ASSEMBLY AT ONE SCALE

Prerequisites. From Chapter 1: the functions G and G_χ of (1) and (3), Theorem 1.2.3, and Sections 1.4 and 1.5. From Chapter 2: Problem 2.1.7 and Estimate B (Section 2.2). From Chapter 4: the notation q, d, n, H, Φ, $P_{t,w}(a)$, $M_w(a)$, c_{χ_d}, $\delta = |T_2|/n$, $R_{q,d}$, Theorems 4.2.1 and 4.4.1, and Section 4.5. From Chapter 5: the bound (1), Theorem 5.2.1 and Problem 5.3.1. From Chapter 8: Theorem 8.4.2 and Section 8.5. Mertens' estimate $\sum_{n\le x}\Lambda(n)/n = \log x + O(1)$. From Chapters 36, 37, 43, 47 and 52: the deliverable stated at the end of each, which Sections 58.2 and 58.5 match with the inputs, and the problems cited below.

Chapter 8 balanced an error $x/m + m$ for a model chain whose inputs were given. What are the inputs in Alpöge's argument [Alp], what error must each achieve, and which of them pays for each term of the final bound? We state what the Lean formalization [Lean] proves. The expositions of Adriazola [Adr], Zhang [Zh, §§4 and 6] and Rubinstein [Rub, §§I.5–I.8] are cited as pointers; they agree with it except where noted. Chapter 59 records the formalization's accounting in full. Throughout, $0 < \varepsilon < \frac{1}{52}$, the range in which [Adr, Thms. 5.10 and 5.12] construct the auxiliary primes of Section 58.5; no problem uses this bound. Let

$$y := x^{\varepsilon}, \qquad b := e^{y}, \qquad a := xb.$$

We treat ζ, so that the averaging of Chapter 4 is over all of H; for $L(s,\psi)$ see Section 4.5 there and [Adr, §6].

58.1. The accounting

The data are: m lifts; a set $T \subset H$ of N_T tops with weights $a_t > 0$, and $A := \sum_t a_t$; nodes $w_j \colon \Phi \to (0,\infty)$ for $1 \le j \le m$; a finite set $\mathcal{W}$ of *words*, and for $u \in \mathcal{W}$ a source lattice N_u of rank d_u, a base lattice M_u, an auxiliary prime T_u, and a bound h_u for the height of the period map $B_u \colon N_u \to R^\vee$, where $R^\vee$ is the target. These objects are defined in Chapter 35. All lattices are hermitian lattices over the ring of integers of a coefficient field $L \supset E$ (over $\mathcal{O}_E$ in [Adr, Def. 18.9]), with the normalized slopes of Chapter 6. Let $S := \sum_{t,j} P_{t,w_j}(a_t)$, let Γ be the total gain of the lifts at the inert primes, net of the colength of the Hecke image, $\mathcal{L}$ their common local terms and E_u their local terms at the level and twisted places. Let $\varphi(t) := t\log t - t$, $v := \frac{1}{2}\,\mathrm{rd}_E$, $\log\Omega := \gamma + \log(\pi/v) - c_{\chi_d}$ (for $E = \mathbb{Q}(\mathrm{i})$, with χ_{-4} in place of χ_d, $v = 1$ and Ω is the Ω of Problem 12.3.7), let $g := \log(g_0 g_1)$ for two level primes g_0, g_1, and write $\langle X\rangle := \sum_u d_u X_u / \sum_u d_u$. The inputs are seven

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inequalities, for every $u \in \mathcal{W}$, with numbers $e_1, \dots, e_7 \geq 0$:

$$S \leq \Gamma + e_1, \tag{A1}$$

$$\mu(M_u) + \Gamma + \mathcal{L} - E_u - e_2 \leq \mu(N_u), \tag{A2}$$

$$\langle \mu(N) - h \rangle - e_3 \leq \mu(R^\vee), \tag{A3}$$

$$\mu(R^\vee) + h_u - \mathcal{L} \leq \textstyle\sum_t \big(m\varphi(a_t) - m a_t \log \Omega - a_t \log T_u \big) + e_4, \tag{A4}$$

$$E_u \leq mAg + e_5, \tag{A5}$$

$$\log T_u \geq \textstyle\sum_j \frac{1}{f} \sum_{\rho \in \Phi} \log\big(w_j(\rho) v/\pi \big) + mg - e_6, \tag{A6}$$

$$\mu(M_u) \geq -e_7, \tag{A7}$$

where $f := |\Phi| = 2n$. In the formalization Γ is a weighted sum, over the inert places, of the gains $\sum_j \mathrm{gain}_j(p)$ less the exponent $\tau(p)$ of the Hecke image (Theorem 42.4.1 and Chapter 41). So Γ already deducts a benchmark colength $\tau(p)$, with the gain set to zero at the primes that are dropped. The actual colength is bounded in terms of $\tau(p) + \mathrm{io}(p)$ (Theorem 41.4.1(b)); its excess over the benchmark is the number I of Chapter 59, which the formalization bounds as it bounds e_1, and which enters e_2.

Problem 58.1.1. $\Gamma \leq \mu(N_u) - \mathcal{L} + E_u + e_2 + e_7$ for every u.

Problem 58.1.2. $\Gamma \leq \mu(R^\vee) - \mathcal{L} + \langle h + E \rangle + e_2 + e_3 + e_7$.

Problem 58.1.3. Let Σ_u be the right side of (A4). Then $\Gamma \leq \langle \Sigma + E \rangle + e_2 + e_3 + e_7$: the heights cancel.

Problem 58.1.4.

$$\Sigma_u + E_u \leq \sum_j \sum_t a_t \big(\log a_t - 1 - \gamma + c_{\chi_d} - \langle \log w_j \rangle_\Phi \big) + e_4 + e_5 + A e_6,$$

where $\langle \cdot \rangle_\Phi$ is the mean over Φ: the level terms mAg cancel, and so do $\log v$ and $\log \pi$.

The sum in Problem 58.1.4 is $\sum_{t,j} M_{w_j}(a_t)$. Problems 58.1.1–58.1.4 and (A1) prove the following [Adr, Prop. 3.1]; with Theorem 4.2.1 it bounds a sum of the functions G_θ of Chapter 4.

Theorem 58.1.1 (accounting)**.** *Under* (A1)–(A7)*,*

$$\sum_{t \in T} \sum_{j=1}^{m} \big(P_{t,w_j}(a_t) - M_{w_j}(a_t) \big) \leq e := e_1 + e_2 + e_3 + e_4 + e_5 + A e_6 + e_7.$$

Three cancellations occur, as in Stepanov's count (Chapter 8): the heights between (A3) and (A4), the level terms between (A5) and (A6), and the linear terms. Euler's constant and c_{χ_d} enter M_w through the prime sum, and $\log \Omega$ through the period, by the Chowla–Selberg formula (Theorem 20.3.1); Problem 58.1.4 is where they meet.

58.2. The inputs and their errors

What must each e_i be? Put $N_2 := |T_2| = \delta n$ and

$$B_0 := 2nmby, \qquad N_c := \frac{N_2\, k\, b\, y^2}{n^2},$$

where k is the dimension of the first hermitian space of the chain [Zh, §6.1]. In the following, $C \geq 1$ and $C_H \geq 0$ depend on ε but not on x. The data satisfy $N_T \geq 1$, $1 \leq m \leq k \leq b$, $mb \leq a$ and $a \leq a_t \leq a + b$ for $t \in T$ [Adr, Def. 18.5, Lemma 18.15] (the bound $k \leq b$ holds once x is large, since k is at most a power of x and $b = e^y$), and $1 \leq |\mathcal{W}| \leq 2^{k-3}$ [Adr, Def. 18.9(a)]; so $a_t \leq 2a$ and $|a_t/b - x| \leq 1$. Let

$$U := 8Cy\big(N_T a(1 + m^2/k) + mB_0\big);$$

Problem 58.2.2 shows that it bounds the spread defined below. The errors are those of [Adr, Def. 18.9 and Prop. 18.14], and the earlier chapters of this part deliver them:

- (A1) and (A2), with $e_1 \leq CN_T mb$ and $e_2 \leq Cy\big(N_T mb + N_T a\, m^2/k + a\big)$: Theorems 37.5.1 and 37.5.2, with Problems 37.5.4 and 37.5.5;
- (A3), with $e_3 = \log|\mathcal{W}| + 8\log(1 + x^{C_H})(U + b) + 12b$: Theorem 43.5.1 and Problem 43.5.3;
- (A4), with $e_4 \leq Cy^2\big(N_T a + m(B_0 + N_c)\big)$: Theorem 47.2.2, where the term mN_c is the cost of the congruence defects, Theorem 52.4.5;
- (A5), with $e_5 \leq CmB_0$: Theorem 52.1.1 and Problem 52.1.3;
- (A6), with $e_6 \leq Cy(1 + m^2/k)$, and (A7), with $e_7 \leq CyN_T a(1 + m/k)$: Theorems 47.3.1 and 47.4.1, with Problem 47.4.2.

Section 58.7 records the status of each.

The formalization assumes more than (A1)–(A7). For all $u, u' \in \mathcal{W}$ it also has

$$\mu_{\max}(N_u) \leq \mu_{\max}(M_u) + \Gamma + \mathcal{L} - E_u + e_2^+, \qquad (\mathrm{A2}^+)$$

$$E_u \geq mAg - e_5, \qquad (\mathrm{A5}^-)$$

$$\mu_{\max}(M_u) \leq e_7^+, \qquad (\mathrm{A7}^+)$$

$$\big|(h_u + A\log T_u) - (h_{u'} + A\log T_{u'})\big| \leq e_H, \qquad (\mathrm{H})$$

$$|\log T_u - \log T_{u'}| \leq e_T, \qquad (\mathrm{A6}')$$

with $e_2^+ \leq CmB_0$, $e_7^+ \leq CyN_T a$, $e_H \leq Cy(N_T a + mB_0)$ and $e_T \leq C\log k$, and an upper bound for $\log T_u$ (Chapter 59). These are further clauses of the statements that give (A2), (A4), (A5), (A6) and (A7); $(\mathrm{A5}^-)$ is the other half of Theorem 52.1.1. With them the spread

$$u_F := \max_u\big(\mu_{\max}(N_u) - h_u\big) - \langle \mu(N) - h\rangle,$$

which enters (A3) through e_3, is bounded by a proved lemma; [Adr, Lemma 18.13] states the bound.

Problem 58.2.1. Assume (A2), (A5), (A7) and $(\mathrm{A2}^+)$–$(\mathrm{A6}')$, and $\mu(N_u) \leq \mu_{\max}(N_u)$ for every u. Then $0 \leq u_F \leq e_2 + e_2^+ + 2e_5 + e_7 + e_7^+ + e_H + Ae_T$.

Problem 58.2.2. With the errors of this section and $y \geq 1$, $u_F \leq U$.

Problem 58.2.3. If $\mathcal{W}$ is nonempty, $|\mathcal{W}| \leq 2^{k-3}$ and $k \leq b$, then $e_3 \leq 8(1 + C_H)\, y\,(U + b) + 13b$ for large x.

Problem 58.2.4. Let $\mathcal{B} := N_T a(1 + m^2/k) + m(B_0 + N_c)$, and suppose that the data satisfy the conditions stated at the start of this section and that $y \geq 1$. Then $e \leq C_1 y^2 \mathcal{B}$ for large x, with C_1 depending on C and C_H only.

Theorem 58.2.1. *Under the conditions on the data and the inputs* (A1)–(A7) *and* (A2$^+$)–(A6′) *with the errors of this section, for large* x, $\sum_{t,j}\big(P_{t,w_j}(a_t) - M_{w_j}(a_t)\big) \le x^{(e_D+e_L+3)\varepsilon}\,\mathcal{B}$, *where* $e_D := 2$ *and* $e_L := 0$.

This is the bound the formalization proves at each translate. The numbers e_D and e_L are the powers of y in the errors of (A4), on the term $m(B_0 + N_c)$, and of (A1); the formalization fixes them before ε (Section 58.5). Problem 58.2.4 gives more, $x^{3\varepsilon}\mathcal{B}$ for large x; we keep the formalization's exponent, so that the exponents below are the ones it proves. The bound is [Zh, (4.11)], [Adr, Thm. A], and [Rub, §I.5, (M)] before normalization. Reading Problem 58.2.4 backwards shows which input pays for which term of $\mathcal{B}$. The term $N_T a$ collects costs incurred once for each top, not once for each lift: the base lattice (e_7), the precision $\log(a\,\mathrm{rd}_E) \le 2y$ of the linear term of the norms (e_4), and the auxiliary prime (e_6). The term $N_T a\,m^2/k$ is the colength and size of the auxiliary prime (e_2, e_6). The term mB_0 collects the local losses of the m lifts (e_5, and e_4); each is of size B_0, which grows with m because the nodes of the jth lift grow with j, as the cost of the kth identity did in Chapter 8. The term mN_c comes only from the congruence defects, through e_4. The spread U and hence e_3 contain all but the last.

58.3. Normalization

The averaging of Chapter 4 is over translates. We write $\tau \in H$ for the translate index, called u there, since u indexes words here; so the nodes of the jth lift at the translate τ are $w_{\tau,j}\colon \Phi \to (0,\infty)$. Before averaging we need, for one x:

- (V1) for every $\tau \in H$, data as in Section 58.2 with the nodes $w_{\tau,1}, \dots, w_{\tau,m}$, and with the same m, k, n, T, T_2 and a_t for all τ, satisfying the conditions stated there, (A1)–(A7) and (A2$^+$)–(A6′) with the errors stated there, with the same C, C_H and threshold for "large x"; the words $\mathcal{W}$ and the lattices may depend on τ;
- (V2) for every $t \in T$ and every j, all hypotheses of Theorem 4.4.1 for $a = a_t$ and the family $(w_{\tau,j})_{\tau\in H}$: $|a_t/b - x| \le 1$, the thick and thin conditions (2) with numbers Δ and X, and the thin-scale bound $|G_\theta(y)| \le \vartheta y$ for all θ and $y \ge X$;
- (V3) $2f(x+1)\vartheta \le 1/x$ and $6f(x+1)(\Delta+1) \le b/x$.

(V3) is the form of the cost conditions in the formalization. We use Chebyshev's bound $\sum_{r\le y}\Lambda(r) \le 3y$ for $y \ge 0$ [RS].

Problem 58.3.1. Under (V3), the last two terms of Theorem 4.4.1 add up to at most $4/(\delta x)$.

Let $Q := N_T m b$: N_T tops, m lifts, weight b.

Problem 58.3.2. Under (V1)–(V3), for large x, $Q\,\frac{\delta}{2}\big(R_{q,d}(x) - \mathcal{K}(x)\big) \le x^{(e_D+e_L+3)\varepsilon}\,\mathcal{B} + 2Q/x$.

The factor $Q\delta/2$ counts the gain: N_T tops, m lifts, weight b, at a proportion $\delta/2$ of the places. What remains is to divide $\mathcal{B}$ by it. The formalization does not form the quotient. It bounds the four terms of $\mathcal{B}$ one at a time by Q times a power of x, and uses only the inequalities

$$\begin{gathered} a \le 2xb, \quad N_T \ge \tfrac{1}{2}x^{-\varepsilon/8}n, \quad x^{-\varepsilon} \le \delta, \quad N_2 \le n, \\ x^{1/2-\varepsilon} \le m \le x^{1/2+\varepsilon/2}, \quad xm \le k \le n^2 x^{1/2-\varepsilon}. \end{gathered} \tag{S}$$

In Problems 58.3.3–58.3.5 assume (S), $x \ge 1$, $\varepsilon > 0$, and let $Y := x^{1/2+13\varepsilon/8}$.

Problem 58.3.3. $N_T a \le 2YQ$ and $N_T a\, m^2/k \le 2Q$.

Problem 58.3.4. $mB_0 \le 4YQ$.

Problem 58.3.5. $mN_c \le 2YQ$. Hence $\mathcal{B} \le 10YQ$.

Theorem 58.3.1 (normalization)**.** *Let $x \ge 3$, $\varepsilon > 0$, $C \ge 0$ and $x^{3\varepsilon/8} \ge 26$. Let n, N_T, N_2, m be positive integers and $b > 0$, and assume* (S)*. If a real number G satisfies $Q\,\frac{\delta}{2}\big(G - 4x^{1/2+\varepsilon}\big) \le x^{C\varepsilon}\,\mathcal{B} + 2Q/x$, then $G \le x^{1/2+(C+3)\varepsilon}$.*

Problem 58.3.6. Under the hypotheses of Theorem 58.3.1, $\delta(G - 4x^{1/2+\varepsilon}) \le 22x^{C\varepsilon}Y$, and $G \le 26x^{1/2+(C+21/8)\varepsilon}$. This proves the theorem.

Theorem 58.3.1 is [Zh, Lemma 6.1], with the same constants, as the formalization proves it. Why the inequalities (S)? Dividing $\mathcal{B}$ by the gain shows which input pays for which term. This is how [Adr, §4.3] and [Rub, §I.5, (M), and §I.6, Thm. A] state the normalization; the arithmetic is correct, but it is not how the formalization argues.

Problem 58.3.7. $$\frac{\mathcal{B}}{N_T mb\delta/2} = \frac{2x}{\delta}\Big(\frac{1}{m} + \frac{m}{k}\Big) + \frac{4nmy}{N_T\delta} + \frac{2ky^2}{nN_T}.$$

Problem 58.3.8. If $\delta \ge x^{-\varepsilon}$ and $N_T \ge \frac{1}{2}x^{-\varepsilon/8}n$, the quotient of Problem 58.3.7 is at most $8x^{17\varepsilon/8}\big(x/m + xm/k + m + k/n^2\big)$.

So x/m is paid at the ends, xm/k at the auxiliary primes, m by the local losses of the steps, and k/n^2 by the congruence defects: the four bounds of Problems 58.3.3–58.3.5, in order. Adriazola writes the fourth term as $kx^{c\varepsilon}/(nN_T)$ [Adr, §4.3], the same up to $n/N_T \le 2x^{\varepsilon/8}$. Rubinstein [Rub, §I.8] describes x/m and m in the same way, and calls xm/k and k/n^2 the costs of the auxiliary parameters k and n; for the last term we follow [Adr, §§16–18]. The hypothesis $\delta \ge x^{-\varepsilon}$ is (1), which Behrend's set of columns satisfies (Theorem 5.2.1, and Problem 36.4.1 at the scale of the argument). The hypothesis $N_T \ge \frac{1}{2}x^{-\varepsilon/8}n$ holds with high probability for the random tops of density $\delta_1 = x^{-\varepsilon/8}$ (Problem 5.3.1, and Theorem 36.4.2(iv)).

Why must the constants in (A4) be exact, when a long chain absorbs so much else? Change one input and follow it.

Problem 58.3.9. Let every $a_t = xb$, and keep all inputs as they are except that the right side of (A4) carries an additional ηmA, $\eta > 0$, as it would if $\log\Omega$ were known only to within η. (i) The bound of Theorem 58.1.1 increases by ηmA, and the bound for $R_{q,d}(x) - \mathcal{K}(x)$ given by Problem 58.3.2 by $2\eta x/\delta$, whatever m is; this is at most $C\sqrt{x}$ exactly when $\eta \le \frac{1}{2}C\delta x^{-1/2}$. (ii) If instead the additional term is ηA, a cost paid once for each top rather than once for each lift, that bound increases by $2\eta x/(m\delta)$.

So a long chain divides a cost paid at its ends by m, but an error in the linear term is paid by every lift, and dividing by the m lifts that pay it gains nothing. Since $x^{-\varepsilon} \le \delta \le 1$, an error η in $\log\Omega$ is harmless if $\eta \le \frac{1}{2}Cx^{-1/2-\varepsilon}$, and only if $\eta \le \frac{1}{2}Cx^{-1/2}$. This is why $\log\Omega$ must cancel exactly against γ and c_{χ_d} in Problem 58.1.4, and why the Chowla–Selberg formula, not an estimate of the period, is used.

58.4. The exponents

The normalization needs of m, k and n only the inequalities (S). The formalization chooses them when it builds the family of data, as [Adr, Prop. 5.15] does. Why $m \approx x^{1/2}$, $k \approx x^{9/2}$ and $n \approx x^2$? The four terms of Problem 58.3.8 explain it. Put $\delta_1 := x^{-\varepsilon/8}$.

Problem 58.4.1. If $\sqrt{x\delta_1} \le m \le 2\sqrt{x\delta_1}$, then $x/m \le x^{1/2+\varepsilon/16}$ and $m \le 2x^{1/2-\varepsilon/16}$. If instead $m = x^{1/2+\kappa}$, then $x/m + m \ge x^{1/2+|\kappa|}$.

Thus m balances x/m against m. The rank hypothesis behind (A3) makes k at least about $m^5 n$ (Problem 43.4.5), and $k/n^2 \le x^{1/2}$ then makes n about x^2; Chapter 59 works this out, and checks that the choices of [Adr, Prop. 5.15], which the formalization makes, satisfy (S). The terms xm/k and k/n^2 never compete with the balance. It remains to bound the constant $\mathcal{K}(x)$ of Problem 58.3.2, which contains $|c_\chi|$ for $\chi = \chi_q, \chi_d, \chi_q\chi_d$, uniformly in q and d.

Problem 58.4.2. Let χ be a primitive nonprincipal character with $|G_\chi(x)| \le x$. Then $|c_\chi| \le 1 + \sum_{n \le x} \Lambda(n)/n$. Hence, if Estimate B ((W1) below) holds at x, then $\mathcal{K}(x) \le C'' \log x$ for $x \ge 3$, with C'' absolute.

In the formalization the bound $|L'/L(1,\chi)| \le C_L \log x$ is one of the conditions on the data at x, not derived from Estimate B as here. By Problems 2.1.7, 58.3.2 and 58.4.2, and Theorem 58.3.1 with $C = e_D + e_L + 3$:

Theorem 58.4.1 (Estimate A)**.** *Assume (V1)–(V3),* (S)*, and (W1) of Section 58.5. For large x, $G(x) \le x^{1/2+(e_D+e_L+6)\varepsilon}$.*

The formalization proves the bound $x^{1/2+(e_D+e_L+7)\varepsilon} = x^{1/2+9\varepsilon}$ for the positive combination of G and the $G_{\psi'}$ attached to a character ψ of order o (Section 58.5), one ε more. After averaging over the subgroup A_0, the characters not trivial on A_0 contribute zero, by orthogonality, and $4o$ characters remain. Those that induce the level give the positive combination; Estimate B bounds the others by $4o\, x^{1/2+\varepsilon}$ in total, and this is what the last ε pays for (Chapter 59).

If the local loss of the jth lift grew like j^2 instead of j, the term m would become m^2, and the balance $x/m + m^2$ would give only the exponent $\frac{2}{3}$ (Problem 8.5.4; [Alp, step 14]).

58.5. The window, and the zeros

Theorem 1.2.3 needs $G(x) \le x^\theta$ at every point of $[Y, Y^{3/2}]$, for arbitrarily large Y. What must hold uniformly as x runs through such a window? The form of [Adr, §18] is this. For every small ε there is $J = J(\varepsilon) \ge 2$ such that for arbitrarily large Y there are $Y_0 \le Y$ with $Y^2 \le Y_0^J$, and primes $q \equiv 3 \pmod 8$, $d \equiv 7 \pmod{16}$, $q \mid d+1$, chosen at the scale Y_0 with $\log q \le Y_0^{\varepsilon/5}$ and $\log d \le Y_0^{\varepsilon/5} \log^2 Y_0$, such that at every $x \in [Y, Y^{3/2}]$:

- (W1) Estimate B: $|G_\chi(x)| \le x^{1/2+\varepsilon}$ for $\chi = \chi_q, \chi_d, \chi_q\chi_d$;
- (W2) the thin-scale bound $|G_\theta(y')| \le y' \exp(-c_6 x^{\varepsilon/8})$, with $c_6 > 0$ independent of x, for the $4n$ characters θ of Chapter 4 and all $y' \ge \frac{1}{3}\exp(x^{\varepsilon/3})$;
- (W3) $\sum_{1 \le i < \lfloor x^{400} \rfloor} \sum_\ell h_{\ell,i}(E) \log \ell \le \exp(\frac{1}{2}x^\varepsilon)$ for the field E of Chapter 4 at x, where $h_{\ell,i}(E)$ is the length of $\mathrm{Sha}_\ell(E,i)$ as in Chapter 30 (there the twist is called e; here e is the error of Theorem 58.1.1);

(W4) data n, m, k, T, T_2, weights, nodes and lattices satisfying (V1)–(V3) of Section 58.3, with $X = \frac{1}{3}\exp(x^{\varepsilon/3})$ and $\vartheta = \exp(-c_6 x^{\varepsilon/8})$ as in (W2), and with (S); the constants C, C_H and the threshold for "large x" do not depend on x, Y, Y_0, q or d.

The data of (W4) depend on x; the pair q, d depends only on Y_0. Chapter 36 delivers (W1) (Problems 36.3.1 and 36.3.2) and (W2) (Problem 36.3.3). (W3) is Theorem 52.4.3, and it is what fixes the position of the window. (W4) is the deliverables of Chapters 37–52 at every x of the window, with the data of Theorem 36.4.2.

Problem 58.5.1. If $Y_0 \le Y$ and $Y^2 \le Y_0^J$, then $[Y, Y^{3/2}] \subset [Y_0, Y_0^J]$. For x in this window and $n \le 2x^{2+8\varepsilon}$, $\log(4nqd) \le 3x^{\varepsilon/5}\log^2 x$ once x is large, while the thin scale $\frac{1}{3}\exp(x^{\varepsilon/3})$ has logarithm $x^{\varepsilon/3} - \log 3$.

Problem 58.5.2. Under (W1)–(W4), $G(x) \le x^{1/2+(e_D+e_L+6)\varepsilon}$ for every x in the window, once Y is large.

Problem 58.5.3. Let $0 < c(\varepsilon) < \frac{1}{2}$ for small $\varepsilon > 0$, and suppose that for every small ε there are arbitrarily large Y with $G(x) \le x^{1/2+c(\varepsilon)}$ for all $x \in [Y, Y^{3/2}]$. Then $\zeta(s) \neq 0$ for $\operatorname{Re} s > \frac{1}{2} + \liminf_{\varepsilon\to 0} c(\varepsilon)$.

By Problem 58.5.2, $c(\varepsilon) = (e_D + e_L + 6)\varepsilon$ for large Y; the formalization's statement has $c(\varepsilon) = (e_D + e_L + 7)\varepsilon = 9\varepsilon$. Problem 58.5.3 then gives the Riemann hypothesis. This is why e_D and e_L must not depend on ε, while J, C, C_H and the threshold for x may. The formalization chooses in this order: first $e_D = 2$, $e_L = 0$ and a bound ε_1; then, for each $\varepsilon < \varepsilon_1$, a single family of data; then the constants of the record and a threshold x_0. In the proof a design threshold is fixed before the family, which is chosen large relative to it; the scalar constants of the record are found after the family. At every $x \ge x_0$ in the set of scales of that family it proves (W4). That set contains $[Y, Y^{3/2}]$ for arbitrarily large Y. Given s with $\operatorname{Re} s > \frac{1}{2}$, it takes ε with $(e_D + e_L + 7)\varepsilon \le \frac{1}{2}\min(\operatorname{Re} s - \frac{1}{2}, \frac{1}{2})$.

Theorem 58.5.1. *Assume (W1)–(W4). Then $\zeta(s) \neq 0$ for $\operatorname{Re} s > \frac{1}{2}$.*

For $L(s, \psi)$ one averages over $A_0 = \ker\psi$ (Section 4.5) and bounds the positive combination G^ψ of Section 1.4 on windows. The interval version is Theorem 1.5.1: a bound for G^ψ on windows excludes the zeros of ζ and of every $L(s, \psi_j)$, real or not; a real zero is excluded by Problem 1.5.6. Chapter 36 works out the averaging for $\psi = \chi_8$, and Section 58.6 closes the example. Everything in this chapter, from (W1)–(W4) to Theorem 58.5.1, is bookkeeping.

58.6. The example $\psi = \chi_8$

Take the example $\psi = \chi_8$ at the end of Chapter 36, with R^ψ and $G^\psi = G + \frac{1}{7}G_{\chi_8}$ as in Problem 36.5.2. This is the formalization's combination for a character of order $o = 2$, whose weights are $1/(3o+1)$. How does the argument exclude a zero of $L(s, \chi_8)$?

Problem 58.6.1. Suppose that for every small $\varepsilon > 0$ there are arbitrarily large Y such that, at every $x \in [Y, Y^{3/2}]$, $R^\psi(x) \le x^{1/2+c\varepsilon}$, with c independent of ε, and the six characters of Problem 36.5.3 satisfy Estimate B. (i) Then $\zeta(s) \neq 0$ and $L(s, \chi_8) \neq 0$ for $\operatorname{Re} s > \frac{1}{2}$. (ii) If $\zeta(s)^7 L(s, \chi_8)$ had a real zero $\beta > \frac{1}{2}$ of multiplicity μ, and no zero of larger real part, the transform of G^ψ would have

residue $\mu/(7\beta(1-\beta)) > 0$ at β. The argument of Problem 1.5.6, with this residue, then excludes β.

58.7. The status of the inputs

The Lean formalization [Lean] proves the final theorem. Its intermediate statements are organized differently from those of [Adr], and we have not checked that they are the hypotheses of [Adr] named below; where a difference is known, from a comparison of the statements, the ledger records it. Each input is *classical*, with a reference; *proved in* [Adr]; or *a hypothesis of* [Adr]: the formalization proves the final theorem, and the corresponding step in [Adr] is the hypothesis named, assumed there, with no published human-readable proof. Where a chapter of this part develops the proof that the formalization gives for an input, the ledger names it and the gaps it records; these accounts follow the formalization, and [Adr] is cited for comparison. The accounting, the normalization, the exponents and the window are proved in this chapter, on the models of Chapters 6, 8, 9 and 15. Of these, the normalization (Theorem 58.3.1), the per-translate bound (Theorem 58.2.1) and the order of choices are as the formalization proves them. The accounting from (A1)–(A7), with the seven errors of [Adr], is expository: the formalization assumes the eleven inequalities of its record (Chapter 59). The bound 8ε for ζ is our deduction; the formalization's statement is 9ε for the positive combination. For each input we name the deliverable that supplies it.

(A1), (A2): Theorems 37.5.1 and 37.5.2, for the lifts of Theorem 37.3.3. The formalization proves the final theorem by a different arrangement at this step (Chapter 42). Its counterpart of (A1) is an inequality between explicit numbers at each inert prime, Theorem 42.4.1, proved with the chain of lifts replaced by the zero chain. The divisibility of the lifts on the Hecke image is Theorem 42.2.1, proved on a separate branch from an input on the lifts, and the two meet only in the source lattices N_u, whose slopes give (A2). On the divisibility branch the formalization proves conditions (a)–(c) of Theorem 37.3.3: the normalized kernel law, for kernels built from one theta function of the CM curve normalized to take the value 1 at the origin (Chapter 38), and the unit scalars and the dual frames at the odd positions (Chapter 39). The Hecke image is controlled by a nondegeneracy condition at one edge (Chapter 40) and its colength bound (Chapter 41). These chapters give a human-readable account, with these gaps: the proofs of the nondegeneracy theorems for small primes and for primes of class I are large and not developed (Chapter 40); we have not followed the covering lemma behind the colength bound (Chapter 41), the proof of the input on the lifts, part (ii) of Theorem 42.4.1, or how the size conditions give the bound for the rejected primes (Chapter 42); nor the proof that the values at the origin entering the one-copy step are p-integral, the computation with coset averages for lattices standard at p, or the passage to the torus-averaged exponent (Chapter 38); and we have not matched the formalization's description of the lattices at the odd positions with the dual model weights of [Adr, §11.1], nor compared its scalars with those of [Adr, Def. 11.23] (Chapter 39). [Adr, Thms. 12.27 and 12.21] derive (A1) and (A2) from [Adr, Hyps. 11.22, 11.38, 11.42 and 12.12], assumed there; [Adr, Prop. 11.27] proves [Adr, Hyp. 11.22] for its own step normalization. The three engines of [Adr, Hyp. 12.12] are not separate statements in the formalization. In the formalization Γ is net of the colength of the Hecke image, and (A2) and (A2$^+$) come from the source lattices by an exact computation (Chapter 59). The sublattice of M_u at level T_u has slope $\mu(M_u) - f\log(T_u-1)/(T_u-2)$ and rank

$(1-1/(T_u-1))^f$ times that of M_u. The slope of N_u differs from that of the Hecke image twisted by the gains by at most a multiple of X''; the formalization proves this from the divisibility branch.

(A3)**:** Theorem 43.5.1. The bound for the spread is Problem 58.2.1, from the further inputs, as in the formalization; [Adr, Lemma 18.13] states it. The frame theorem and its application to the period maps are proved in [Adr, Thms. 13.8 and 13.15]. The Gram conditions, [Adr, Thms. 14.1–14.3], are derived there from hypotheses of [Adr, §14], assumed there. The formalization does not state [Adr, Hyp. 14.10], a period map with a seesaw relation; it states and proves the Bessel identity for the period maps directly (Theorem 43.4.3). Chapters 44, 45 and 46 give a human-readable account: Bessel–Poincaré series and their Gram operators, the seesaw as a transport of linear relations, and the identity. The gaps they record: we have not followed the proofs of the decay statement and of the four bounds behind the off-diagonal estimate, which use the Siegel–Weil formula in its convergent range, nor checked that the terms of the expansion of the Gram operator are the double cosets of the model (Chapter 44); we have not traced how the constructed family is shown to satisfy the laws of the seesaw package (Chapter 45); and we have not traced how the classification of discrete series used for the multiplicity statement is supplied, nor read the proof of the unfolding step (Chapter 46).

(A4)**,** (A6)**,** (A7)**:** Theorems 47.2.2, 47.3.1 and 47.4.1. The CM period is classical [CS] (Theorem 20.3.1), with a correction proved in [Adr, Props. 15.3 and 15.6]; the mass formula is classical [GHY]; (A7) is proved in [Adr, Thm. 15.33]. For the height and target bounds of (A4), the corresponding steps in [Adr] are [Adr, Hyps. 15.10–15.12, 15.16 and 15.18], assumed there. The formalization proves the covolume estimate of Theorem 47.2.2 for the lattices of forms of the lifts, and Chapters 49, 50 and 51 give a human-readable account: reduction to a lower slope bound for holomorphic forms by linear algebra and descent, the period through the flat line, and the slope bound on covers by an effective arithmetic Hilbert–Samuel inequality. The gaps they record: we have not traced how the lattices of forms correspond to the spaces of the theta correspondence (Chapter 49), nor the anchor bound for the flat line (Chapter 50); we have not read the proof for the last lift, the tower-and-volume argument, or below resolution of singularities and the Poincaré–Lelong formula, and we have not checked the normalization of the height (Chapter 51); and of the step that turns the bounds for the lattices of the lifts into the bound for the target we have read the statements only. For (A6), Theorem 47.3.1 states the input in the form the formalization proves: the rank of the source and the dimension budget are given exactly by the mass formula, and the auxiliary prime is bounded below through critical levels. Chapter 48 gives a human-readable account, with these gaps: we have read the statements of the mass ratio and of the two counts in the formalization but not their proofs, we have not checked the Vandermonde count at a top by hand, and the formalization uses the lower bound for $\log T_u$ in the accounting, as in Problem 58.1.4; we have not traced how it constructs a prime in its window. The corresponding step in [Adr] is the inequality [Adr, Hyp. 15.23], assumed there. See also the remark below.

(A5)**, and** mN_c **in** (A4)**:** Theorems 52.1.1 and 52.4.5. Theorem 52.1.1 states (A5) and (A5$^-$) in the form the formalization proves, with the local exponents given by formulas. Chapter 53 gives a human-readable account, with these gaps: we have not traced how the family the formalization constructs is shown to satisfy the size conditions, we have not re-derived the row exponents and the Fock degree, we have not followed the congruence behind the bound for the primes of class I whose Frobenius label $h_p \neq 0$ has odd order, and we have not traced how the orbit losses enter the gain inequality at the inert primes. The corresponding

steps in [Adr] are [Adr, Hyps. 16.13 and 16.14], assumed there, from which [Adr, Thm. 16.6] derives (A5). The congruence defects, Theorem 52.4.5, are derived in [Adr, Thm. 17.37] from [Adr, Hyps. 17.24 and 17.36] and (W3), assumed there.

(V1)–(V3), (W1), (W2): Chapter 36. (V1) is the inputs of Section 58.2 for every translate; the rest is proved in [Adr, §§4–6] from classical theorems [Gal], [Pa], [Dav, Ch. 14]. (V3) is the formalization's form of the cost conditions, and the normalization is Theorem 58.3.1, [Zh, Lemma 6.1] as the formalization proves it.

(W3): Theorem 52.4.3, from the Selmer bound on windows, Theorem 52.4.1. Over $\mathbb{Q}$ the odd twists are classical [MW]; the bound for the even twists is not known even over $\mathbb{Q}$ (Section 30.2). The Galois representations are classical [HT], [Shi], [CH], [Car1]. Over E, the corresponding step in [Adr] is [Adr, Hyp. 17.11], assumed there; as [Adr, §17.4] notes, it implies more towards Vandiver's conjecture than any published result. The formalization proves the Selmer bound on windows, and Chapter 57 outlines the proof (Theorem 57.1.1): the minus part is bounded at every scale by a one-sided Euler-system argument with Gauss sums, without the main conjecture (Chapter 55); the plus part by congruences, case by case, given a bound on fixed layers at finitely many bad primes (Chapter 56); and a window of scales is chosen on which those layers and the prime 2 contribute little, by Iwasawa theory along cyclotomic towers (Chapter 54). The gaps they record: we have not read the proofs of the bound for H^2 and of the Ferrero–Washington step, nor identified the Iwasawa module behind the exceptional twists (Chapter 54); we have not transcribed the condition at ℓ in the Stickelberger bound or read its proof (Chapter 55); we have not read the treatment of the bad primes, the cases that use forms on GSp_4, or how Kato's bound is established (Chapter 56); and we have not followed how the constants are made independent of q and d, nor the passage to ranges of scales with exponent $3/2$ (Chapter 57).

(W4): The deliverables of Chapters 37–52 at every x of the window, with constants independent of x, Y, Y_0, q, d. The corresponding step in [Adr] is [Adr, Hyp. 18.10], assumed there for every admissible datum; even given the hypotheses of [Adr, §§11–17], [Adr, §18.3] does not derive it. The formalization proves a weaker statement, which is what the window argument of this chapter uses (Section 42.5 of Chapter 42): for each primitive character and every small ε there is one family of data, and x_0, such that at every $x \geq x_0$ in its window set a chain record exists; the record at x ties together one member of the family for each translate, sharing the frame, m, f, q, d and the top weights, with $e_D = 2$ and $e_L = 0$ fixed before ε, a design threshold fixed before the family, and the scalar constants of the record found after it (Section 58.5).

The public sources and the formalization differ on where the arithmetic geometry enters; see the remark at the end of Chapter 47.

Hints

§58.1.

58.1.1: Use (A7) in (A2).

58.1.2: Take the weighted mean of Problem 58.1.1 over u, and write $\mu(N_u) = (\mu(N_u) - h_u) + h_u$.

58.1.3: The weighted mean of (A4).

58.1.4: Use (A5), and (A6) multiplied by $A \geq 0$. Then use the definition of $\log \Omega$.

§58.2.

58.2.1: For the upper bound compare a word u with each word u', using (A2$^+$) and (A7$^+$) for u and (A2) and (A7) for u'; then take the weighted mean over u'.
58.2.2: $N_T mb \le N_T a$, $a \le N_T a$, $m \le k \le b$, $A \le 2N_T a$ and $\log k \le y$.
58.2.3: Problem 43.5.1 and $k \le b$; and $\log(1 + x^{C_H}) \le (1 + C_H)\log x \le (1 + C_H)y$.
58.2.4: Bound each e_i separately, using $b \le a \le N_T a$, $A \le 2N_T a$ and Problem 58.2.3.

§58.3.

58.3.1: $8n = 4f$, and $f \ge 2$.
58.3.2: Theorem 4.4.1 for each pair (t, j), Theorem 58.2.1 for each τ, and Problem 58.3.1; average over the $N_T m$ pairs.
58.3.3: $x \le x^{1/2+\varepsilon} m$, and $m^2/k \le m/x$.
58.3.4: $n \le 2x^{\varepsilon/8} N_T$.
58.3.5: $N_2 \le n$ and $k \le n^2 x^{1/2-\varepsilon}$.
58.3.6: Problem 58.3.5, $4/x \le 2$ and $1/\delta \le x^{\varepsilon}$.
58.3.7: $N_2 = \delta n$ and $a = xb$.
58.3.8: $n/N_T \le 2x^{\varepsilon/8}$ and $1/\delta \le x^{\varepsilon}$.
58.3.9: Follow the extra term through Problem 58.1.3 and Theorem 58.1.1, then divide by $Q\delta/2$ as in Problem 58.3.2; here $A = N_T xb$.

§58.4.

58.4.1: Compare each of x/m and m with $x^{1/2}$.
58.4.2: Solve (3) for c_χ. For $\mathcal{K}(x)$ use Mertens' estimate.

§58.5.

58.5.1: $Y^{3/2} \le Y^2$ and $Y_0 \le x$.
58.5.2: Theorem 58.4.1 at each x; by (W4) its threshold does not depend on x.
58.5.3: Theorem 1.2.3 for each ε.

§58.6.

58.6.1: Problem 36.5.3 and Theorem 1.5.1, as in Problem 58.5.3; for (ii), compute the residue as in Problem 1.5.2, and note that Problem 1.5.3 uses only its sign.

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