THE OBJECTS OF THE CONSTRUCTION

Prerequisites. From Chapter 4: the field E, the groups H and Γ, the CM type Φ, cells, and Problems 4.1.1 and 4.1.3. From Chapter 21: class sets, the groups Γ_L, forms and their norm (Section 21.3), the masses M_ν and numbers κ_ν of (2) there, Problem 21.2.5, the Weyl dimension (4) there and Problem 21.4.2. From Chapter 23: the lift (1) there and Problem 23.5.1. From Chapter 6: degrees and slopes, Problems 6.3.2, 6.3.6 and 6.3.7; slopes below are the normalized slopes $\mu^\sharp$ there, written μ. Hermitian forms over a CM field and their signatures; Hilbert's reciprocity law [Ne, Ch. VI, §8].

Chapters 37–58 speak of words, tops, nodes, terminal primes, the lattices M_u, N_u, R and the period maps B_u. What are they? This chapter defines them, in the form in which the Lean formalization [Lean] uses them, and checks that the ranks, signatures and weights fit together. We follow [Rub, §III.10], [Zh, §4.1] and [Adr, §7, Defs. 6.1, 11.5–11.10, 18.5, 18.9]. Fix the scale x, $0 < \varepsilon < \frac{1}{52}$, $b := \exp(x^\varepsilon)$, $a := xb$ and $b' := a\exp(-x^{\varepsilon/3})$, as in Chapter 58.

35.1. The field and its places

Let $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$ be primes with $q \mid d+1$, let $n \geq 4$ be a power of 2, $N := 4n$, and let E, $K = \mathbb{Q}(\sqrt{-d})$, H, $\Gamma = \mathrm{Gal}(E/\mathbb{Q})$ and $\Phi = \mathrm{Gal}(E/K)$ be as in Chapter 4. Let F be the maximal real subfield of E, and $f := |\Phi| = 2n$. The elements $\rho \in \Phi$ are the *places*; $\rho = (s, e_1, 1)$ is the cell (v, s) with $v = \frac{1}{2}(1 - e_1)$.

Three subsets of Φ are singled out. A set $\mathcal{T} \subset H$ of *tops* gives the *top places* $(0, t)$, $t \in \mathcal{T}$. A set $T_2 \subset H$ and a *translate* $\tau \in H$ give the *thick places* $(1, \tau s)$, $s \in T_2$. The other places are *thin*. (Chapter 58 writes T for $\mathcal{T}$.) The translate is fixed until Section 35.5.

Let $p \nmid 2qd$ be inert in K, and put $\varphi_p := c\,\mathrm{Frob}_p$, where $c = ([1], -1, -1)$ is complex conjugation.

Problem 35.1.1. $\varphi_p \in \Phi$. Either $\mathrm{Frob}_p = c$, or $c \notin \langle \mathrm{Frob}_p \rangle$ and φ_p has the order of Frob_p, which is even.

Problem 35.1.2. Multiplication by φ_p sends the cell (v, s) to $\big(v + \frac{1}{2}(1 + \chi_q(p)) \bmod 2,\ s[p]\big)$. So the orbits stay in one row when $\chi_q(p) = -1$ and alternate between the rows when $\chi_q(p) = 1$.

The primes inert in K with $\chi_q(p) = 1$ are called *signal* primes, and those with $\chi_q(p) = -1$ primes of *inert class*. When φ_p has order 2ν, an orbit $\{\rho\varphi_p^s : s \in \mathbb{Z}/2\nu\}$ of φ_p on Φ is an orbit with its *positions* s in the sense of [Adr, §11.1]; its even positions are the places $\rho\,\mathrm{Frob}_p^s$, its odd positions the places $\rho\,\mathrm{Frob}_p^s\,c$. Chapter 37 works along these orbits.

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Problem 35.1.3. For $q = 11$, $d = 1319$, $n = 4$, the orbit of the place $(0, [1])$ is $(0, [1]), (0, [3]), (0, [7]), (0, [5])$ for $p = 13$, and $(0, [1]), (1, [7])$ for $p = 23$. For $p = 17$, $\mathrm{Frob}_p = c$.

In the Lean formalization. The field is a record `Host` with required properties (Galois group, CM type `cmType`, Frobenius elements), not a field given by generators; for ζ it is the field E above, by an inference from the definitions in [Rub, §III.10, §9, Remark (1)]. Tops are `Host.TT`, top and thick places `Host.topPlaces` and `Host.thickPlaces`, the translate `Host.u0`, the classes of primes `primeClass` (`splitK`, `signal`, `inertClass`).

35.2. Ranks and hermitian spaces

The construction has three integers k, m, t_0, subject to

$$\begin{gathered} k, m \text{ even}, \qquad t_0 \text{ odd}, \qquad 2 \le m, \qquad m + 3 \le k, \\ k + m + 4 \le t_0, \qquad 3m < t_0, \qquad m \le f \le k, \qquad m^2 \le N_m, \end{gathered} \tag{1}$$

where $N_j := k + j(t_0 + 1)$ for $0 \le j \le m$. Put $r := k + m - 1$. In the argument $m \approx x^{1/2}$ is the number of lifts and $k \approx x^{9/2}$ (Chapter 58); t_0 is a power of x as well [Rub, §III.10, §3]. Hermitian spaces are over E; at each $\rho \in \Phi$ a hermitian space has a signature, the numbers of positive and negative eigenvalues of ρ applied to a Gram matrix. The construction uses the following spaces.

- (S1) X_j for $0 \le j \le m$, positive definite of rank N_j.
- (S2) A vector $z \in X_m$ with $\langle z, z\rangle \neq 0$, and $X' := z^{\perp} \subset X_m$.
- (S3) Y_j for $0 \le j \le m$, of rank $r + jt_0$ and signature $\big(r + jt_0 - \lfloor \frac{m-j}{2} \rfloor, \lfloor \frac{m-j}{2} \rfloor\big)$ at every ρ; isometric embeddings $X_j \to Y_j$ for $j < m$, and an isometry $X' \to Y_m$.
- (S4) A decomposition $Y_0 = X_0 \perp Ez_0 \perp P_1 \perp \cdots \perp P_{m/2-1}$ into the image of X_0, a line with $\langle z_0, z_0\rangle = -T$ for a prime T (Section 35.5), and hyperbolic planes P_i.
- (S5) W_j for $1 \le j \le m$, of rank r and signature $(k + j - 1, m - j)$ at every ρ, isometric to Y_0 over F_v for every finite place v of F.

The forms of Section 35.4 live on the definite groups $U(X_j)$; the spaces Y_j and W_j carry the periods (Chapters 43 and 47).

Problem 35.2.1. For $n = 4$ the smallest ranks allowed by (1) are $(m, k, t_0) = (2, 8, 15)$. Then $r = 9$ and $N_0, N_1, N_2 = 8, 24, 40$.

Problem 35.2.2. Under (1), r is odd, every N_j is even, $N_m = (r + mt_0) + 1$, and $\mathrm{rk}\, Y_j - N_j = m - 1 - j$.

Problem 35.2.3. For $j < m$ the orthogonal complement of X_j in Y_j has rank $m - 1 - j$ and signature $\big(\lceil \frac{m-j}{2} \rceil - 1, \lfloor \frac{m-j}{2} \rfloor\big)$ at every ρ. For $0 \le j \le m$, $U(Y_j)$ is compact at the archimedean places exactly when $j \ge m - 1$.

The codimension $m - 1 - j$ is odd for even j and even for odd j; accordingly the periods between $U(X_j)$ and $U(Y_j)$ are Bessel periods for even j and Fourier–Jacobi periods for odd j [Zh, §4.1]. At $j = m$ the inclusion reverses: $Y_m \cong X'$ has codimension one in X_m.

Problem 35.2.4. The space Y_0 of (S4) has rank r and signature $(k + \frac{m}{2} - 1, \frac{m}{2})$, as (S3) requires; this is the signature of $W_{m/2}$. The space W_m is positive definite.

Problem 35.2.5. For $(m,k,t_0)=(2,8,15)$ the spaces Y_0,Y_1,Y_2 have ranks $9,24,39$ and signatures $(8,1),(24,0),(39,0)$; W_1,W_2 have signatures $(8,1),(9,0)$; there are no hyperbolic planes in (S4).

The local condition in (S5) constrains the signatures. An isometry g multiplies the Gram determinant by the norm $\det g\,\overline{\det g}$. So if W_j and Y_0 are isometric at every finite place of F, the quotient $\delta\in F^\times$ of their Gram determinants is a norm from E_v at every finite v. By Hilbert's reciprocity law for the quadratic extension E/F, δ is then negative at an even number of real places of F.

Problem 35.2.6. The sign of δ at the real place of F below ρ is $(-1)^{m/2-j}$, the same at all f places. Since f is even, the parity condition holds for every j.

The spaces are data of the records of the formalization, which requires (S1)–(S5) of them and constructs such records at every large scale (Section 35.5); [Rub, §III.10, Hyp. O12] assumes them.

In the Lean formalization. The ranks are `Ranks` (with `t` for t_0, `N`, `r`, `rankY`), the spaces `Spaces`: `X`, `X'`, `z`, `Y`, `iXY`, `eYm`, `zm`, `hypE`, `hypF`, `W`, and definite spaces `V j` of rank $r+t_0$ not used here. The signatures are `sigYD`, `sigY0`, `sigW` with `D7.Y_sigval`, `W_sigval`, `baseSpace_isometric_graded`; Problems 35.2.2 and 35.2.4 are `FactROdd`, `FactNEven`, `FactXm`, `FactYm`, `RankBaseSpaceEq`, `SignatureBaseMiddleEq`, `FactWmDef`. The record `Inhab.Rt` has the ranks of Problem 35.2.1 with larger odd t_0.

35.3. Weights, nodes and words

At each place ρ the construction fixes numbers $\xi_1,\dots,\xi_k\in\mathbb{Z}+\frac12$ and integers $\xi'_1,\dots,\xi'_{k-1},\theta_1,\dots,\theta_m,w_1,\dots,w_m$, all depending on ρ, with

$$\xi_1>\xi'_1>\xi_2>\cdots>\xi'_{k-1}>\xi_k>-\theta_1,\qquad w_j-\theta_j\ge\tfrac{t_0+3}{2},\qquad \theta_{j+1}-w_j\ge\tfrac{t_0+3}{2}. \tag{2}$$

The *weight of level* j at ρ is $\lambda^{(j)}=\lambda^{(j)}(\rho)\in\mathbb{Z}^{N_j}$: first $\lambda^{(j)}_i=\xi_i+i-\frac{k+1}{2}$ for $1\le i\le k$, then, for $l=1,\dots,j$, a block of t_0+1 entries equal to $-w_l+\frac{k}{2}+(2l-1)\frac{t_0+1}{2}$. The numbers $w_j(\rho)$ are the *nodes*. For a weight λ of rank N' the shifted numbers $\lambda_i+\frac{N'+1}{2}-i$ are its *Harish-Chandra parameters*.

Problem 35.3.1. Under (2) every $\lambda^{(j)}$ is integral and nonincreasing.

Problem 35.3.2. The Harish-Chandra parameters of $\lambda^{(j)}$ are $\xi_1,\dots,\xi_k$ together with the blocks $B_l:=\{-w_l+\frac{t_0}{2}-c:0\le c\le t_0\}$, $l\le j$, all translated by $j\frac{t_0+1}{2}$. Thus the j-th lift keeps the old parameters, up to a common translation, and adds the block of t_0+1 consecutive numbers centred at $-w_j$.

At the top place $\rho_t=(0,t)$ of a top t there are integers a_t, the *top weights*, and b_t, with $\xi_1(\rho_t)=a_t+\frac{k-1}{2}$ and $\xi_k(\rho_t)=-b_t-\frac{k-1}{2}$, where $a\le a_t\le a(1+x^{-13}/k)$ and $b'\le b_t\le(1+x^{-2})b'$ [Adr, Def. 18.5].

Problem 35.3.3. $\lambda^{(0)}(\rho_t)$ begins with a_t and ends with $-b_t$.

So the top weight $a_t\approx xb$ is the length of the first row of the weight at the top place. The nodes lie in windows of a common width Δ: $b\le w_j\le b+\Delta+1$ at thick places, $b_t+b'/x\le w_j\le b_t+b'/x+\Delta$ at the top place of t, and $w''\le w_j\le w''+\Delta$ at thin places, where $w'':=b'/(xn)$. This is the thick–thin condition (2) of Chapter 4,

and the nodes are the weights $w_j\colon \Phi \to (0,\infty)$ of the prime sums $P_{t,w_j}(a_t)$ of Chapter 58. At places other than top places $\xi_1 - \xi'_1$ and all ξ'_i are bounded by a constant multiple of a size parameter S_c [Rub, §III.10, Hyp. O24].

The *words* vary the middle of the parameter. There is an integer $\Delta_{\mathcal{W}} \geq 1$ with $\xi'_l - \xi'_{l+1} = 4\Delta_{\mathcal{W}}$ at every place. A word u is a string of letters $\epsilon_1, \dots, \epsilon_{k-2} \in \{0,1\}$; its parameters are ξ_1, ξ_k, the ξ', θ, w above, and

$$\xi_{l+1}(u) = \xi'_l - \tfrac{1}{2} - \epsilon_l \Delta_{\mathcal{W}} \qquad (1 \leq l \leq k-2),$$

the same letters at every place. A finite set $\mathcal{W}$ of words with $|\mathcal{W}| \leq 2^{k-3}$ is chosen; Chapter 43 explains why there must be many (Problem 43.4.5).

Problem 35.3.4. Every word satisfies (2), and distinct words have distinct weights $\lambda^{(0)}(\rho)$ at every place.

By the branching rule for $U(k) \supset U(k-1)$, whose condition is interlacing of Harish-Chandra parameters, the representations of all words at ρ contain the representation of $U(k-1)$ with parameter ξ': the words are weights of rank k above one fixed weight of rank $k-1$ [Rub, §III.10, §6].

How does a lift change the number of forms? Work in the model of Chapter 21, with one place and $\mathbb{Z}[\mathrm{i}]$ in place of $\mathcal{O}_E$. For a nonincreasing $\lambda \in \mathbb{Z}^{N'}$ write $z_i = \lambda_i + N' - i$, so that $D_{N'}(z)$ is its Weyl dimension; this depends only on the differences of the λ_i.

Problem 35.3.5. Let $\lambda \in \mathbb{Z}^{N'}$ be nonincreasing, $c \leq \lambda_{N'}$ an integer, and λ^+ the weight λ followed by t_0+1 entries c, with z^+ defined in rank $N'+t_0+1$. Then

$$V_{N'+t_0+1}(z^+) = V_{N'}(z) \prod_{c'=1}^{t_0} c'! \prod_{i=1}^{N'} \prod_{l=1}^{t_0+1} \big(\lambda_i - c + N' - i + l\big).$$

Problem 35.3.6. In the situation of Problem 35.3.5, with the masses M_ν of Chapter 21,

$$\frac{M_{N'+t_0+1} D_{N'+t_0+1}(z^+)}{M_{N'} D_{N'}(z)} = \prod_{c'=1}^{t_0} c'! \prod_{i=1}^{N'} \prod_{l=1}^{t_0+1} \big(\lambda_i - c + N' - i + l\big) \prod_{\nu=N'+1}^{N'+t_0+1} \pi^{-\nu} \kappa_\nu.$$

For $\lambda = (3,1)$, $t_0 = 1$, $c = 0$ this ratio is $5/64$.

So the count of forms changes at each lift by a product of differences among the new parameters and between them and the old ones; the factor $\prod_{c'} c'!$ is the Vandermonde product of the new block, and no factorial from the denominators of the Weyl formula is left over. Over E the Weyl factor is a product over Φ, and the mass ratio is an exact formula for unramified CM fields, used in Chapter 47.

In the Lean formalization. The parameters are `HCParam` with fields `u`, `v`, `b`, `e` (our ξ, ξ', θ, w) and the gaps `gap_be`, `gap_eb`; the weights are `MemberForms.lam` with `lam_eq`; the nodes `wj`; the top weights $\mathtt{SizeData}_2\mathtt{.aT}$, `bT` with `top_u`; the windows `win_thick`, `win_top`, `win_thin`; the words `SData.letter`, `u_letter`, `cloud_gap` (spacing `Del`). Problem 35.3.1 is `factHCMono` with the integrality facts `FactSInt`, `FactEInt`.

35.4. Forms and the lifts

For each level j the construction has a finite set Cl_j of lattices in X_j, representatives of the classes of a genus with level structure, and their finite stabilizers Γ_Λ. A *form of level j* assigns to $\Lambda \in \mathrm{Cl}_j$ a vector of the representation of weight $(\lambda^{(j)}(\rho))_{\rho\in\Phi}$ fixed by Γ_Λ, with the norm $\sum_\Lambda |\Gamma_\Lambda|^{-1}\|\varphi(\Lambda)\|^2$, as in Section 21.3 of Chapter 21. The j-th *lift* K_j, from level $j-1$ to level j, is

$$(K_j\varphi)(\Lambda') = \sum_{\Lambda\in\mathrm{Cl}_{j-1}} \frac{\epsilon_j(\Lambda,\Lambda')}{|\Gamma_\Lambda|}\,\mathbf{K}_j(\Lambda,\Lambda')\,\varphi(\Lambda), \qquad |\epsilon_j(\Lambda,\Lambda')| = 1. \tag{3}$$

This is (1) of Chapter 23 with unit scalars. The block $\mathbf{K}_j(\Lambda,\Lambda')$ is built from Taylor coefficients at 0 of a theta function of the lattice $\mathrm{Hom}_{\mathcal{O}_E}(\Lambda,\Lambda')$, shifted by a fractional ideal and averaged over Heisenberg translates at the level primes, on $V = \prod_{\rho\in\Phi} M_{N_j\times N_{j-1}}(\mathbb{C})$ [Adr, §7, Def. 11.10]; this is the *kernel* of the lift. The *chain* is the composite $K_m\cdots K_1$. Let D_Γ be a common multiple of the orders $|\Gamma_\Lambda|$, $\Lambda\in\mathrm{Cl}_j$, $j<m$.

Problem 35.4.1. $\mathrm{Hom}_{\mathcal{O}_E}(\Lambda,\Lambda')$ is a free $\mathbb{Z}$-module of rank $2fN_{j-1}N_j$, and its theta function is a function of $fN_{j-1}N_j$ complex variables; for $(m,k,t_0)=(2,8,15)$ and $j=1$ there are 1536 of them.

Problem 35.4.2. $(K_m\cdots K_1\varphi)(\Lambda_m)$ is the sum over $\Lambda_j\in\mathrm{Cl}_j$, $j<m$, of $\prod_{j<m}|\Gamma_{\Lambda_j}|^{-1}\prod_{j=1}^m \epsilon_j(\Lambda_{j-1},\Lambda_j)\cdot\mathbf{K}_m(\Lambda_{m-1},\Lambda_m)\cdots\mathbf{K}_1(\Lambda_0,\Lambda_1)\varphi(\Lambda_0)$. Hence D_Γ^m clears the stabilizer denominators of the chain.

In [Adr] the lifts are [Adr, Def. 11.10], and D_Γ may be taken to be Minkowski's bound $M(2fN_m)$ for the orders of finite subgroups of $\mathrm{GL}_{2fN_m}(\mathbb{Z})$ [Adr, Prop. 11.14]; see Problem 37.3.1 of Chapter 37. The integrality of the kernels needed there rests in [Adr] on the hypotheses (N1)–(N3) of Chapter 37, assumed there; the formalization proves the final theorem without assuming them.

In the Lean formalization. The chain is `ColumnChain`: classes `cls`, stabilizers `stab`, weights `wt` $= 1/|$`stab`$|$, the common multiple `DGam` with `dvd_DGam`, the block `Kmap` $=$`rhoDet`$\cdot$`rhoDet'`$\cdot($`divDe`$\circ$`unf`$\circ$`sharp`$)$, the lift `K` and the composite `col`. Here `rhoDet`, `rhoDet'` are the unit scalars, and `sharp`, `divDe` rescale coordinates by powers with exponents $w_j - N_j/2$ and $w_j - N_{j-1}/2$. The block `unf` is abstract; its relation to theta functions is the separate statement `ThetaX`, which we have not followed. The forms of level 0 are `Mu0`, and the inner product is `MemberForms.ipC`.

35.5. Terminal primes, lattices and period maps

Each word $u\in\mathcal{W}$ has its own lattice classes and lifts, and a *terminal prime* $T_u = \varpi_u\bar{\varpi}_u$ with $\varpi_u\in\mathcal{O}_K$ and $\varpi_u\equiv 1$ modulo $Nqd(g_0g_1)^2$, where g_0,g_1 are the level primes (Chapter 58). The prime T of (S4) is T_{u_0} for a base word u_0. The terminal primes are chosen in windows fixed by dimensions of spaces of forms, so that $|\log T_u - \log T_{u'}| = O(\log k)$ [Adr, Def. 18.9 (K7)]; Chapter 47 computes their size. (Chapter 58 calls T_u the auxiliary prime.)

Problem 35.5.1. $T_u\equiv 1 \pmod{Nqd}$, so T_u splits completely in E. There are f primes of F above T_u; each has residue field $\mathbb{F}_{T_u}$ and splits in E.

At each prime v of F above T_u the level structure of a lattice $\Lambda\in\mathrm{Cl}_j$ contains an $\mathbb{F}_{T_u}$-vector space of dimension N_j with a nonzero linear form, and $\alpha\in\mathbb{F}_{T_u}^\times$ acts

on lattices with level structure, and on forms, by rescaling that form. A form of level 0 is *ramified at* T_u if $\sum_{\alpha\in\mathbb{F}_{T_u}^\times} \alpha\cdot\varphi = 0$ for each of the f primes v.

The lattices are over the ring of integers of a *coefficient field* $L \supset E$, a number field that is Galois over $\mathbb{Q}$ and totally complex, with a fixed embedding in $\mathbb{C}$ (Chapter 43 writes K for it). For a place w of L let $\mathfrak{p}_w$, e_w be its prime and ramification index, and $\ell_w := \log N\mathfrak{p}_w/[L:\mathbb{Q}]$, as in Chapter 52.

(M) The *base lattice* M_u is an $\mathcal{O}_L$-lattice spanning an L-structure of the space of forms of level 0 of the word u, with the norm of Section 35.4 at the fixed embedding. Let M_u^{ram} be its intersection with the ramified forms, of rank d_u.

(N) An $\mathcal{O}_L$-lattice N'_u of rank d_u in $M_u^{\mathrm{ram}}\otimes L$, with the same norms, such that $N'_{u,w}\subseteq M_{u,w}^{\mathrm{ram}}$ at every $w\nmid T_u$, with equality above the primes that are not inert in K and above the inert primes at which no gain is taken; and the *source lattice* $N_u := I_u N'_u$, with $I_u := \prod_w \mathfrak{p}_w^{-\lfloor e_w\sigma_u(w)\rfloor}$ for numbers $\sigma_u(w)\geq 0$, the *gains*, nonzero at finitely many w.

(R) The *top lattice* R, an $\mathcal{O}_L$-lattice spanning an L-structure of a finite-dimensional space $\mathcal{E}$ of automorphic forms, the same for all words, and its dual $R^\vee := \mathrm{Hom}_{\mathcal{O}_L}(R,\mathcal{O}_L)\subset \mathrm{Hom}_L(R\otimes L, L)$, whose unit lattices are the dual unit lattices of Chapter 6.

(B) The *period map* $B_u\colon N_u\otimes L\to \mathrm{Hom}_L(R\otimes L, L)$. Over $\mathbb{C}$, $B_u(\varphi)(G)$ lifts φ along the chain $K_m\cdots K_1$, restricts the result to the lattices of $X' = z^\perp$, projects it to one weight of rank N_m-1, and pairs it with the form on $X'\cong Y_m$ attached to $G\in\mathcal{E}$.

The record requires $B_u(N_u)\subseteq R^\vee$ and $h(B_u)\leq h_u$ for numbers h_u (Chapter 43). Chapter 37 proves the inclusion, with the gains $\sigma_u(w)$ at the inert primes; Chapter 43 identifies B_u with a Bessel period; Chapter 47 bounds h_u. At the places above T_u the formalization imposes a separate condition on N'_u, and it defines $\mathcal{E}$ and R place by place from integral structures that depend on twisting data; we have not traced either, nor the projection in (B).

Problem 35.5.2. Let $N'\subseteq M$ be $\mathcal{O}_L$-lattices of rank d in one space, with the same archimedean norms, and len_w the length of $(M/N')\otimes\mathcal{O}_{L,w}$. Then $\mu(N') = \mu(M) - \frac{1}{d}\sum_w 2\ell_w\,\mathrm{len}_w$.

Problem 35.5.3. For integers n_w, almost all zero, $\mu\big(\prod_w \mathfrak{p}_w^{-n_w}N\big) = \mu(N) + \sum_w 2\ell_w n_w$. If $n_w = e_w\sigma_p$ for every $w\mid p$, the contribution of p is $2\sigma_p\log p$.

Combining Problems 35.5.2 and 35.5.3,

$$\mu(N_u) = \mu(M_u^{\mathrm{ram}}) - \frac{1}{d_u}\sum_{w\nmid T_u} 2\ell_w\,\mathrm{len}_w(M_u^{\mathrm{ram}}/N'_u) + \sum_w 2\ell_w\lfloor e_w\sigma_u(w)\rfloor$$

when $N'_{u,w} = M_{u,w}^{\mathrm{ram}}$ above T_u. The last sum is the gain, about $\sum_p 2\sigma_p\log p$; the middle one is the loss at the inert primes. This is the shape of (A2) of Chapter 58, whose left side has $\mu(M_u)$ in place of $\mu(M_u^{\mathrm{ram}})$; we have not traced the comparison of these two slopes in the formalization.

Problem 35.5.4. $B_u(N_u)\subseteq R^\vee$ if and only if $B_u(\varphi)(y)\in\prod_w \mathfrak{p}_w^{\lfloor e_w\sigma_u(w)\rfloor}$ for all $\varphi\in N'_u$ and $y\in R$.

So the inclusion $B_u(N_u)\subseteq R^\vee$ says that the periods of the lattice N'_u are divisible, at each place w, by $\mathfrak{p}_w^{\lfloor e_w\sigma_u(w)\rfloor}$; the gains are measured in units of v_p.

The record. For one scale and one translate, the objects $\mathcal{W}$, T_u, M_u, N_u, h_u, R, together with the local terms $E_{u,j}$, the common local term $\mathcal{L}$, the total gain Γ and the index loss, form a *chain record* [Adr, Def. 18.9] if they satisfy the eight conditions (K1)–(K8) listed there. These are the inequalities (A1)–(A7) of Chapter 58 with explicit errors. The formalization proves the final theorem; the corresponding step in [Adr], the existence of chain records for every admissible datum, is [Adr, Hyp. 18.10], assumed there. The formalization establishes records only for the data of one family class that it selects, which is weaker than the quantifiers of [Adr, Hyp. 18.10] and suffices for the final theorem.

In the Lean formalization. T_u is `ColumnsAtT.T`, with `D7.T_norm` (the congruence), `D7.T_win` (the window, of relative width $1/(fk)$ around `Tstar`), and `D7.cOf_eq`; the f primes above T_u are `vT` with residue fields `Fv`, the action is `rhoT` (`MemberForms.dil`), and the ramified forms are `Mram`. L is `Rows.Data.LL`; lattices are `HLat`. M_u is `(D7.fm u).lat 0`, M_u^{ram} is `DV.MramL`, N_u' is $\texttt{SizeData}_2\texttt{.NuLat}$, with the conditions `DV.NuDef`, and $N_u =$`D7.div.Nu u` by `DV.NprimeDef`; the gains are $\texttt{SizeData}_2\texttt{.sg}$ and $2\ell_w e_w\sigma$ is $\texttt{SizeData}_2\texttt{.sz w }\sigma$. R, B_u, h_u are `DivData.RR`, `Bk`, `h`; `Rows.BuDefQ` and `D7.BuRational` give (B); $B_u(N_u) \subseteq R^\vee$ is `DivData.OneMapIntoCC`, a clause of `LinkDIV_at`. A family of records over the scales is `FamP`; the interface statements on it are `LinkG`, `LinkS`, `LinkDIV_at`, `LinkMu` and `LinkN_at` in `GlobalTheoremInputs`, and `ChainTied` with `TieLocal` ties them to the abstract record of [Adr, Def. 18.9] (rank d_u, T_u, $\mu(N_u)$, $\mu_{\max}(N_u)$, h_u, $\mu(M_u)$, $\mu_{\max}(M_u)$, $\mu(R^\vee)$, the local terms). The selections and analytic inputs from which these statements are derived are the package `HypsAlt`, on one family class.

Hints

35.1.1: Φ has index two and contains neither c nor Frob_p. If $c = \mathrm{Frob}_p^i$, compare the three components; H is a 2-group. Then compute $(c\,\mathrm{Frob}_p)^s$ for even and odd s.

35.1.2: Problem 4.1.1; compare Problem 4.1.3.

35.1.3: Compute $[p]$, $\chi_{11}(p)$, $\chi_{1319}(p)$ and use Problem 35.1.2.

35.2.1: Take each of m, k, t_0 as small as the constraints allow, in this order.

35.2.3: The image of X_j is positive definite, so the complement carries all negative directions. For the last part use (S3) directly.

35.2.4: A hyperbolic plane has signature $(1,1)$ at every place.

35.2.6: At a real place the sign of the Gram determinant is (-1) to the number of negative eigenvalues.

35.3.1: Between half-integers and integers, a strict inequality gains $\frac{1}{2}$. Where the blocks begin use $\xi_k > -\theta_1$ and $w_1 - \theta_1 \geq \frac{t_0+3}{2}$; between two blocks use $w_{l+1} - w_l \geq t_0 + 3$.

35.3.2: For an entry of the block l the position is $i = k + (l-1)(t_0+1) + c$, $1 \leq c \leq t_0 + 1$.

35.3.4: $\frac{1}{2} + \Delta_{\mathcal{W}} < 4\Delta_{\mathcal{W}}$.

35.3.5: Sort the pairs of indices into old–old, new–new and old–new.

35.3.6: Problem 21.4.2 with (4), and Problem 21.2.5 for the value.

35.4.1: Over $\mathbb{R}$, $\mathrm{Hom}_E(X_{j-1}, X_j) \otimes \mathbb{R}$ is the product over $\rho \in \Phi$ of complex matrix spaces.

35.4.2: Induction on m, as in Problem 23.5.1.

35.5.1: $\varpi_u\bar{\varpi}_u$ is congruent to 1 modulo the ideal generated by Nqd; then Problem 4.1.1.

35.5.2: Only the unit lattices at the finite places change. At each w use elementary divisors of N'_w in M_w, as in Problem 6.3.2.

35.5.3: Problems 6.3.6 and 6.3.7.

35.5.4: $B_u(I_uN'_u) = I_uB_u(N'_u)$; localize, where R is free.

References

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