LOCAL TERMS, CONGRUENCE DEFECTS AND SELMER WINDOWS

Prerequisites. Chapter 6 reads a local norm from a lattice exponent (Problem 6.5.1). We use the eigenspaces of cyclotomic class groups from Chapter 29, Theorems 29.2.1 and 29.2.2, and the strict Selmer groups over $\mathbb{Q}$ from Chapter 30, Theorems 30.1.1 and 30.1.2. Chapter 31, Problem 31.2.2, computes a projection of the weight-twelve lattice. Theorems 32.2.1 and 33.4.1 construct a residual extension and bound a cotangent length, respectively. We also recall the local correspondence and the Galois representations of regular algebraic polarized cusp forms (Theorems 34.1.1 and 34.2.1).

Why does a split prime contribute exactly half its logarithm before one sums the local exponents? And what happens when projecting onto a Hecke eigenspace introduces a denominator? The first calculation gives a main term for each lift. To bound the denominator of the projection, the argument uses the orders of strict Selmer groups over a varying CM field.

52.1. The two level primes

Fix $0 < \varepsilon < 1/52$ and a large scale x. Write $y = x^{\varepsilon}$, $b = e^{y}$. There are n columns, $f = 2n$ elements in the CM type Φ of the host field E, and m lifts. Write $B_0 := fmby$ and let $A = \sum_t a_t$ be the total weight of the tops. For a finite place $w \mid p$ of a Galois coefficient field $L \supset E$, use the integer valuation v_w, ramification index e_w, residue degree f_w, and weight $\ell_w = f_w \log p/[L:\mathbb{Q}]$. Thus $\sum_{w|p} \ell_w e_w = \log p$.

The two distinct level primes $g_\iota = \pi_\iota \bar{\pi}_\iota$, $\iota = 0, 1$, split in the imaginary quadratic subfield K of E. Set $g = \log(g_0 g_1)$. Let u index a chain of lifts. At its j-th lift the exponents $n_{u,j}(\rho)$ have sum $n_{u,j}$, and the polynomial model has degree Δ_j and shift $\kappa_j \geq 0$. Let D_Γ be a positive common multiple of the stabilizer orders in the lifts. At $w \mid g_\iota$ the local exponent is [Adr, Def. 16.5]

$$E_{u,j}(w) = v_{g_\iota}(D_\Gamma) - \tfrac{1}{2}\Delta_j + \kappa_j + e_w^{-1} \sum_{\rho \in \Phi} n_{u,j}(\rho) v_w(\tau_\rho \bar{\pi}_\iota),$$

where τ_ρ extends ρ to L. Write $T_u = \varpi_u \bar{\varpi}_u$ for the terminal prime, distinct from g_0, g_1, d. At $w \mid T_u$ the exponent is $\varepsilon_u^T \geq 0$ when $v_w(\tau_\rho \varpi_u) > 0$ for every $\rho \in \Phi$, and zero otherwise. At the remaining primes it is $v_p(D_\Gamma)$, with an additional error ε_j^d at d or $\varepsilon_{j,p}^{\mathrm{in}}$ at the remaining primes in the inert loss set P^{in}. Write $E_{u,j} := \sum_w 2\ell_w e_w E_{u,j}(w)$ and $E_u = \sum_{j=1}^{m} E_{u,j}$.

Problem 52.1.1. For $K = \mathbb{Q}(\mathrm{i})$ and $5 = (2+\mathrm{i})(2-\mathrm{i})$, the places above 5 contribute 0 and $\frac{1}{2}\log 5$ to the weighted valuation of $2-\mathrm{i}$ in K. In general $\sum_{w|g_\iota} \ell_w v_w(\tau_\rho \bar{\pi}_\iota) = \frac{1}{2}\log g_\iota$, independently of L and ρ.

Problem 52.1.2. The contribution of g_ι to $E_{u,j}$ is $(2v_{g_\iota}(D_\Gamma) - \Delta_j + 2\kappa_j + n_{u,j}) \log g_\iota$.

Date: November 25, 2026.

Theorem 52.1.1 (local terms)**.** *Let the exponents $E_{u,j}(w)$ at the places other than g_0, g_1 be given by the formulas of Section 53.1 of Chapter 53, and let the data satisfy the size conditions (S1)–(S5) there and Lemmas 53.2.1, 53.3.1 and 53.4.1. There are C and x_0 such that $|E_{u,j} - Ag| \le CB_0$ for every datum at a scale $x \ge x_0$, every u and every $1 \le j \le m$.*

Simplest incarnation: the local exponent of a pairing in Problem 6.5.1. *Status*: proved in the Lean formalization [Lean], for the family it constructs; Chapter 53 states the formulas and the lemmas, and outlines the proofs (Theorem 53.5.1). The formalization proves the final theorem; the corresponding step in [Adr] is [Adr, Thm. 16.6, Cor. 16.7], derived there for data satisfying the conditions (E1)–(E6) of [Adr, §16.2]. These require $|n_{u,j} - \Delta_j - A| \le \delta_j$ and bound

$$(\delta_j + 2\kappa_j)g, \quad 2\varepsilon_u^T \log T_u, \quad 2\log D_\Gamma, \quad 2|\varepsilon_j^d| \log d, \quad \sum_{p \in P^{\mathrm{in}}} 2|\varepsilon_{j,p}^{\mathrm{in}}| \log p$$

by constant multiples of B_0. The size estimates and the bound for inert losses having an explicit formula are proved in [Adr, Lemmas 16.10–16.12]; the remaining orbit losses and the degrees and sizes for the constructed lifts are [Adr, Hyps. 16.13–16.14], assumed there. In the formalization the numbers κ_j, ε_u^T, ε_j^d and $\varepsilon_{j,p}^{\mathrm{in}}$ are given by formulas, and the orbit losses are part of $\varepsilon_{j,p}^{\mathrm{in}}$.

Problem 52.1.3. Summation gives $|E_u - mAg| \le e_5 = CmB_0$, hence (A5), $E_u \le mAg + e_5$. With n, x fixed this allowance is proportional to m^2.

52.2. The denominator of a projection

At a place w where the local type of the lift differs from the standard type, let $O = \mathcal{O}_{L_w}$ and $\ell = p_w$. For each maximal ideal $\mathfrak{m}$ of the Hecke algebra acting on integral forms, let $\Lambda_{\mathfrak{m}}$ be the localized lattice. Its rational span splits into the eigenspaces V_G occurring in the lift and the complementary eigenspaces V_B. Let π_G project along V_B. The congruence defect of [Adr, Def. 17.1, Prop. 17.30] is

$$\eta_j(w) = \frac{1}{e_w} \sum_{\mathfrak{m}} \mathrm{length}_O \frac{\pi_G \Lambda_{\mathfrak{m}}}{\Lambda_{\mathfrak{m}} \cap V_G}.$$

Here $\mathrm{length}_O(O/\ell O) = e_w$. At a place of standard type put $\eta_j(w) = 0$.

Problem 52.2.1. In the classical model take $\Lambda = M_{12}(\mathbb{Z}) \otimes \mathbb{Z}_{691}$, $V_G = \mathbb{Q}_{691}\Delta$ and $V_B = \mathbb{Q}_{691}E_{12}$. The defect is 1, and its weighted contribution $\ell_w e_w \eta$ over $\mathbb{Q}$ is $\log 691$.

Theorem 52.2.1 (Fitting inequality, [Adr, Prop. 17.27])**.** *Let O be a complete discrete valuation ring with fraction field K_w, and $\mathbb{T} \subset \prod_{z \in X} O$ an O-algebra on a finite set X. Let δ_{z_0} be the least valuation of a nonzero entry at z_0 of an element supported there, or infinity if there is none. Let P be an a'-by-s matrix over an O-algebra R', $a' \ge s$, with representations ρ_z over O. Assume (F1)–(F4). For every s-by-s matrix Q over R', the function $z \mapsto \det \rho_z(Q)$ belongs to $\mathbb{T}$. Each $\rho_z(P)$ for $z \ne z_0$ has nonzero rational kernel. The representation ρ_{z_0} is block upper triangular, with diagonal blocks ρ_i of sizes d_i whose joint reduction maps onto the product of the full matrix algebras. The residue field has size $> s\sum_i d_i$. Then*

$$\delta_{z_0} \le \sum_i \mathrm{length}_O \ker\big(\rho_i(P) : (K_w/O)^{sd_i} \longrightarrow (K_w/O)^{a' d_i}\big).$$

Simplest incarnation: the two eigenvalues in the example following Theorem 33.4.1. *Status*: proved in [Adr].

Problem 52.2.2. Let $O = \mathbb{Z}_{691}$ and $R' = \mathbb{T} = \{(v_1, v_2) \in O^2 : v_1 - v_2 \in 691O\}$, with its two coordinate representations. For the one-entry matrix $P = ((691, 0))$ and z_0 the first coordinate, both sides of Theorem 52.2.1 are 1. The graded cotangent module in the weight-twelve example following Theorem 33.4.1 thus has the same length.

52.3. Strict Selmer groups and the two signs

For a number field F use $W_i = (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(-i)$ with cyclotomic action to the power $-i$. The group $\mathrm{Sha}_\ell(F, i)$ consists of continuous $H^1(F, W_i)$ classes whose restrictions to inertia at every finite place and to the decomposition group at every place over ℓ vanish. For a finite ℓ-group U put $\lambda(U) = \mathrm{length}_{\mathbb{Z}_\ell}(U) \log \ell = \log |U|$. The sum used in the window [Adr, Def. 17.2, Hyp. 17.11] is

$$V(F, t) = \sum_{\ell} \sum_{1 \le i < t} \lambda(\mathrm{Sha}_\ell(F, i)).$$

Theorem 52.3.1 (restriction and class groups, [Adr, Props. 17.7–17.8])**.** *Let ℓ be odd, $i \ge 1$, and $(\ell - 1) \nmid i$. For $E \subset \mathbb{Q}(\mu_N)$, $\ell \nmid N$, restriction embeds* $\mathrm{Sha}_\ell(\mathbb{Q}, i)$ *into* $\mathrm{Sha}_\ell(E, i)$ *with Galois-invariant image. Let A_ℓ be the ℓ-part of the class group of $\mathbb{Q}(\mu_\ell)$, and ω the Teichmüller character. The dimension of* $\mathrm{Sha}_\ell(\mathbb{Q}, i)[\ell]$ *is the ℓ-rank of $A_\ell^{(\omega^{-i})}$.*

Simplest incarnation: Theorem 30.1.1 identifies the two sides by duality. *Status*: classical.

Problem 52.3.1. In the Artin description of Theorem 30.1.1, an equivariant homomorphism $\phi : A_\ell \to W_i[\ell]$ satisfies $\phi(\sigma z) = \omega(\sigma)^{-i}\phi(z)$. After choosing a basis of $W_i[\ell]$, its possible values give precisely $\mathrm{Hom}(A_\ell^{(\omega^{-i})}, \mathbb{F}_\ell)$. For even i the character is even.

Problem 52.3.2. For positive odd i, Theorem 30.1.2 gives $\sum_{\ell \text{ odd}} \lambda(\mathrm{Sha}_\ell(\mathbb{Q}, i)) \le \log |\operatorname{num}(B_{i+1}/(i+1))| \le (i+1)\log(i+1) + 3$. For $t = \lfloor x^{401} \rfloor$ the sum over odd $1 \le i < t$ is $O(t^2 \log(2t)) = o(\sqrt{b})$.

For comparison, statement (V) of [Adr, §17.4] asserts the existence of constants C, P_0 with the following property. For every even $i \ge 2$ and finite set S of primes $\ell \ge 5$ with $(\ell - 1) \nmid 2i$, the groups below are finite and

$$\sum_{\ell \in S} \lambda(\mathrm{Sha}_\ell(\mathbb{Q}, i)) \le C(i+1)^{P_0}.$$

Problem 52.3.3. Assuming (V), $A_\ell^{(\omega^{-i})} = 0$ once $\log \ell > C(i+1)^{P_0}$ for these i, ℓ [Adr, Cor. 17.15]. The threshold is polynomial in i for the logarithm of the prime.

Kurihara proved vanishing at $i = 2$. For $i+1 = \nu$ odd, Soulé's sufficient threshold for $\log \ell$ is $\nu^{224\nu^4}$. The estimates of de la Torre Durán give $\exp(O(\nu^2 \log \nu))$ in the same comparison [Adr, §17.4, Rem. 17.16], [Kur, So2, DT]. The odd-twist estimate in Problem 52.3.2 concerns $\mathbb{Q}$: over $\mathbb{Q}$ the odd twists are classical (Theorem 30.1.2), while the bound for the even twists is not known even over $\mathbb{Q}$. The window requires both signs over E, and includes $\ell = 2$.

Problem 52.3.4. In the nonsplit residual extension of Theorem 32.2.1, write the diagonal characters as χ_1, χ_2 and the upper entry as $b_{12}(\sigma)$. Then $b_{12}(\sigma)/\chi_2(\sigma)$ represents a nonzero class with coefficients in $\chi_1\chi_2^{-1}$. For the absolute Galois group of F, if this coefficient line is $W_i[\ell]$, the image of the class in $H^1(F, W_i)$ belongs to $\mathrm{Sha}_\ell(F, i)$ precisely when that image satisfies the two local conditions.

In (G1) of [Adr, §17.4] a Selmer class gives a tangent vector at the Eisenstein point. Passing from such vectors to an automorphic count also requires (G3) and (G4). The comparison there with Allen–Newton–Thorne uses the distinction between individually conjugate self-dual constituents and a pair exchanged by conjugate duality. Newton–Thorne's adjoint vanishing requires an enormous image. Kato's vanishing applies at the reducible modular points.

52.4. The window and the congruence bound

Fix a primitive character ψ of conductor M and order o, as in Chapter 36, whose letters we keep: e is the exponent of the 2-power tower. For an integer $e \geq 0$ set $N = \mathrm{lcm}(2^{e+2}, M, q, d)$. Write $E(\psi, e, q, d)$ for the subfield of $\mathbb{Q}(\mu_N)$ fixed by classes $a \equiv \pm 1 \bmod 2^{e+2}$ with $\psi(a) = 1$ and $(\frac{a}{q}) = (\frac{a}{d}) = 1$. A window pair at Y_0 consists of primes $q \equiv 3 \bmod 8$, $d \equiv 7 \bmod 16$ with $z = Y_0^{\varepsilon/5}$,

$$z/2 < \log q \leq z, \qquad \tfrac{1}{2} z \log^2 Y_0 < \log d \leq z \log^2 Y_0,$$

such that q and every odd prime $p \leq z$ divide $d + 1$ [Adr, Defs. 17.9–17.10].

Theorem 52.4.1 (Selmer window)**.** *Write $\alpha = 2 + 8\varepsilon$. For each ψ, ε and integer-valued function $e \geq 0$ with $x^\alpha \leq 2^{e(x)} \leq 2x^\alpha$ for $x \geq 1$, there are $J_1 \geq 1, X_0$ such that for every $X \geq X_0$ and every window pair at $Y_0 = X^{1/J_1}$ there is $W \in [Y_0, \sqrt{X}]$ with the following property. For every $W \leq x \leq W^2$, with $E = E(\psi, e(x), q, d)$ and $t = \lfloor x^{401} \rfloor$, all the groups defining $V(E, t)$ are finite and $V(E, t) \leq \sqrt{b}$.*

Simplest incarnation: the odd twists over $\mathbb{Q}$ in Problem 30.2.4. *Status*: proved in the Lean formalization [Lean], as Theorem 57.1.1 of Chapter 57, which outlines the proof: the minus part is bounded at every scale (Chapter 55), the plus part given a bound on fixed layers at finitely many bad primes (Chapter 56), and a window of scales is chosen on which those layers and the prime 2 contribute little (Chapter 54). The corresponding step in [Adr] is [Adr, Hyp. 17.11], assumed there. As [Adr, §17.4] notes, the statement implies more towards Vandiver's conjecture than any published result.

Theorem 52.4.2 (the two signs, [Adr, Thm. 17.12])**.** *For odd ℓ let $P_\pm$ sum λ of the $c = \pm(-1)^i$ eigenspaces over these primes and $1 \leq i < t$, with c complex conjugation. With $P_2 := \sum_{1 \leq i < t} \mathrm{length}_{\mathbb{Z}_2} \mathrm{Sha}_2(E, i)$, the two idempotents $(1 \pm c)/2$ at odd primes give $V(E, t) = (\log 2) P_2 + P_+ + P_-$. Write $F_E = \mathrm{cond}(E)$ and $d_E = [E : \mathbb{Q}]$. The following two estimates, including finiteness, imply the conclusion of Theorem 52.4.1.*

(I) *Fix ψ, ε, and functions e, t with $e(x) \geq 2$, $2^{e(x)} \leq x^4$, $2 \leq t(x) \leq x^{401}$ for $x \geq 2$. There are $J_1, C, P_0', X_1 > 0$ such that for $X \geq X_1$ and primes q, d in the indicated residue classes, prime to M, with*

$$q, d > \max(P_0', 2X^{401} + 1), \qquad \log(qd) \leq X^{\varepsilon/(4J_1)},$$

there is $W \in [X^{1/J_1}, \sqrt{X}]$ *such that for* $W \le x \le W^2$,

$$P_2 \le e^{x^{4\varepsilon/5}}, \qquad P_+ \le (F_E t)^C + C e^{x^{4\varepsilon/5}}.$$

Here $E = E(\psi, e(x), q, d)$ *and* $t = t(x)$.

(II) *For all such fields and* $t \ge 2$, *with an absolute constant* C_-,

$$P_- \le 3 d_E t^2 \log(t F_E) + C_- \sum_{\substack{\ell \mid o \\ \ell \text{ odd}}} \ell d_E^2 t^2 \log^2(t \ell F_E).$$

Simplest incarnation: the minus eigenspaces of Theorem 29.2.2. *Status*: proved in [Adr] as a reduction. For the restriction of a rational class, c acts trivially, so even twists occur in P_+ and odd twists in P_-. The estimates (I) and (II) are inputs of this reduction, not proved in [Adr]. The formalization proves (II) as Theorem 55.2.3, at every scale, by a one-sided Euler-system bound with Gauss sums in the form of Kolyvagin and Rubin; the main conjecture is not used. It proves the plus estimate of (I) as Theorem 56.4.1, given a bound on fixed layers that holds on the windows of Chapter 57. The generic plus estimate and (V) both use the character estimate (CorGK), for $K = \mathbb{Q}(\sqrt{-3})$ in the deduction of (V) [Adr, Props. 17.14, 17.19].

Problem 52.4.1. For e as in Theorem 52.4.1, $x \ge Y_0$ and $t = \lfloor x^{401} \rfloor$, the inequalities for the window pair give $\log(F_E t) \le 2x^{\varepsilon/4}$ for all sufficiently large x. Consequently each fixed power of $F_E t$ and $e^{x^{4\varepsilon/5}}$ is $o(\sqrt{b})$.

Problem 52.4.2. Fix ψ, ε, e as in Theorem 52.4.1 and assume its conclusion. Suppose window pairs exist at X^{1/J_1} for unbounded X. Then for arbitrarily large x the sum of $\lambda(\mathrm{Sha}_\ell(\mathbb{Q}, i))$ over odd $\ell \nmid M q_x d_x$ and $1 \le i < \lfloor x^{401} \rfloor$ with $(\ell - 1) \nmid i$ is at most $\sqrt{b}$. The two excluded window primes may vary with x [Adr, Prop. 17.22].

What uniform estimate would an order formula for these groups have to supply? Burns–Greither and Huber–Kings treat equivariant special values for abelian fields and Dirichlet characters [BuG, HK].

Problem 52.4.3. Suppose that for a given E, t every $\mathrm{Sha}_\ell(E, i)$, $1 \le i < t$, has order $\ell^{v_\ell(a_{\ell,i})}$, with nonzero $a_{\ell,i} \in \mathbb{Z}_\ell$. Then the numerical condition in Theorem 52.4.1 is exactly $\sum_\ell \sum_{1 \le i < t} v_\ell(a_{\ell,i}) \log \ell \le \sqrt{b}$ at each scale in its window. Finiteness of the individual groups alone gives no such upper bound.

Theorem 52.4.3 (rescaled window, [Adr, Props. 18.3–18.4])**.** *Assume the conclusion of Theorem 52.4.1. In the construction* $n = c_* 2^{e(x)} \in [x^\alpha, 2x^\alpha]$ *with* $c_* \mid o$ *eventually constant. There is* $J \ge 2$ *and there are arbitrarily large* Y *with base* $Y_0 \le Y$, $Y^2 \le Y_0^J$ *and a window pair at* Y_0 *such that throughout* $Y \le x \le Y^{3/2}$,

$$V(E, \lfloor x^{400} \rfloor) \le \sqrt{b}, \qquad E = E(\psi, e(x), q, d).$$

All groups in this sum are finite. With a sufficiently large base the same bound holds for all integer twists $1 \le i < x^{400}$, *as in (W3).*

Simplest incarnation: the single-field sum in Problem 30.2.4. *Status*: derived in [Adr, Props. 18.3–18.4] from [Adr, Hyp. 17.11], the statement of Theorem 52.4.1, assumed there; with Theorem 52.4.1 as the formalization proves it, the derivation is Problem 52.4.4. The formalization makes a similar passage to ranges of scales with exponent 3/2 (Chapter 57). The prime selections in [Adr, Thms. 5.10, 5.12, Prop. 18.4] supply pairs at arbitrarily large bases.

Problem 52.4.4. Write $\theta = c_*^{1/\alpha}$ and apply Theorem 52.4.1 with exponent $e(\theta s)$ for sufficiently large s. For $Y = \theta W$ and $Y_0 \geq \max(3, 2\theta^{400})$, one has $Y^2 \leq Y_0^{J_1+1}$. If $Y \leq x \leq Y^{3/2}$, then $s = x/\theta \in [W, W^2]$ and $2x^{400} \leq s^{401}$. Thus every integer $i < x^{400}$ lies below $\lfloor s^{401} \rfloor$.

Let k be the rank parameter for the lifts, T_2 the distinguished set of columns, and $N_c = |T_2|kby^2/n^2$. Let t_r count the copies of the auxiliary space added at each lift. The ranks in the construction give $t_r \leq x^{400} - 1$ [Adr, Lemma 18.2]. Write d_j^0 for the rank of lifted forms of standard type at step j.

Theorem 52.4.4 (congruences at the twisted places)**.** *There are constants C_N, x_0 such that for every admissible datum at $x \geq x_0$ and every j, if the groups in $V(E, t_r)$ are finite and the sum converges, then*

$$\sum_{w \text{ twisted}} \ell_w e_w \eta_j(w) \leq d_j^0 \{C_N(yB_0 + (\log x)^2 N_c) + V(E, t_r)\}.$$

Simplest incarnation: the projection denominator in Problem 31.2.2. Theorem 31.4.1 gives the classical cuspidal comparison. *Status*: in [Adr] this is [Adr, Hyp. 17.24], assumed there, and [Adr] outlines a proof as follows. The formalization proves the final theorem; the corresponding step in [Adr] is this hypothesis. The lattice quotient is a sum of projection defects, bounded using the congruence exponent of each eigensystem with its multiplicity [Adr, Props. 17.25, 17.30]. After removing eigensystems outside the chosen local types, the per-component inequalities and their weighted sum yield the congruence bound, under the multiplicity and size bounds in [Adr, Prop. 17.34, Hyp. 17.36]. The Galois representations of Theorem 34.2.1, Poitou–Tate duality and Berger's potential semistability theorem enter the comparison in [Adr, §17.5]. The estimate for lifting crystalline classes with Wach modules, the other local estimates and the integral comparison with theta series are among the steps that [Adr, §18.7] lists as not proved there.

Problem 52.4.5. Assume the conclusion of Theorem 52.4.4, (W3), $d_j^0 \leq D_0$ and $(\log x)^2 \leq y$. Then $\sum_w \ell_w e_w \eta_j(w) \leq (C_N + 1)D_0 y(B_0 + N_c)$ [Adr, Thm. 17.37]. Dividing by the positive rank bound D_0 and summing m lifts puts mN_c, with the common factor y, in the height allowance. The formula for e_5 in Problem 52.1.3 uses only B_0.

Theorem 52.4.5 (cost of the congruence defects, [Adr, Thm. 17.37])**.** *On an admissible window satisfying (W3) and the conclusion of Theorem 52.4.4, if $d_j^0 \leq D_0$ for all j and $(\log x)^2 \leq y$, each lift has weighted congruence defect at most $(C_N + 1)D_0 y(B_0 + N_c)$.*

Simplest incarnation: the single summand at 691 in Problem 31.2.2. *Status*: derived in [Adr, Thm. 17.37] from (W3) and [Adr, Hyp. 17.24], the statement of Theorem 52.4.4, assumed there; the step from these is Problem 52.4.5.
Deliverable. For the chain record of [Adr, Def. 18.9, Prop. 18.14], Theorem 52.1.1 gives (K5), and Problem 52.1.3 gives (A5). Theorem 52.4.5 supplies the congruence contribution mN_c in the error of (K4), hence of (A4), under the component and rank bounds. Theorem 52.4.3 supplies the independent window input (W3) from Theorem 52.4.1, whose proof in the formalization Chapter 57 outlines. In [Adr] these rest on [Adr, Hyps. 16.13, 16.14, 17.11, 17.24 and 17.36], assumed there; the formalization proves the final theorem, and Chapter 53 develops its proof of Theorem 52.1.1 (Section 58.7 of Chapter 58).

Hints

§52.1.

52.1.1: Use the norm from L to $\mathbb{Q}$ of an element of K.

52.1.2: Apply the two sums over $w \mid g_\iota$ to the two parts of the local exponent.

52.1.3: Keep B_0 in the sum until its last occurrence.

§52.2.

52.2.1: Localize the two lattices in Problem 31.2.2.

52.2.2: Find the elements supported on the first coordinate. Use (K_w/O)[691] on the other side.

§52.3.

52.3.1: Apply equivariance to each character summand of $A_\ell/\ell A_\ell$.

52.3.2: Use Problems 30.2.4 and 30.2.5, then compare logarithms.

52.3.3: Apply (V) to a singleton prime.

52.3.4: Multiply two upper triangular matrices and then change the basis of the extension.

§52.4.

52.4.1: Use $F_E \le N \le 2^{e+2}Mqd$ and $\varepsilon/4 - \varepsilon/5 > 0$.

52.4.2: Take $x = W$ and use Theorem 52.3.1 prime by prime.

52.4.3: Test the size condition on a single cyclic group.

52.4.4: Compare θ and θ^{400} with the lower endpoint of the window.

52.4.5: Compare $\sqrt{b}$ with B_0 before summing over j.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

[BuG] D. Burns and C. Greither, *On the equivariant Tamagawa number conjecture for Tate motives*, Invent. Math. **153** (2003), 303–359.

[DT] A. de la Torre Durán, *Perfect forms and torsion bounds in algebraic K-theory over totally real and CM fields*, arXiv:2611.06300.

[HK] A. Huber and G. Kings, *Bloch–Kato conjecture and main conjecture of Iwasawa theory for Dirichlet characters*, arXiv:math/0101071.

[Kur] M. Kurihara, *Some remarks on conjectures about cyclotomic fields and K-groups of $\mathbb{Z}$*, Compositio Math. **81** (1992), 223–236.

[Lean] The Lean community, *riemann-hypothesis*, a Lean formalization of Alpöge's argument, `https://github.com/leanprover-community/riemann-hypothesis`.

[So2] C. Soulé, *Perfect forms and the Vandiver conjecture*, J. Reine Angew. Math. **517** (1999), 209–221.