HEIGHTS, THE CM PERIOD AND THE AUXILIARY PRIME

Prerequisites. The period calculation in Chapters 12 and 20, the mass and the squared norm of a lift in Chapters 21 and 23, the degrees of lattices of sections in Chapters 26, 27 and 28, the Siegel–Weil formula of Chapter 22, the exceptional zero in Chapter 9, and the nodes of Chapter 36. We use squared operator norms and slopes equal to twice the arithmetic degree divided by the field degree and the rank [Adr, §15.1]. Theorem 47.3.1 quotes Theorem 48.4.1 of the later Chapter 48, which proves it from the counts; the remarks on the formalization refer to Chapters 49, 50, 51 and 58 and are not used in the problems. The theorems are quoted from [Adr, §15], which records their status; every definition and bound that a problem uses is stated here.

47.1. The period in a squared norm

Which linear term accompanies $a\log a - a$ in the height of a theta lift? For a complex lattice L write $\mathcal{A}(L) = \operatorname{area}(L)|\Delta(L)|^{1/6}$, where $\Delta = g_2^3 - 27g_3^2$. This quantity is invariant under homothety. Let $d \equiv 7 \pmod{16}$ be prime, $K = \mathbb{Q}(\sqrt{-d})$, and χ_d the primitive character of discriminant $-d$. Let $E \supset K$ be a CM field whose relative different is $\mathfrak{b}\bar{\mathfrak{b}}$, and let Φ consist of the $f = [E : K]$ embeddings extending a fixed complex embedding of K. Write $v = \mathrm{rd}_E/2$, where $\mathrm{rd}_E = |\operatorname{disc} E|^{1/(2f)}$, and $c_{\chi_d} = L'(1,\chi_d)/L(1,\chi_d)$.

Theorem 47.1.1 (CM period)**.** *Let L_C represent each of the h_K ideal classes of K as a complex lattice. At each $\tau \in \Phi$, the normalized arithmetic degree $\log \Omega_\tau$ of the coordinate line of periods $\mathcal{L}_\tau$ over a field of definition F means $2\widehat{\deg}(\mathcal{L}_\tau)/[F:\mathbb{Q}]$* [Adr, Def. 15.5]*. Its value is*

$$\log \Omega_\tau = \frac{1}{h_K}\sum_C \log\frac{\mathcal{A}(L_C)}{\pi} - \log\frac{\mathrm{rd}_E}{\sqrt{d}} = \gamma + \log\frac{\pi}{v} - c_{\chi_d} =: \log\Omega.$$

Simplest incarnation: Problem 20.3.5 computes the class mean by Kronecker's limit formula (Theorem 20.1.1). *Status*: the class mean is classical, by Kronecker's limit formula; the relative-different correction is proved in [Adr, Props. 15.3 and 15.6].

Problem 47.1.1. For $E = K$ the correction is zero and $\log\Omega = \gamma + \log(2\pi/\sqrt{d}) - c_{\chi_d}$. Using $\mathcal{A}(\mathbb{Z}+\mathbb{Z}\tau) = 4\pi^2\operatorname{Im}(\tau)|\eta(\tau)|^4$ and the Gamma product of Theorem 20.3.1 gives $\log\Omega = -\frac{1}{2}\log d + \frac{w_K}{2h_K}\sum_{j=1}^{d-1}\chi_d(j)\log\Gamma(j/d)$, where w_K counts the roots of unity in K.

Problem 47.1.2. In the Gaussian example, $\Omega_{\mathrm{i}} = 4\pi|\eta(\mathrm{i})|^4 = \Gamma(1/4)^4/(4\pi^2)$ (Problem 12.3.7). Thus $\log\Omega_{\mathrm{i}} = 1.4763\ldots$, whereas the bound for a single Gaussian lift in Theorem 23.3.1 is $\log\|\Theta_{(a)}\|^2 \le a\log a - a - a\log(2\pi) + O(\log a)$. Multiplying its squared norm by $(2\pi/\Omega_{\mathrm{i}})^a$ changes the last linear term to $-a\log\Omega_{\mathrm{i}}$.

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47.2. The height and the target slope

Fix one translate and its nodes throughout. For large x and fixed $0 < \varepsilon < 1/52$ write $y = x^{\varepsilon}$, $b = e^{y}$, $a = xb$, $\Lambda_a = \log(a \operatorname{rd}_E)$ and $\varphi(z) = z \log z - z$, with $\varphi(0) = 0$. The chain has m lifts, initial rank k, node weights $w_{j\rho} > 0$ ($1 \le j \le m$, $\rho \in \Phi$), and N_T distinguished embeddings (the tops $t \in T$) with weights a_t and sum $A = \sum_t a_t$. A word u specifies a level prime T_u, a source lattice N_u and a period map $B_u = P_u K_m \cdots K_1 : N_u \to R^{\vee}$, where P_u is the final pairing and R is the top lattice. Over the field of definition, $\operatorname{ht}(B_u)$ is the mean of $\log \|B_u\|^2$ over the archimedean places.

Theorem 47.2.1 (Height of the composite)**.** *For a fixed word u, let $s_j = s_j(u)$ bound the mean log squared norm of K_j. In the frames of the chain the specified bound is*

$$s_j = \sum_{\rho,i} \big(\varphi(\nu_{j\rho i}) - \nu_{j\rho i} \log \Omega\big) + C_0 f N^2 (\Lambda_a + m^3 y),$$

where $C_0 \ge 1$ is independent of the word and $\nu_{j\rho i}$ are the N row weights at ρ. Write $h_{j\rho}$ for the log hook constant of its frame, $e^{\max}_{\lambda,j}$ for its error allowance, $e^{\max}_P$ for the allowance in $\log \|P_u\|^2 \le -A \log T_u + e^{\max}_P$, and $\langle \log \kappa \rangle$ for the mean log scalar of the target metric. Use the allowances of [Adr, Def. 15.14]*. Under the kernel, pairing and frame bounds of* [Adr, Hyps. 15.10–15.12]*, for $B_u \ne 0$,*

$$\operatorname{ht}(B_u) \le h_u := -A \log T_u + e^{\max}_P - \langle \log \kappa \rangle + \sum_j \big(s_j - \sum_{\rho} h_{j\rho} + e^{\max}_{\lambda,j}\big).$$

Simplest incarnation: the squared norm in Theorem 23.3.1 has one factorial term. *Status*: derived in [Adr, Prop. 15.9, Cor. 15.15] from the kernel, pairing and frame bounds for the chain, [Adr, Hyps. 15.10–15.12], assumed there. The formalization [Lean] proves the final theorem; the corresponding step in [Adr] is these hypotheses, assumed there.

Problem 47.2.1. Let u, u' be words. Suppose that in h_u and $h_{u'}$ all terms other than $-A \log T_u$ and the summands $\varphi(\nu) - \nu \log \Omega$ of the s_j agree, that for each j and ρ at most k of these summands differ, and that the two weights of each differing pair lie in $[1, M]$ and differ by at most S. Then $|(h_u + A \log T_u) - (h_{u'} + A \log T_{u'})| \le mfkS(\log M + |\log \Omega|)$. If moreover $kS \le b$ and $\log M + |\log \Omega| \le (14 + 6/\varepsilon)y$, this is at most $(20 + 6/\varepsilon)(N_T a \Lambda_a + m B_0)$, where $B_0 = fmby$.

For the heights of its selected record, [Adr, §18.7, The height at one index] derives the hypotheses of Problem 47.2.1, with $M = 6x^6 b$. This gives the variation bound of [Adr, Prop. 18.11(vi)] with $C_{\mathrm{var}} = 20 + 6/\varepsilon$, and hence (K6). Applying it to the heights of Theorem 47.2.1 requires identifying their row sums, hook factors and mean of $\log \kappa$ with those of the record.

The target slope is bounded through the covolumes of lattices M_j of forms on unitary groups of signatures $(p_j, q_j) = (k + j - 1, m - j)$. Let $D = \operatorname{rk} R$, $n = f/2$, N_2 the size of T_2, the set indexing the thick weights, $B_0 = fmby$ and $N_c = N_2 kby^2/n^2$. Write $\mathcal{L}$ for the sum of the normalized local lengths at the twisted places, as in [Adr, §15.4].

Theorem 47.2.2 (Covolumes and the height inequality)**.** *The covolume estimate on the chain, with* $\mathrm{lcv} = -2\widehat{\deg}/[F:\mathbb{Q}]$ *over the field of definition* F*, is*

$$\frac{\mathrm{lcv}(M_j)}{D} \le \sum_{\rho}(p_j - q_j)w_{j\rho}\log\Omega + CB_0\log\log x.$$

For sufficiently small ε *and each admissible family there are* C, x_0 *for which this holds at every* $x \ge x_0$*, uniformly in the point, lift and level. After passage between levels and the local comparisons, the target bound of* [Adr, Hyp. 15.16] *reads*

$$\mu(R^\vee) - \langle \log\kappa\rangle \le \sum_{j,\rho}(p_j - q_j)w_{j\rho}\log\Omega + \mathcal{L} + 1 + C_0 b + C_0 my(B_0 + N_c).$$

With Theorem 47.2.1, the kernel weight relations and the size conditions of [Adr, Hyp. 15.18, Thm. 15.19]*, this gives (A4):*

$$\mu(R^\vee) + h_u - \mathcal{L} \le m\sum_t \big(\varphi(a_t) - a_t\log\Omega\big) - A\log T_u + e_4,$$
$$e_4 = (50C_0 + 10)\big(N_T a\Lambda_a + my(B_0 + N_c)\big).$$

Simplest incarnation: for theta sections on an elliptic curve J with a unit-discriminant model, Theorem 28.1.2 gives the slope $-h_F(J) + \frac{1}{2}\log(k/(2\pi^2))$ in weight k, with h_F the Faltings height. *Status*: derived in [Adr, Thms. 15.19 and 15.70] from [Adr, Hyps. 15.10–15.12, 15.16, 15.18, 15.58–15.59 and 15.69], assumed there. The formalization proves the final theorem; the corresponding step in [Adr] is these hypotheses, assumed there.

The tops are embeddings, $T \subseteq \Phi$, and we write $a_\rho = a_t$ at the top $\rho = t$. The kernel weight relations [Adr, Hyp. 15.18] say that the row weights of s_j at ρ have sum

$$\sum_{i=1}^{N}\nu_{j\rho i} = \mathbf{1}_T(\rho)a_\rho + kw_{j\rho} + \delta_{j\rho},$$

where $\mathbf{1}_T(\rho)$ is 1 at a top and 0 otherwise, and $\delta_{j\rho}$ is an error term bounded there in terms of the sizes of the chain.

Problem 47.2.2. In $\sum_j s_j$ plus the covolume terms $\sum_{j,\rho}(p_j - q_j)w_{j\rho}\log\Omega$ of Theorem 47.2.2, the coefficient of $w_{j\rho}\log\Omega$ is $2j - 1 - m$, of absolute value at most $m - 1$, and the top terms over all lifts contribute $-mA\log\Omega$. The remaining terms $-\delta_{j\rho}\log\Omega$ are at most $|\delta_{j\rho}|\Lambda_a$ in absolute value when $|\log\Omega| \le \Lambda_a$.

The remaining term $\sum_j(2j-1-m)w_{j\rho}\log\Omega$ of Problem 47.2.2 is not small term by term. Since $\sum_{j=1}^{m}(2j - 1 - m) = 0$, it equals $\sum_j(2j - 1 - m)(w_{j\rho} - b)\log\Omega$, and it is absorbed in e_4 only because the nodes at the thick cells lie within Δ of b (Theorem 36.4.2(ii)); at the other cells they are at most $2b'$.

Problem 47.2.3. If $\Lambda_a \le 2y$ as in [Adr, Lemma 18.15(iii)] and $y \ge 1$, then $e_4 \le (100C_0 + 20)y\big(N_T a + m(B_0 + N_c)\big)$, and hence $e_4 \le (100C_0 + 20)y^2\big(N_T a + m(B_0 + N_c)\big)$, the form used in Chapter 58.

Problem 47.2.4. On the fixed projective line, Theorem 26.1.1 gives $\deg H^0(\mathcal{O}(D)) = D^2/4 + O(D\log D)$. For the two points cut out by $F(X,Y) = X(X - HY)$, H a positive integer, Theorem 27.2.1 gives $\deg\Lambda_D = \frac{D}{2}\log(1 + H^2) - \log H$ for $D \ge 1$, where Λ_D is the lattice of sections on these points with the evaluation metric. The constant term is unbounded in absolute value as H varies.

For the chain, [Adr, Thms. 15.46 and 15.51] reduce the norm estimates to the normalized kernel, theta-null mean and class counts [Adr, Hyps. 15.42–15.44], and to the branching norm [Adr, Hyp. 15.50] together with factorization of the final pairing, its evaluation bound and the stated weight and size conditions. Siegel–Weil expresses a genus average by a product of local densities in Theorem 22.2.1. With the doubling method it gives the Rallis inner product formula in the convergent range. For the selected record, [Adr, §15.13] specifies the Petersson forms and bounds the local and L-value corrections. Applying [Adr, Prop. 15.52] to the period chain requires identifying the row and lift parts and their forms on each active type, and comparing the frame weights with the hook products; items (i) and (ii) of that proposition for the lifts of the chain are assumed there. For the target slope, [Adr, Thm. 15.70] uses passage, the base-side comparison and local estimates. The passage uses near-constant multiplicities [Adr, Hyp. 15.56]. The covolume deduction [Adr, Thm. 15.66] assumes (I1)–(I17) and a uniform mean slope on deep covers [Adr, Hyp. 15.69]. Arithmetic Hilbert–Samuel on a fixed model (Theorem 26.2.1) gives asymptotics there. The component estimates needed to pass to a mean slope are spelled out in [Adr, Lemma 15.62]. The kernel weight relations remain a separate hypothesis [Adr, Hyp. 15.18].

47.3. Counting forms and choosing the prime

How large a prime can the final pairing use? Let $m \geq 2$ be even, $r = k + m - 1$ and $g = \log(g_0 g_1)$ for the two level primes g_0, g_1. For a shifted highest weight z of length s let $V_s(z) = \prod_{i<i'}(z_i - z_{i'})$ and $D_s(z) = V_s(z)/\prod_{i=0}^{s-1} i!$. For the genus of the standard rank-s hermitian lattice write $\mathrm{mass}(s)$ for the sum of reciprocal stabilizer orders. Let I_Π, I_Σ be the source and target level indices at g_0, g_1. For the column and row parameters $z_{u\rho}^{\mathrm{col}}, z_\rho^{\mathrm{row}}$ define

$$d_u^* = \mathrm{mass}(k) I_\Pi \prod_\rho D_k(z_{u\rho}^{\mathrm{col}}), \qquad X_u = (D^*/d_u^*)^{1/(fk)},$$

where D^* is the target dimension budget. The exponent fk comes from the source dimension $d_u^*(T_u^k - 1)^f$ for trivial stabilizers.

Theorem 47.3.1 (The dimension budget and the level)**.** *Suppose that the rank of the source lattice and the dimension budget are given exactly by the mass formula,*

$$\mathrm{rk}\, N_u = d_u^*\,(T_u^k - 1)^f, \qquad D^* = \mathrm{mass}(r)\, I_\Sigma \prod_\rho D_r(z_\rho^{\mathrm{row}}),$$

and that the auxiliary prime is chosen so that $C_0^{-1}\Gamma_0^{-2} D^* \leq \mathrm{rk}\, N_u \leq C_0 \Gamma_0^{-2} D^*$, with $C_0 \geq 1$ and the frame factor $\Gamma_0 \geq 1$ of the mean slope inequality satisfying $\log \Gamma_0 \leq k/8$. Let the row and column parameters, the nodes $b_{\rho j}$ and the level primes satisfy the conditions of Theorem 48.4.1 and Problem 48.4.2, with the constant C_u there. Then

$$\log T_u \geq \sum_{j=1}^{m} \frac{1}{f} \sum_{\rho \in \Phi} \log \frac{b_{\rho j} v}{\pi} + mg - e_6, \qquad e_6 = (3 + C_u) y + \log 5 + \frac{1}{f}\Big(\frac{\log(2C_0)}{k} + \frac{1}{4}\Big).$$

Simplest incarnation: Theorem 21.4.1 cancels the Weyl factorials in the Gaussian mass, leaving $V_s(z)\pi^{-s(s+1)/2}$ times the local factors. *Status*: in the Lean formalization [Lean] the two counts are proved from the mass formula and Harish-Chandra's formal degrees, and the lower bound for T_u follows from the lemmas it proves on

critical levels. Chapter 48 develops that proof; Theorem 48.4.1 is our combination of those lemmas, and we have not traced where the formalization uses it. The formalization proves the final theorem; the corresponding step in [Adr] is the dimension budget [Adr, Hyp. 15.23], assumed there, from which [Adr, Prop. 15.25, Thm. 15.26] derive (A6). The mass and formal degree must use the same measures [Adr, §18.7].

In [Adr] the budget is the inequality

$$D^* \geq (2/3)^f \,\mathrm{mass}(r) I_\Sigma \prod_\rho D_r(z_\rho^{\mathrm{row}}),$$

and (A6) is stated with the nodes $w_{j\rho}$: under the row and column interlacing, mass-ratio and size conditions of [Adr, Prop. 15.25], and for $T_u \geq Z_u := X_u \Gamma_0^{-2/(fk)}$ with $\log \Gamma_0 \leq fk$, one has

$$\log T_u \geq \sum_{j=1}^{m} \frac{1}{f} \sum_{\rho \in \Phi} \log \frac{w_{j\rho} v}{\pi} + mg - e_6', \qquad e_6' = C_6 y(1 + m^2/k) + 2.$$

The nodes satisfy $b_{\rho j} < w_{j\rho} \leq b_{\rho j}(1 + 1/x)$ [Adr, §15.6, (P4)], so Theorem 47.3.1 gives this form of (A6) with $e_6 + m/x$ in place of e_6', an error of order y. The formalization states the counts exactly, not the inequality of [Adr, Hyp. 15.23].

At each ρ the row parameter is $z_\rho^{\mathrm{row}} = (v_{\rho,0}, \ldots, v_{\rho,k-2}, -b_{\rho 1}, \ldots, -b_{\rho m})$, of length r, with $v_{\rho,i} \geq 0$ and $b_{\rho j} \geq 1$, and the column parameter $z_{u\rho}^{\mathrm{col}}$ has length k. The proof of (A6) in [Adr, Prop. 15.25] takes as an input the ratio of consecutive masses: for $k < M \leq r$,

$$\frac{\mathrm{mass}(M)}{\mathrm{mass}(M-1)} = \frac{\big((M-1)!\,(v/\pi)^M\big)^f}{(2v)^{f/2}}\, L(M), \qquad |\log L(M)| \leq f \log \zeta(M).$$

This is a hypothesis of [Adr, Prop. 15.25], assumed there; for $E = \mathbb{Q}(i)$ and small rank compare the mass of Chapter 21.

Problem 47.3.1. In $\log(D^*/d_u^*)$ the factorials $((M-1)!)^f$, $k < M \leq r$, of the mass ratio $\mathrm{mass}(r)/\mathrm{mass}(k)$ cancel exactly against the denominators of $\prod_\rho D_r(z_\rho^{\mathrm{row}})/D_k(z_{u\rho}^{\mathrm{col}})$. In $V_r(z_\rho^{\mathrm{row}})$ the differences $v_{\rho,i} + b_{\rho j}$ contribute at least $b_{\rho j}^{k-1}$ for each j. After division of logarithms by fk, the coefficient of the mean of $\log b_{\rho j}$ over ρ from these factors is $(k-1)/k$.

Problem 47.3.2. At a level prime g_i the leading powers in the target and source indices are $g_i^{r(r+1)/2+k}$ and $g_i^{k(k+1)/2}$ respectively. Each index lies between one half and one times its leading power [Adr, Lemma 15.21]. Taking both primes at each of the f embeddings gives

$$\frac{1}{fk} \log \frac{I_\Sigma}{I_\Pi} \geq \left(m + \frac{m(m-1)}{2k} \right) g - \frac{2 \log 2}{k}.$$

Problem 47.3.3. Let $\log \Gamma_0 \leq fk$. For $Z_u \leq T_u \leq (1 + 1/(fk)) Z_u$,

$$D^* \Gamma_0^{-2} \leq d_u^* T_u^{fk} \leq e D^* \Gamma_0^{-2}, \qquad \log T_u \geq \log X_u - 2.$$

The required congruence is $T_u = \alpha\bar{\alpha}$ in K, with $\alpha \equiv 1 \pmod{\mathfrak{m}\mathcal{O}_K}$ for $\mathfrak{m} = Q_S q d (g_0 g_1)^2$ [Adr, Rem. 15.27]. Here q is the first auxiliary character modulus and Q_S is the part of the conductor of E prime to qd. The ray class estimate in [Adr, §5.8, (C9)], from [TZ, Thm. 1.4] with Stark's exceptional-zero bound, supplies

such a prime in $(Z,(1+h)Z]$ when $\log(4d\mathfrak{m}^2)\le c\sqrt{\log Z}$ and $e^{-c\sqrt{\log Z}}\le h\le 1$, for an absolute $c>0$. [Adr] keeps the exceptional zero in this estimate; with Theorem 9.3.1 of Chapter 9 the exceptional zero does not occur.

Problem 47.3.4. Take $m\asymp x^{1/2-\varepsilon/16}$, $fk=x^{O(1)}$, $\log(Q_S qd)\ll x^{\varepsilon/5}\log^2 x$, and $(kb)^8\le g_i\le 2(kb)^8$ as in [Adr, Prop. 5.15, Def. 6.1]. If $\log Z_u\gg my$, then $\log(4d\mathfrak{m}^2)\asymp y=o(\sqrt{\log Z_u})$ and $\log(fk)=o(\sqrt{\log Z_u})$. Hence (C9) supplies T_u in the interval of Problem 47.3.3.

Under the counting hypotheses the required lower scale follows already from [Adr, Prop. 15.25]: $w_{j\rho}\ge 1$, $v\ge\pi$, $g\ge 16y$ and $k\ge m$ give $\log Z_u\ge mg-O(y(1+m^2/k))-2$. In the ranks of [Adr, Prop. 5.15], $m/k\to 0$, so the error is $o(my)$.

Problem 47.3.5. For each top t and node j, subtracting the counted node term from the height term gives

$$\varphi(a_t)-a_t\log\Omega-\frac{a_t}{f}\sum_{\rho}\log\frac{w_{j\rho}v}{\pi}=a_t\left(\log a_t-1-\gamma+c_{\chi_d}-\frac{1}{f}\sum_{\rho}\log w_{j\rho}\right).$$

In the mean slope inequality the average of $\mu(R^\vee)+h_u$ is bounded using Theorem 47.2.2 in the same metric. Multiplying Theorem 47.3.1 by $-A$ then contributes $-mAg$, which cancels the $+mAg$ in the local bound (A5) [Adr, Prop. 3.1]. Problem 47.3.5 gives the coefficient that survives.

47.4. The base lattice

Let M_u be the level-zero lattice for the word u. Its coefficient module at ρ is the integral Weyl module of highest weight λ_ρ, with the Fock form normalized to give its highest weight vector squared norm one. For a dominant weight λ of length k, let $|\lambda|=\sum_i\lambda_i$ and subtract the last entry from every entry to obtain the partition ν. Let $\mu_{\mathrm{std}}(\nu)$ be the slope of this normalized Weyl lattice. The Fock model is $\mathbb{Q}[x_{ab}:1\le a,b\le k]$, with GL_k acting on the rows and the Fock form $\{p,q\}=\sum_\alpha p_\alpha q_\alpha\,\alpha!$, where $p=\sum_\alpha p_\alpha x^\alpha$ and $\alpha!=\prod_{a,b}\alpha_{ab}!$; thus the monomials are orthogonal with $\|x^\alpha\|^2=\alpha!$ [Adr, §15.7]. For a partition ν of length at most k the highest weight vector is $hw_\nu=\prod_i\Delta_i^{\nu_i-\nu_{i+1}}$, with Δ_i the leading principal $i\times i$ minor of (x_{ab}) and $\nu_{k+1}=0$. Let $c_\nu=\|hw_\nu\|^2/\prod_i\nu_i!$.

Theorem 47.4.1 (Slope of the base lattice)**.** *In the level-zero genus,*

$$\mu(M_u)=-\frac{\log T_u}{k}\sum_\rho|\lambda_\rho|+\sum_\rho\mu_{\mathrm{std}}(\nu_\rho),\qquad \mu_{\mathrm{std}}(\nu)\ge-|\nu|\log k-\log c_\nu.$$

Under the weight and size bounds of [Adr, Thm. 15.33]*, including* $\log T_u\le C_T my$*, (A7) is* $\mu(M_u)\ge -e_7$*, with* $e_7=C_7N_T a\Lambda_a(1+m/k)$*.*

Simplest incarnation: Problem 26.3.2 computes the slope of a Gaussian invariant line in weight divisible by four. Its Petersson squared norm includes a stabilizer factor $1/4$, giving slope $\log 4$ where the normalized coefficient line has slope zero. *Status*: the displayed formulas and the lower bound are proved in [Adr, Props. 15.30, 15.32, Thm. 15.33].

Problem 47.4.1. For $\nu=(l,h)$ with $l\ge h\ge 0$, so that $hw_\nu=x_{11}^{l-h}(x_{11}x_{22}-x_{12}x_{21})^h$, one has $\|hw_\nu\|^2=(l+1)!\,h!/(l-h+1)$ and $c_\nu=(l+1)/(l-h+1)$. For $(l,h)=(2,1)$ the polynomial $x_{11}(x_{11}x_{22}-x_{12}x_{21})$ has squared Fock norm 3.

For a constant $C_* \geq 1$ the lower inequalities of (K7) and (K8) in [Adr, Def. 18.9] read

$$\log T_u \geq \sum_{j=1}^{m} \frac{1}{f} \sum_{\rho \in \Phi} \log \frac{w_{j\rho} v}{\pi} + mg - C_* \Lambda_a \Big(1 + \frac{m^2}{k}\Big), \qquad \mu(M_u) \geq -C_* N_T a \Lambda_a \Big(1 + \frac{m}{k}\Big).$$

Problem 47.4.2. Since $\Lambda_a = \log x + y + \log \mathrm{rd}_E \geq y \geq 1$, the form of (A6) in [Adr] after Theorem 47.3.1 gives the lower inequality of (K7) for every $C_* \geq C_6 + 2$, and Theorem 47.4.1 gives the lower inequality of (K8) for every $C_* \geq C_7$. The upper bound on $\mu_{\max}(M_u)$ is a separate input to [Adr, Prop. 18.11(viii), Rem. 15.34].

Remark: where the arithmetic geometry enters. The public sources differ here. Zhang [Zh, §4.3] applies the moduli schemes, the arithmetic Hilbert–Samuel theorem, Moriwaki's criterion for small sections and Fargues's estimates for Hodge–Tate maps to the covolumes of the initial lattices, the analogues of the base lattices M_u (his bound $\mu(M_\alpha) \geq -E_{\mathrm{sl}}$). Adriazola bounds the target through the covolumes of the lattices M_j of forms of the m lifts [Adr, Prop. 15.53, Thm. 15.70], with a covolume bound for each j [Adr, Hyp. 15.58]; on the geometric side [Adr, Thm. 15.66] derives the covolume bound on the chain [Adr, Hyp. 15.59] from a uniform mean slope on deep covers [Adr, Hyp. 15.69] and inputs (I1)–(I17). For the base lattices [Adr, Thm. 15.33] gives Theorem 47.4.1 from a computation with Weyl modules. In the formalization the arithmetic geometry is applied to the lattices M_j of forms of the m lifts, on the Shimura varieties of signature (p_j, q_j), which is the covolume estimate in Theorem 47.2.2; a geometry-of-numbers inequality then turns these bounds into the bound for the target (Chapters 49 and 51). The periods also differ: those of [Zh] are periods of CM abelian varieties, after Anderson, with Shimura–Taniyama reciprocity and the integrality theorem of Deligne and Ribet among the results used for them. In the formalization only elliptic curves with complex multiplication by $\mathcal{O}_K$ enter this step: Ω of Theorem 47.1.1 is the Chowla–Selberg period of such a curve, and it reaches the lattices of forms through the degree of one metrized line at points built from two such curves (Chapter 50). The same remark follows the map of the inputs in Chapter 58.

Deliverable. For the chain record of [Adr, Def. 18.9], Theorem 47.2.1 specifies h_u, Theorem 47.2.2 supplies the height inequality in (K4), and Problem 47.2.1 gives (K6) under its hypotheses on the row weights; [Adr, §18.7] derives these for the heights of its selected record, and applying this to the heights of Theorem 47.2.1 requires identifying their row sums, hook factors and mean of $\log \kappa$ with those of the record. Theorems 47.3.1 and 47.4.1 supply the first inequalities of (K7) and (K8). The upper and between-word bounds on $\log T_u$ and the maximal base slope still enter [Adr, Prop. 18.11(vii)–(viii)] separately. These are the contributions to (A4), (A6) and (A7) in [Adr, Prop. 18.14]. Existence of records for chosen data, weights and nodes at every scale of arbitrarily large windows is assumed in [Adr, Hyp. 18.10]; the formalization constructs records on a family that it selects (Section 58.7 of Chapter 58).

Hints

§47.1.

47.1.1: Take logarithms of the product over classes.

47.1.2: Problem 12.3.7; then compare the linear terms of $\log \|\Theta_{(a)}\|^2$ before and after the rescaling.

§47.2.

47.2.1: On $[1, M]$ the derivative of $\varphi(z) - z\log\Omega$ is $\log z - \log\Omega$. For the second bound use $m \geq 1$.

47.2.2: Insert the weight sums into the terms $-\nu_{j\rho i}\log\Omega$ of s_j, and use the signature of M_j.

47.2.3: Substitute $\Lambda_a \leq 2y$ in e_4.

47.2.4: Theorem 26.1.1 for the first claim and Theorem 27.2.1 for the second; then let H grow.

§47.3.

47.3.1: Telescope the mass ratios over $k < M \leq r$, and compare $\prod_{i<r} i!$ with $\prod_{i<k} i!$.

47.3.2: Expand $r(r+1)/2 + k - k(k+1)/2$ with $r = k + m - 1$, and bound each index below by half its leading power.

47.3.3: Use $\log(1+t) \leq t$.

47.3.4: Compare y^2 with my.

47.3.5: Insert the value of $\log\Omega$ from Theorem 47.1.1.

§47.4.

47.4.1: Expand $(x_{11}x_{22} - x_{12}x_{21})^h$ by the binomial theorem; the terms are orthogonal monomials, and the resulting sum over i is a sum of binomial coefficients $\binom{l-i}{l-h}$.

47.4.2: Bound each term of e_6' by a multiple of $\Lambda_a(1 + m^2/k)$.

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