the kernels whose jets define the lifts of the chain. The step kernels of Chapter 37 carry in addition the scalar s_j of the step normalization, which is a unit; this is (N1), treated in Chapter 39. *Status*: proved in the Lean formalization. The Lean proves Theorem 38.3.1 at every admissible prime $p \ge 5$ inert in K, and states the hyperbolic form in the coordinates of the places at the primes whose Frobenius orbits have even length at least 2. Where Frobenius acts on the CM type as complex conjugation, the orbits have length 1, there is no hyperbolic splitting along an orbit, and the Lean uses a weaker shape that gives no gain.

In the Lean formalization. The one-variable function is `Formalism.L2.Th1.F.th0` (the translate of `sgStar` divided by its value at 0), with `s2`. The kernels are the class `IsComb`, built from `Formalism.L2.Ker.Cd.Base.thMM` (characterized by `IsThM`, with κ given by `KapChoice.kappa`), `thZero`, `thMoved` and `eNat`. Theorem 38.3.1 is `Formalism.L2.KerInert.IntInert`, proved as `kernels_integral_inert` through `intInert_of` from `CombFact`,

`PairFact`, `WeakApprox`, `PullFact`, `StdStep`, `TensFact` and `ThFourInt`; the one-copy step is `thFourInt_of` (from `OneCopyLoc`, `FourProd`, `LocInv`, `GlobONB`), with `scale_integrality`, `Formalism.L2.Th1.Top.kappa_integral`, `ThetaFourthTensorPowerProductShape` and `Formalism.L2.Th1.E.FourthPower`. It is repackaged as `GlobalTheory.KerStd.HostAt` by `hostAt_inert`. The rules (R1)–(R3) are `first_level_curve_holds` and `NormRel`; (B1)–(B4) are `B1`–`B4` in `product_zeroth_first_parts`, proved through `biextension_hypotheses_of_level_one`. Theorem 38.3.2, in the coordinates of the places and with the averaged β, is `OrbitPlaces.EvenConditionsInert` (with `ThetaExpansionAllPlaces` and `BeOK`), proved by `GlobalTheory.KerStd.exists_beFam_evenConditionsInert_of_kerEnt`. The good model is `Formalism.L2.Th1.CM.CM2`, proved by `cmGoodModel_of_globalArtinReciprocity` [Lean].

Hints

Section 38.1. 38.1.1. Extend η $\mathbb{R}$-linearly to $\mathbb{C}$ and use Problem 12.1.3. 38.1.2. Problems 12.1.4 and 38.1.1. 38.1.3. Problem 38.1.2, and $\mathrm{Im}(\bar{\omega}\lambda) \in A\mathbb{Z}$. 38.1.4. Differentiate $\log(\sigma_\Lambda(z)/z)$ twice; it is even. 38.1.5. An integral series with constant term 1 has an integral inverse.

Section 38.2. 38.2.1. $T +_{\mathcal{E}} [c]T = [1 + c]T$. 38.2.2. Induction on k, with (R1). 38.2.3. Problem 38.2.2; then (R3) for the diagonal terms and (R2), (R1) for the cross terms. 38.2.4. As in Problem 38.2.3, with the summands ordered $X_1, \dots, X_m, Y_1, \dots, Y_m$. 38.2.5. Expand by sesquilinearity; compute a determinant. 38.2.6. For (ii), $\sum_{\zeta \in \mu_n} \zeta^k$ is n or 0.

Section 38.3. 38.3.1. Look at the coefficient of $\prod_{a \le 4} (x^{(a)})^\mu$. 38.3.3. Compute $h(b'_j, b'_l)$; then substitute. 38.3.4. Write $f \otimes u\ell(X) + f' \otimes u\ell(Y)$ in the basis e_1, e_2 of Problem 38.2.5; the coefficients form a hyperbolic pair. Then Theorem 38.3.1 with the variables of $P^\perp$ set to 0, Problem 38.2.4, and $X_e = \varkappa^{-1}(x_e)$, $Y_e = \varkappa^{-1}(y_e)$.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

[AEC] J. H. Silverman, *The Arithmetic of Elliptic Curves*, 2nd ed., Springer, 2009.

[Lean] The Lean community, *riemann-hypothesis*, a Lean formalization of Alpöge's argument, `https://github.com/leanprover-community/riemann-hypothesis`.

[LT] J. Lubin and J. Tate, Formal complex multiplication in local fields, Ann. of Math. (2) **81** (1965), 380–387.

[Sch] W. Scharlau, *Quadratic and Hermitian Forms*, Springer, 1985.