THE ACCOUNTING IN THE LEAN FORMALIZATION

Prerequisites. Chapter 58, all of it, and its notation. From Chapter 1: the functions G and G_χ of (1) and (3), and Theorems 1.2.3 and 1.5.1. From Chapter 4: Problems 4.4.1 and 4.4.2. From Chapter 43: Theorem 43.4.1 and Problem 43.4.5. From Chapter 42: Section 42.3 and Theorem 42.4.1. Chebyshev's bound $\sum_{n\le y}\Lambda(n)\le 3y$ for $y\ge 0$, as in Chapter 58.

Chapter 58 stated the inputs at one scale in a form close to [Adr], and proved the final theorem from them in the form the Lean formalization [Lean] proves it. What exactly does the formalization assume at one scale, and in what order does it make its choices? This chapter records its accounting, and the problems check that the record gives the statements of Chapter 58. The record has been read in the formalization's own statements; where we have read a statement but not its proof, we say so. In this chapter $c\ge 1$ is a constant of the record, which may depend on ε but not on x, and $s:=\langle\mu(N)-h\rangle$.

59.1. The record at one scale

The formalization's statement at a scale x is one record for all translates $\tau\in H$ at once (`SizeData`$_2$). For each τ it has the words u and their numbers d_u, T_u, $\mu(N_u)$, $\mu_{\max}(N_u)$, h_u, $\mu(M_u)$, $\mu_{\max}(M_u)$ and E_u, and the numbers S, Γ, $\mathcal{L}$, $\mu(R^\vee)$ and $\mu_{\min}(R^\vee)$ of Chapter 58. It also has the excess colength I of the Hecke images, and the parameters δ_F,η_F,e_F of the frame theorem. Its inequalities are those of Chapter 58, inputs (A1)–(A7) and (A2$^+$)–(A6$'$), with the errors written in a few quantities: the precision $L_a\le 2y$ of the linear term of the norms, the number X'' and the weight b' of Chapter 42, and a number L of the record.

- (A1): $S-\Gamma\le cN_Tmb\,x^{e_L\varepsilon}$, and also $I\le cN_Tmb\,x^{e_L\varepsilon}$.
- (A2): the error depends on the word, $e_2(u)=c\big(I+N_Tm(a/k+b')\log T_u+X''\big)+1$; and $e_2^+=cmB_0$.
- (A3), in the form the frame theorem gives: if $s-\mu_{\min}(R^\vee)\le Z$, then

$$s-\log|\mathcal{W}|-(1-\delta_F)^{-1}\big(4u_F\log(1+Z/u_F)+(\eta_F+4\sqrt{e_F})Z\big)\le\mu(R^\vee),$$

  and the record has such a Z with $0<Z\le x^{C_H}b$, and $0\le\delta_F\le\bar\delta_F<1$, $0\le\eta_F\le\bar\eta_F$, $0\le e_F\le\bar e_F\le\frac14$.
- (A4): $e_4=cN_TaL_a+cm(B_0+N_c)x^{e_D\varepsilon}$.
- (A5), (A5$^-$): $|E_u-mAg|\le cmB_0$.
- (A6): $e_6=c(L_a+m^2L_a/k)$; and $\log T_u\le cmL$, $e_T=c\log k$.
- (A7), (A7$^+$): $e_7=cN_TaL_a(1+m/k)$ and $e_7^+=cN_TaL_a$.
- (H): $e_H=c(N_TaL_a+mB_0)$.

The conclusion, for every translate, is the bound of Theorem 58.1.1 with an explicit error (`each_unit_bound_statement`). Its proof is the three cancellations of Chapter 58, with the spread bounded as in Problem 58.2.1 (`u_function_le_mean`).

Date: November 25, 2026.

Problem 59.1.1. Let ℓ_u be the right side of (A6), with $e_6 = c(L_a + m^2 L_a/k)$. Then

$$\sum_t m\big(\varphi(a_t) - a_t \log \Omega\big) - A\ell_u + mAg = \sum_{j,t} M_{w_j}(a_t) + Ac\big(L_a + m^2 L_a/k\big).$$

Problem 59.1.2. If the error e_2 in (A2) depends on the word, Theorem 58.1.1 and Problem 58.2.1 hold with e_2 replaced by $\langle e_2 \rangle$.

Problem 59.1.3. With the errors of the record, $\langle e_2 \rangle \le c\big(cN_T mb\, x^{e_L \varepsilon} + cN_T m^2(a/k + b')L + X''\big) + 1$.

So the upper bound for $\log T_u$ is used only to bound the error of (A2).

Problem 59.1.4. For $Z > 0$ the function $v \mapsto v \log(1 + Z/v)$ is increasing on $(0, \infty)$. Hence, if $0 < u_F \le U$ and $0 < Z \le x^{C_H} b$, then $u_F \log(1 + Z/u_F) \le (U + b)\log(1 + x^{C_H})$.

Problem 59.1.5. Suppose $0 < u_F \le U$, $\bar{\delta}_F \le \frac{1}{2}$ and $(\bar{\eta}_F + 4\sqrt{\bar{e}_F})x^{C_H} \le 6$. Then the form of (A3) in the record implies (A3) with $e_3 = \log|\mathcal{W}| + 8\log(1 + x^{C_H})(U + b) + 12b$.

The formalization obtains $(\bar{\eta}_F + 4\sqrt{\bar{e}_F})x^{C_H} \le 6$ from the construction of the frames. Why does it need (H)?

Problem 59.1.6. Let $\mathcal{W} = \{u, u'\}$ with $d_u = d_{u'}$, let all the slopes $\mu(N)$, $\mu_{\max}(N)$, $\mu(M)$, $\mu_{\max}(M)$ be 0, let $T_u = T_{u'}$, $E_u = E_{u'} = mAg = \Gamma + \mathcal{L}$, and $h_u = 0$, $h_{u'} = D \ge 0$. Then (A2), (A5), (A7), (A2$^+$), (A5$^-$), (A7$^+$) and (A6$'$) hold with all errors 0, and $u_F = D/2$.

So without (H) the spread is not bounded by the other inputs: e_H must be at least D here.

59.2. Where Γ and the slopes of the sources come from

At an inert place w the source lattice is the Hecke image twisted by the gains, $N_{u,w} = \mathfrak{p}_w^{-\lfloor e_w \mathrm{sg}_u(w) \rfloor} NP_w$ (Section 42.3). In the formalization Γ is a weighted sum over the inert places of $\sum_j \mathrm{gain}_j(p) - \tau(p)$: the colength of the Hecke image is paid inside Γ, and appears in both (A1) and (A2) through Γ. Inputs (A2) and (A2$^+$) then follow by linear algebra from three statements.

- The colength bound away from T_u, Theorem 42.4.1(iii); its excess over τ is bounded by a multiple of I.
- The slope of N_u differs from that of its Hecke image, twisted by the gains, by at most a multiple of X''. The error is the rounding of the exponents, and the normalization at T_u, at the level primes and at d. The formalization proves this from the divisibility branch (`linkMu_of_linkDIV`).
- At T_u: the sublattice M_u^H of M_u at level T_u satisfies, exactly,

  $$\mu(M_u^H) = \mu(M_u) - \frac{f \log(T_u - 1)}{T_u - 2}, \qquad \mathrm{rk}\, M_u^H = \Big(1 - \frac{1}{T_u - 1}\Big)^f \mathrm{rk}\, M_u,$$

  and $\mu_{\max}(M_u^H) \le \mu_{\max}(M_u)$ (`adm26`). The places over T_u cost at most $cN_T m(a/k + b') \log T_u$ per unit of rank. We have read this statement, not its proof.

Problem 59.2.1. If $T \ge 3$ and $d' = (1 - 1/(T-1))^f d$, then $0 \le 1 - d'/d \le f/(T-1)$.

Problem 59.2.2. For every integer $T \geq 3$, $\dfrac{f \log(T-1)}{T-2} \leq \dfrac{2f \log T}{T}$.

So passing to level T_u lowers the slope by at most $2f \log T_u/T_u$ and the rank by a proportion at most $f/(T_u - 1)$; both are small, since T_u is large by (A6).

59.3. The exponents

The record writes the powers of x in two of its errors with exponents $e_L\varepsilon$ and $e_D\varepsilon$: in (A1) on $N_T mb$, and in (A4) on $m(B_0 + N_c)$. The formalization proves, for every translate, the bound of Theorem 58.2.1: the error is at most $x^{(e_D+e_L+3)\varepsilon}\mathcal{B}$ for large x. We have read the statement of the step that collects the errors into $\mathcal{B}$, not its proof. It then applies Theorem 58.3.1, which it proves with the constants of [Zh, Lemma 6.1] (`proposition_size_bound`), with $C = e_D + e_L + 3$.

The formalization works with a level: for a primitive character ψ of order o, the primitive characters ψ' inducing the elements of the group generated by ψ. Its function is

$$G_m(y) := G(y) + \frac{1}{3o+1} \sum_{\psi' \neq 1} \operatorname{Re} G_{\psi'}(y),$$

and for $\psi = \chi_8$, $o = 2$, this is the function $G + \frac{1}{7}G_{\chi_8}$ of Chapter 58.

Problem 59.3.1. For $n \leq y$, the coefficient of $\Lambda(n)(y/n - 1)$ in $G_m(y)$ is real and at least $2(o+1)/(3o+1) \geq \frac{2}{3}$.

After averaging over the subgroup A_0, the characters not trivial on A_0 contribute zero, by orthogonality, and $4o$ characters remain. Those that induce the level give G_m, and the formalization bounds $G_m(x)$ by $x^{1/2+(C+3)\varepsilon}$ plus the sum over the others, which Estimate B bounds by $4o\, x^{1/2+\varepsilon}$.

Problem 59.3.2. Let $C \geq 0$, $o \geq 1$, $x \geq 1$, $\varepsilon > 0$ and $1 + 4o \leq x^{\varepsilon}$. Then $x^{1/2+(C+3)\varepsilon} + 4o\, x^{1/2+\varepsilon} \leq x^{1/2+(C+4)\varepsilon}$.

With $C = e_D + e_L + 3$, $e_D = 2$ and $e_L = 0$, this is the formalization's bound

$$G_m(x) \leq x^{1/2+(e_D+e_L+7)\varepsilon} = x^{1/2+9\varepsilon}$$

for every large x in the set of scales of the family; this is `ChebyshevInequalityOnWindows`.

59.4. The costs of the averaging

The formalization does not use Problem 4.4.2. It moves the thick weights to b with a Lipschitz bound in the weight, and states the costs as condition (V3) of Chapter 58 (`CostWindows`).

Problem 59.4.1. For $A > 0$ and $w, w' > 0$, $\left|wG(A/w) - w'G(A/w')\right| \leq 3A|w - w'|/\min(w, w')$. The same holds with G_χ in place of G, for χ primitive and non-principal.

With Problem 4.4.1, the formalization bounds the cost of one pair (t, j) by $2f(x+1)b\vartheta + 6f(x+1)(\Delta+1)$, which (V3) makes at most $2b/x$. It moves a_t/b to x with a bound for the derivative of G_χ that involves $|L'/L(1,\chi)|$, and takes $|L'/L(1,\chi)| \leq C_L \log x$ as a condition on the data at x. In the thin-scale bound, $\vartheta = \exp(-c\, x^{\varepsilon/8})$.

59.5. The choices of m, k and n

The normalization needs only the inequalities (S). Where do the sizes $k \approx x^{9/2}$ and $n \approx x^2$ come from? Let $\delta_1 := x^{-\varepsilon/8}$ as in Chapter 58.

Problem 59.5.1. A real number k with $xm/k \le 1$ and $k/n^2 \le x^{1/2-\varepsilon}$ exists if and only if $n^2 \ge m\,x^{1/2+\varepsilon}$. With $m \ge x^{1/2-\varepsilon/16}$ this forces $n \ge x^{1/2}$.

Since k is a dimension it must also be an integer; at the scale of the argument the interval $[xm, n^2x^{1/2-\varepsilon}]$ is long, so this costs nothing.

The lower bound for k comes from the rank hypothesis. The sources must have total rank at least x^{C_H} times the dimension of the target, each has rank at most C_0 times that dimension divided by $\Gamma_0^2 = \exp(4C_\Gamma m^5 ny)$, the constant of near orthogonality (Theorem 43.4.1), and there are at most 2^{k-3} of them; so $k \ge m^5 n$ for large x (Problem 43.4.5). This bound is needed by (A3), not by the normalization.

Problem 59.5.2. If $k \ge m^5 n$, $k \le n^2 x^{1/2}$ and $m \ge x^{1/2-\varepsilon/16}$, then $n \ge x^{2-5\varepsilon/16}$ and $k \ge x^{9/2-5\varepsilon/8}$.

The choices of [Adr, Prop. 5.15] and [Rub, §I.5] are $m \in [\sqrt{x\delta_1}, 2\sqrt{x\delta_1}]$, $k \in [x^{9/2+10\varepsilon}, 2x^{9/2+10\varepsilon}]$ and $n \in [x^{2+8\varepsilon}, 2x^{2+8\varepsilon}]$, with n a power of 2 times a bounded integer.

Problem 59.5.3. With these choices, x/m, xm/k, m and k/n^2 are at most $x^{1/2+\varepsilon/16}$, $2x^{-3-161\varepsilon/16}$, $2x^{1/2-\varepsilon/16}$ and $2x^{1/2-6\varepsilon}$; for $\varepsilon = \frac{1}{100}$ the exponents are 0.500625, -3.100625, 0.499375 and 0.44. Moreover (S) holds for large x if $\delta \ge x^{-\varepsilon}$, $N_T \ge \frac{1}{2}x^{-\varepsilon/8}n$ and $N_2 \le n$.

Problem 59.5.4. With these choices, $4C_\Gamma m^5 ny \le 256\,C_\Gamma\, x^{9/2+139\varepsilon/16}$. Since $\frac{139}{16} < 10$, the inequality of Problem 43.4.5 holds for large x.

In the formalization these are the choices made when the family is built (`RankVal`).

59.6. The order of choices

The formalization proves the following, for each primitive character ψ (proved by `topDesWin`). There are e_D, e_L and $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1)$ there is a family of data whose set of scales contains $[Y, Y^{3/2}]$ for arbitrarily large Y, and constants and x_0, such that the record of Section 59.1 exists at every $x \ge x_0$ in that set. It takes $e_D = 2$ and $e_L = 0$. In the proof the constants are chosen before the family, which is large relative to them. The record at x ties together one member of the family for each translate. By contrast, [Adr, Hyp. 18.10] asks for the inputs at every admissible datum.

Problem 59.6.1. Let $c_0 \in \mathbb{R}$, $\delta_0 > 0$ and $\varepsilon_1 > 0$. Then $\varepsilon := \min\big(\varepsilon_1/2, \delta_0/(|c_0|+1)\big)$ satisfies $0 < \varepsilon < \varepsilon_1$ and $c_0\varepsilon \le \delta_0$.

Problem 59.6.2. Fix $e_D, e_L \ge 0$ and $\varepsilon_1 > 0$, and suppose that for every $\varepsilon \in (0, \varepsilon_1)$ there are arbitrarily large Y with $G(x) \le x^{1/2+(e_D+e_L+7)\varepsilon}$ for all $x \in [Y, Y^{3/2}]$. Let $\operatorname{Re} s > \frac{1}{2}$ and $\theta := \frac{1}{2} + \frac{1}{2}\min(\operatorname{Re} s - \frac{1}{2}, \frac{1}{2})$. Then there are arbitrarily large Y with $G(x) \le x^\theta$ on $[Y, Y^{3/2}]$, and $\theta < \operatorname{Re} s$; so $\zeta(s) \ne 0$.

This is the formalization's last step (`level_of_tiedWin`), for G_m and Theorem 1.5.1 in place of G and Theorem 1.2.3. The choice of ε depends on s only through $e_D + e_L + 7$, which is why e_D and e_L come before ε.

Hints

§59.1.

59.1.1: Expand φ and $\log\Omega$, and use $|\Phi| = f$, as in Problem 58.1.4.
59.1.2: In Problem 58.1.1 keep $e_2(u)$, and take the weighted mean.
59.1.3: Problem 59.1.2, the bound for I, and $\log T_u \le cmL$.
59.1.4: The derivative is $\log(1+t) - t/(1+t)$ with $t = Z/v$. Then compare with $v = U + b$.
59.1.5: Problem 59.1.4, $(1-\delta_F)^{-1} \le 2$ and $\sqrt{e_F} \le \sqrt{\overline{e}_F}$.
59.1.6: $\max_u(\mu_{\max}(N_u) - h_u) = 0$.

§59.2.

59.2.1: Bernoulli's inequality.
59.2.2: For $T \ge 4$, $T \le 2(T-2)$; check $T = 3$ directly.

§59.3.

59.3.1: There are $o - 1$ characters $\psi' \ne 1$, and $|\psi'(n)| \le 1$.
59.3.2: $x^{1/2+\varepsilon} \le x^{1/2+(C+3)\varepsilon}$.

§59.4.

59.4.1: The function $w \mapsto wG(A/w)$ is continuous, and away from the points A/n its derivative is $A/w - \sum_{n \le A/w} \Lambda(n)$.

§59.5.

59.5.2: Eliminate k.
59.5.4: $m \le 2x^{1/2-\varepsilon/16}$ and $n \le 2x^{2+8\varepsilon}$.

§59.6.

59.6.1: $c_0 \le |c_0|$.
59.6.2: Problem 59.6.1 with $c_0 = e_D + e_L + 7$, and Theorem 1.2.3.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

[Lean] The Lean community, *riemann-hypothesis*, a Lean formalization of Alpöge's argument, `https://github.com/leanprover-community/riemann-hypothesis`.

[Rub] *The Alpöge–Claude proof of the Riemann hypothesis explained for the Riemann zeta function*, written by AI models at the behest of M. O. Rubinstein, working draft, 21 November 2026, `https://www.math.uwaterloo.ca/~mrubinst/alpoge_claude_grh/zeta-paper.pdf`.

[Zh] R. Zhang, *A preliminary note on Alpöge's proposed proof of the generalized Riemann hypothesis*, preprint, 2026, hexagon:2611.00809.