SLOPES, FRAMES AND BESSEL–POINCARÉ SERIES

Prerequisites. From Chapter 6: normed spaces, degrees, slopes and heights (Section 6.3), the normalized slopes $\mu^\sharp$ and heights $h^\sharp$, and Theorem 6.4.1. From Chapter 7: Theorems 7.1.1, 7.2.1 and 7.3.1, with the notation of Sections 7.2 and 7.3. From Chapter 10: Section 10.1. From Chapter 11: the Poincaré series, Problems 11.1.1, 11.3.1 and 11.4.2, and Theorem 11.3.1. Bernoulli's inequality. The statements in the text are quoted from Adriazola's reconstruction [Adr] of Alpöge's argument [Alp], with its numbering, and the sizes m, k, n, $b = e^y$, $y = x^\varepsilon$, $a = xb$ are those of [Adr, Prop. 5.15]: $m \le 2x^{1/2-\varepsilon/16}$, $n \le 2x^{2+8\varepsilon}$ and $k \le 2x^{9/2+10\varepsilon}$. No problem needs more of [Adr] than is stated here.

Chapter 7 bounded the mean slope of a target by a family of maps whose images fill it, given a Gram condition at one complex place; Chapters 10 and 11 found such conditions for Poincaré series on $\mathrm{SL}_2(\mathbb{Z})$. What does the argument need of these, for lattices of rank about $x^{9/2}$ over a CM field, and which of the needed statements are proved? This chapter states them in the form used, as numbered theorems quoted from [Adr]. After each we name its simplest incarnation in the earlier chapters and its status: *classical*, with a reference; *proved in* [Adr] (from published results); or *derived in* [Adr] *from hypotheses*, named there and assumed there. The Lean formalization [Lean] proves the final theorem; we have not checked that its intermediate statements are these hypotheses, and Section 58.7 of Chapter 58 records the differences that are known. The problems specialize the statements, check their exponents, and deduce the form in which Chapter 58 uses them. Chapters 44, 45 and 46 develop the inputs of this chapter from the Lean code: Bessel–Poincaré series and their Gram operators, the seesaw, and the Bessel identity.

43.1. Slopes over a totally complex field

Let K be a totally complex number field of degree n_K, and Σ_K its set of complex places; for $v \in \Sigma_K$ fix an embedding σ_v in the class v. A *hermitian lattice* $\bar{L}$ over $\mathcal{O}_K$ is a finitely generated $\mathcal{O}_K$-submodule L spanning a K-vector space V, with a hermitian norm on $V_v := V \otimes_{K,\sigma_v} \mathbb{C}$ for each v. It is the normed space of Chapter 6 whose unit lattice at a finite place w is $L \otimes \mathcal{O}_{K,w}$. We use the slope μ and height ht of [Adr, (13.1), (13.2)], with $\mathrm{ht}(\varphi) := \frac{2}{n_K} \sum_{v \in \Sigma_K} \log \|\varphi_v\|_{\mathrm{op}}^2$. They are the $\mu^\sharp$ and $h_\infty^\sharp$ of Section 6.3, that is, $2/n_K$ times the slope and the archimedean height there; for K imaginary quadratic they coincide with them. So $\mu(\mathcal{O}_K) = 0$, dividing all norms by $c > 0$ raises μ by $2 \log c$, and Theorems 43.1.1–43.3.1 are unchanged if slopes, heights and the numbers s, h_i, u_F, H_0 are multiplied by a common positive constant. The sets of slopes, $\mu_{\max}$ and $\mu_{\min}$ are as in Chapter 7.

Theorem 43.1.1 ([Adr, Prop. 13.1])**.** *Let $\bar{L}$ be a hermitian lattice over $\mathcal{O}_K$ of rank $\ell \ge 1$. There are real numbers $x_1 \ge \cdots \ge x_\ell$, the slopes of $\bar{L}$, and for each real θ a saturated submodule $L_{\ge\theta} \subseteq L$, such that: the number of j with $x_j \ge \theta$ is $\operatorname{rk} L_{\ge\theta}$; $x_1 + \cdots + x_\ell = \ell\mu(L)$; $\mu_{\min}(L) \le x_j \le \mu_{\max}(L)$; every nonzero submodule N with*

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$\mu_{\min}(N) \geq \theta$ *lies in* $L_{\geq\theta}$*; and* $\mu_{\min}(L_{\geq\theta}) \geq \theta$ *if* $L_{\geq\theta} \neq 0$*. Moreover* $\mu(L^\vee) = -\mu(L)$ *and* $\mu_{\min}(L^\vee) = -\mu_{\max}(L)$*.*

Simplest incarnation: Theorem 7.1.1 and the truncations $V^{\geq\lambda}$ after it. *Status*: classical, by Stuhler [St] and Grayson [Gr] over $\mathbb{Z}$, and by Bost [Bo1] and Gaudron [Ga] over rings of integers.

Theorem 43.1.2 ([Adr, Prop. 13.3])**.** *Let* $\bar{A}$ *be a hermitian lattice over* $\mathcal{O}_K$ *in* V*, let* V' *carry hermitian norms at the places of* Σ_K*, and let* $\varphi\colon V \to V'$ *be* K*-linear with* $\varphi(A) \neq 0$*. Give* $\varphi(A)$ *the norms of* V'*. Then* $\mu_{\min}(\varphi(A)) \geq \mu_{\min}(A) - \mathrm{ht}(\varphi)$*.*

Simplest incarnation: Theorem 6.4.1 and Problem 7.1.6. *Status*: classical (Bost [Bo1]); [Adr] gives the proof in its normalization.

Problem 43.1.1. Let K be imaginary quadratic, let $\mu_1 > \mu_2$, and let $L := \mathcal{O}_K e_1 \oplus \mathcal{O}_K e_2$ with $e_1 \perp e_2$ and $\|e_i\| = e^{-\mu_i/2}$. Then $\mu_{\max}(L) = \mu_1$, $\mu_{\min}(L) = \mu_2$ and $\mu(L) = \frac{1}{2}(\mu_1 + \mu_2)$. Let $c_1, c_2 \in K^\times$, let $\varphi := \mathrm{diag}(c_1, c_2)\colon K^2 \to K^2$ with the standard norm on the target, and $\lambda_i := \mu_i + 2\log|c_i|$. Then $\mathrm{ht}(\varphi) = \max(\lambda_1, \lambda_2)$ and $\mu_{\min}(\varphi(L)) = \min(\mu_1 - \lambda_1, \mu_2 - \lambda_2)$, so Theorem 43.1.2 is an equality for φ if and only if $\lambda_2 \geq \lambda_1$.

So the slope inequality is sharp when the map stretches the part of small slope at least as much as the rest.

43.2. The frame theorem

For one map, Theorem 43.1.2 sees only $\mu_{\min}$ and $\mu_{\max}$. For a family, Chapter 7 passed to the mean slope. The form used is the following; for $u = 0$ we read $u\log(1 + H_0/u)$ as 0. For a K-linear map B between spaces with fixed bases, $[B]_v$ is its matrix with σ_v applied to the entries.

Theorem 43.2.1 (frame theorem, [Adr, Thm. 13.8])**.** *Let* I *be a nonempty finite set, and for* $i \in I$ *let* $\bar{N}_i$ *be a hermitian lattice over* $\mathcal{O}_K$ *of rank* d_i *in* V_i*, with* $\Delta := \sum_i d_i > 0$*. Let* $\bar{C}$ *be a hermitian lattice of rank* D *in* W*. Let* $B_i\colon V_i \to W$ *be* K*-linear with* $B_i(N_i) \subseteq C$*, and* h_i *real with* $\mathrm{ht}(B_i) \leq h_i$ *whenever* $B_i \neq 0$*. Fix* $v \in \Sigma_K$*, bases of the* V_i *and* W*, invertible complex matrices* Y_i *of size* d_i *and* Y' *of size* D*, an integer* $D_1 \geq 0$*, and complex* $(D + D_1) \times d_i$ *matrices* A_i *whose first* D *rows form* $Y'^{-1}[B_i]_v Y_i$*. Let* $\gamma_i > 0$ *and* $Q := \sum_i \gamma_i A_i A_i^*$*. Let* s*,* $u_F \geq 0$*,* $H_0 \geq 0$*,* $0 \leq \epsilon \leq \frac{1}{4}$*,* $\eta \geq 0$ *and* $0 \leq \delta_R < 1$ *satisfy*

(i) $s\Delta \leq \sum_i d_i\big(\mu(N_i) - h_i\big)$*;*
(ii) $\mu_{\max}(N_i) - h_i \leq s + u_F$ *for every* i*;*
(iii) $s - \mu_{\min}(C) \leq H_0$*;*
(iv) $(1 - \delta_R)(D + D_1) \leq D$*;*
(v) $\sum_i \|\gamma_i A_i^* A_i - 1\|_{\mathrm{HS}}^2 \leq \epsilon\Delta$*;*
(vi) $\mathrm{tr}(Q^2) \leq (1 + \eta)(\mathrm{tr}\, Q)^2/(D + D_1)$*.*

Then

$$\mu(C) \geq s - \frac{4u_F \log(1 + H_0/u_F) + (\eta + 4\sqrt{\epsilon})H_0}{1 - \delta_R}.$$

(In [Adr] the numbers u_F, H_0, ϵ, δ_R are called u, H, e, δ.) *Simplest incarnation*: Theorem 7.2.1, with the rank hypothesis of Section 7.2 obtained from Theorem 7.3.1 with $\theta = (\eta + 4\sqrt{\epsilon})/(1 - \delta_R)$ and $c = 4/(1 - \delta_R)$, as at the end of Section 7.3; here (v) and (vi) are the Gram conditions of that section, with $D' = D + D_1$. *Status*: proved in [Adr], from Theorems 43.1.1 and 43.1.2 by the argument of Chapter 7.

Problem 43.2.1. Let $\bar{N}$ and $\bar{C}$ have the same rank, with $\bar{N}$ semistable, and let $B\colon V \to W$ be bijective with $B(N) \subseteq C$. Then Theorem 43.2.1 applies with $I = \{1\}$, $h_1 = \mathrm{ht}(B)$, $s = \mu(N) - \mathrm{ht}(B)$, $u_F = 0$, $D_1 = 0$, $A_1 = 1$, $\gamma_1 = 1$, $\epsilon = \eta = \delta_R = 0$ and any $H_0 \geq \max(0, s - \mu_{\min}(C))$, and gives $\mu(C) \geq \mu(N) - \mathrm{ht}(B)$, the conclusion of Problem 7.2.3.

For a single map the theorem is weaker than Theorem 43.1.2 [Adr, Rem. 13.9]; its gain is the passage from $\mu_{\max}(C)$ to $\mu(C)$ for a redundant family. Conditions (v) and (vi) are checked through the Gram operators of larger, uncompressed maps.

Theorem 43.2.2 ([Adr, Prop. 13.11])**.** *Let I be a nonempty finite set and $\tilde{D} \geq 0$ an integer; for $i \in I$ let $d_i^S \geq d_i \geq 0$ be integers, P_i a complex $\tilde{D} \times d_i^S$ matrix, V_i a complex $d_i^S \times d_i$ matrix with $V_i^* V_i = 1$, and $\gamma_i > 0$. Put $K_{ij} := \sqrt{\gamma_i \gamma_j}\, P_i^* P_j$, $A_i := P_i V_i$, $Q := \sum_i \gamma_i A_i A_i^*$, $\Delta_S := \sum_i d_i^S$ and $\Delta := \sum_i d_i$. Let $D^* > 0$, $0 < \sigma \leq \frac{1}{20}$, $-1 \leq \eta_0 \leq \frac{\sigma}{2}$, $\eta_1 \leq \frac{\sigma}{2}$, $e_0 \geq 0$ with $\sqrt{e_0} \leq \frac{\sigma}{2}$, and $0 \leq \delta_T \leq \frac{1}{4}$, and suppose*

(G1) $\|K_{ij}\|_{\mathrm{HS}}^2 \leq (1+\eta_0) d_i^S d_j^S / D^*$ *for $i \neq j$;*
(G2) $\|K_{ii} - 1\|_{\mathrm{HS}}^2 \leq e_0 d_i^S$ *for every i;*
(G3) $\tilde{D} \leq (1+\eta_1) D^*$*;*
(G4) $D^* \leq \sigma \Delta_S$*;*
(G5) $\sum_i (d_i^S - d_i) \leq \delta_T \Delta_S$.

Then $\Delta \geq (1-\delta_T)\Delta_S > 0$, $\tilde{D} \geq 1$, $\sum_i \|\gamma_i A_i^* A_i - 1\|_{\mathrm{HS}}^2 \leq 2e_0\Delta$ and $\mathrm{tr}(Q^2) \leq (1 + 5\sigma + 3\sqrt{\delta_T})(\mathrm{tr}\, Q)^2/\tilde{D}$.

(In [Adr], δ_T is called δ'.) *Simplest incarnation*: Problem 10.1.4 and Theorem 10.1.2. *Status*: proved in [Adr]; it is linear algebra.

Problem 43.2.2. In the setting of Theorem 10.1.2, with $\eta_0 \geq 0$, take $\tilde{D} = D^* = D$, P_i the matrix of B_i in orthonormal bases, $V_i = 1$, $d_i^S = d_i = r_i$, $e_0 = \eta_1 = \delta_T = 0$ and $\sigma := \max(2\eta_0, D/R)$. If $\sigma \leq \frac{1}{20}$, then (G1)–(G5) hold, and Problem 10.1.4 gives the conclusion of Theorem 43.2.2 with the factor $1 + 5\sigma$ replaced by the smaller $1 + \eta_0 + D/R$.

So the condition $D^* \leq \sigma\Delta_S$ is the term D/R of Theorem 10.1.2: the sources must have total rank much larger than the target.

43.3. The period maps and the mean slope

In the argument the target is the dual $R^\vee$ of a lattice R of automorphic forms at the top of the chain, of rank $D_{\mathrm{rows}} \geq 1$ in a K-vector space Y, and the sources are lattices N_u of rank $d_u \geq 1$ in spaces V_u, indexed by a finite set $\mathcal{W}$ of *words*; the *period maps* are K-linear maps $B_u\colon V_u \to Y^*$ [Adr, §13.4]. Here K is a totally complex Galois field containing the field E of Chapter 4. Their archimedean structure at one place is the following [Adr, Def. 13.13]. A *Bessel realization* of $(B_u)_{u\in\mathcal{W}}$ at $v \in \Sigma_K$ consists of: a Hilbert space $\mathcal{H}$ of finite dimension D_{all}, with inner product conjugate-linear in the first variable, and a conjugate-linear isometric embedding $\iota_0\colon Y_v \to \mathcal{H}$, where Y_v has the norm of R at v; for each u, a Hilbert space C_u of dimension d_u^S, a linear map $P_u\colon C_u \to \mathcal{H}$, a number $c_u > 0$, an inner product on $V_{u,v}$ and a linear isometric embedding $\iota_u\colon V_{u,v} \to C_u$; and unitary operators U of Y_v and U_u of $V_{u,v}$, and numbers $\alpha_u \neq 0$, such that

$$(B_{u,v}\varphi)(F) = \alpha_u \left\langle \iota_0(UF),\, P_u \iota_u U_u \varphi \right\rangle_{\mathcal{H}} \qquad (\varphi \in V_{u,v},\ F \in Y_v). \tag{1}$$

Its *Gram operators* are $K_{uu'} := (c_u c_{u'})^{-1/2} P_u^* P_{u'} \colon C_{u'} \to C_u$. In the construction $\mathcal{H}$ is a space of automorphic forms on a unitary group, C_u the space of column forms of the word u, ι_u the inclusion of the forms ramified at the auxiliary prime T_u, P_u the map to Bessel Poincaré series and c_u its normalizing volume.

Problem 43.3.1. Write κ for the weight k and P_ℓ for the series P_m of Chapter 11, and T_κ for its Kloosterman term. Let $\mathcal{H} = Y_v = S_\kappa$ (a model only: there is no field K or lattice R here), with $\langle F, G\rangle_{\mathcal{H}}$ the Petersson product $\langle G, F\rangle$, let $\iota_0(F)(z) := \overline{F(-\bar{z})}$, and for $\ell \geq 1$ let $V_\ell = C_\ell = \mathbb{C}$, $\iota_\ell = 1$, $P_\ell(1) := P_\ell$, $B_\ell(\varphi)(F) := \varphi\, a_F(\ell)$ and $c_\ell := \Gamma(\kappa-1)(4\pi\ell)^{1-\kappa}$. Then (1) holds with $U = U_\ell = 1$ and $\alpha_\ell = 1/c_\ell$, and $K_{\ell\ell'} = \delta(\ell,\ell') + T_\kappa(\ell,\ell')$.

So (1) generalizes the unfolding of Problem 11.1.1: a Fourier coefficient becomes a Bessel period.

Theorem 43.3.1 ([Adr, Thm. 13.15])**.** *Let R, N_u, B_u be as above, and h_u real. Put*

$$s := \frac{\sum_u d_u\big(\mu(N_u) - h_u\big)}{\sum_u d_u}, \qquad u_F := \max_{u \in \mathcal{W}} \big(\mu_{\max}(N_u) - h_u\big) - s.$$

Let $D^* > 0$, $H_0 > 0$, $0 < \sigma \leq \frac{1}{20}$, $-1 \leq \eta_0 \leq \frac{\sigma}{2}$, $\eta_1 \leq \frac{\sigma}{2}$, $0 \leq e_0 \leq \frac{1}{8}$ with $\sqrt{e_0} \leq \frac{\sigma}{2}$, $0 \leq \delta_T \leq \frac{1}{4}$ and $0 \leq \delta_R < 1$, and suppose:

(a) *$B_u(N_u) \subseteq R^\vee$ for every u;*
(b) *$\mathrm{ht}(B_u) \leq h_u$ whenever $B_u \neq 0$, for the norms of $\bar{N}_u$ and $\bar{R}^\vee$;*
(c) *$s + \mu_{\max}(R) \leq H_0$;*
(d) *(B_u) has a Bessel realization at some $v \in \Sigma_K$ with $D_{\mathrm{all}} \geq D_{\mathrm{rows}}$ and $(1-\delta_R) D_{\mathrm{all}} \leq D_{\mathrm{rows}}$;*
(e) *$\|K_{uu'}\|_{\mathrm{HS}}^2 \leq (1+\eta_0) d_u^S d_{u'}^S / D^*$ for $u \neq u'$, $\|K_{uu} - 1\|_{\mathrm{HS}}^2 \leq e_0 d_u^S$, and $D_{\mathrm{all}} \leq (1+\eta_1) D^*$;*
(f) *$D^* \leq \sigma \sum_u d_u^S$ and $\sum_u (d_u^S - d_u) \leq \delta_T \sum_u d_u^S$.*

Then $s - e_3 \leq \mu(R^\vee)$, where

$$e_3 := \frac{4u_F \log(1 + H_0/u_F) + \big(5\sigma + 3\sqrt{\delta_T} + 4\sqrt{2e_0}\big) H_0}{1 - \delta_R} \geq 0.$$

This is the input (A3) of [Adr, Prop. 3.1], with the weights d_u. *Simplest incarnation*: Theorem 7.2.1 for the maps B_u, with the rank hypothesis from Theorem 7.3.1. *Status*: proved in [Adr], from Theorems 43.1.1, 43.2.1 and 43.2.2: by [Adr, Lemma 13.14] the realization gives matrices $A_u = P_u V_u$ as in Theorem 43.2.1, with $D = D_{\mathrm{rows}}$ and $D_1 = D_{\mathrm{all}} - D_{\mathrm{rows}}$, so that (d) is condition (iv) there. That the construction satisfies (a)–(f) rests in [Adr] on hypotheses, assumed there. Inclusion (a) is the divisibility of the chain [Adr, §§11–12, Lemma 13.20], and h_u in (b) is defined in [Adr, Def. 15.14]. Hypothesis (c) enters only through the choice of H_0, and Section 43.5 returns to it. Hypotheses (d)–(f) are the subject of the next section.

43.4. Bessel Poincaré series

The automorphic side is as follows [Adr, §14.1]. Let G be the unitary group of a hermitian space over the CM field E; the totally real subfield of E has degree $2n$. The space $\mathcal{H}$ consists of the automorphic forms on G with a fixed discrete series at the archimedean places and fixed by the level idempotent of the construction, with

the Petersson product; $D_{\mathrm{all}} = \dim \mathcal{H}$, and D^* is the formal degree of the discrete series times the volume of the automorphic quotient of G times the value at 1 of the level idempotent, as in Section 11.4. For $y = x^{\varepsilon}$ put

$$\Gamma_0 := \exp\bigl(2C_\Gamma\, m^5 n y\bigr).$$

Theorem 43.4.1 (near orthogonality, [Adr, Thm. 14.1])**.** *There are absolute constants $C_0 \geq 1$ and $C_\Gamma > 0$ such that for every ε, every x large in terms of ε, and every family of data of* [Adr, Part II] *at the scale x whose words satisfy*

$$C_0^{-1} \leq d_u^S \Gamma_0^2 / D^* \leq C_0 \qquad (u \in \mathcal{W}), \tag{2}$$

the Gram operators satisfy (e) of Theorem 43.3.1 with $e_0 \geq 0$ and $\eta_0, \eta_1, \sqrt{e_0} \leq \exp(-x^{\varepsilon/8})$.

Simplest incarnation: Theorem 11.3.1, through Problem 11.3.1. *Status*: derived in [Adr, Thm. 14.79] from [Adr, Hyps. 14.71–14.77]: the kernel expansion, the main row, the loops, the nongeneric and degenerate terms, the multiplicity formula and the sizes. Of these, [Adr, Hyp. 14.76] is deduced from the label dictionary and the pre-trace inputs, [Adr, Hyps. 14.59 and 14.61], by [Adr, Cor. 14.63] under the size conditions [Adr, (14.20)], and the two exponential size conditions of [Adr, Hyp. 14.77] are [Adr, Lemma 14.78]; the others are assumed there [Adr, §18.7]. The formalization proves the final theorem; the corresponding steps in [Adr] are these hypotheses.

Problem 43.4.1. In the setting of Problem 43.3.1, let $1 \leq M \leq (\kappa - 1)/(4\pi e)$, where $e = 2.718\ldots$ as in Theorem 11.3.1, let $\mathcal{W} = \{1, \ldots, M\}$, so that $d_\ell^S = 1$, and let $D^* := (\kappa - 1)/12$. Then (e) of Theorem 43.3.1 holds with $e_0 = \kappa^2 4^{-\kappa}$, $\eta_1 = 13/(\kappa - 1)$ and $1 + \eta_0 = (\kappa - 1)\kappa^2 4^{-\kappa}/12$. But $D^* \geq \frac{\pi e}{3} \sum_\ell d_\ell^S$, with $\frac{\pi e}{3} = 2.846\ldots$, so the first half of (f) fails for every $\sigma \leq \frac{1}{20}$.

This is the obstruction at the end of Chapter 11. In the argument each word contributes a column space of dimension about $D^* \Gamma_0^{-2}$ by (2), nearly orthogonal to the others, and the words are so many that their dimensions add up to more than $x^{C_H} D^*$.

Theorem 43.4.2 (share of irreducible rows, [Adr, Thm. 14.2])**.** *Let $J \geq 1$ be the fixed integer of the construction. Let D_{rows} be the dimension of the span of the image of $R \otimes_{K,\sigma_v} \mathbb{C}$ in $\mathcal{H}$ under the row embedding, and define δ_R by $D_{\mathrm{rows}} = (1 - \delta_R) D_{\mathrm{all}}$. For every family of data of* [Adr, Part II] *and every x large in terms of the family constants, $0 \leq \delta_R \leq \exp(-x^{\varepsilon/(7J)})$.*

Simplest incarnation: Problem 11.4.2, the dimension of S_k as formal degree times covolume. *Status*: derived in [Adr, Thm. 14.46] from [Adr, Hyps. 14.42–14.44] and the sizes [Adr, Lemma 14.64]. Of these, [Adr, Hyp. 14.42] is proved [Adr, Cor. 14.57] through Gelfand–Graev representations [DL, §10], [Cart, Ch. 8]; [Adr, Hyp. 14.43] follows from [Adr, Hyps. 14.59 and 14.61] by [Adr, Cor. 14.63]; [Adr, Hyp. 14.44] is assumed there.

Theorem 43.4.3 (Bessel identity, [Adr, Thm. 14.3])**.** *For every family of data of* [Adr, Part II] *and every large x, the period maps have a Bessel realization* (1) *at v, with ι_0 the row embedding, ι_u the column embedding, and P_u the Bessel Poincaré series. Moreover $B_{u,v}(\varphi \otimes 1)(\xi \otimes 1) = \sigma_v\bigl(B_u(\varphi)(\xi)\bigr)$ for $\varphi \in V_u$ and $\xi \in Y$.*

Simplest incarnation: Problem 11.1.1, as Problem 43.3.1 shows. *Status*: derived in [Adr, Thm. 14.15], given the data [Adr, (P1)–(P5)] of the construction and [Adr, Hyps. 14.10–14.12]: the seesaw relation, the dictionaries and rationality. The seesaw relation [Adr, Hyp. 14.10] follows from the laws of the theta tower and its dictionaries, [Adr, Hyps. 14.17 and 14.21], by [Adr, Cor. 14.22, Rem. 14.23]; the double-step identity in [Adr, Hyp. 14.17] follows from the laws of the Weil representation, [Adr, Hyps. 14.27 and 14.28], by [Adr, Thm. 14.29]; and [Adr, Lemma 14.24] reduces rationality to three period identities. The hypotheses at the ends of these reductions are assumed there, except that for its selected record [Adr, §14.2.4, (14.15)] derives rationality at σ_v; applying this to the periods in the seesaw requires comparing them on the rational row and column bases. Here the formalization diverges from [Adr]: it does not state [Adr, Hyp. 14.10], a period map with a seesaw relation, but states the Bessel identity for the period maps B_u directly, as in this theorem, and proves the final theorem through it.

So Theorems 43.4.2 and 43.4.3 give (d) of Theorem 43.3.1, and Theorem 43.4.1 gives (e). The second half of (f) counts ramified columns.

Problem 43.4.2. Let $T \geq 2$ and $d^S \geq 0$, and $d := d^S\big(1 - 1/(T-1)\big)^{2n}$. Then $d^S - d \leq (4n/T)\, d^S$. Hence, if $d_u = d_u^S(1 - 1/(T_u - 1))^{2n}$ and $T_u \geq b$ for every u, the second half of (f) holds with $\delta_T = 4n/b$, and $d_u > \frac{1}{2} d_u^S$ once $b > 8n$ and $d_u^S > 0$.

The proof of Theorem 43.4.3 needs one nonzero pairing of a Poincaré series with a theta lift.

Problem 43.4.3. Let $P\colon C \to \mathcal{H}$ be a linear map of finite-dimensional inner product spaces, $c > 0$, and $K := c^{-1} P^* P$. Then $\dim \ker P \leq \|K - 1\|_{\mathrm{HS}}^2$. Hence, if $\|K - 1\|_{\mathrm{HS}}^2 \leq e_0 \dim C$ with $e_0 \leq \frac{1}{2}$, then P does not vanish on any subspace of C of dimension greater than $\frac{1}{2} \dim C$.

By Problems 43.4.2 and 43.4.3, the diagonal clause of Theorem 43.4.1 gives $\varphi_0 \in V_{u,v}$ with $P_u \iota_u U_u \varphi_0 \neq 0$ [Adr, Lemma 14.14]. Theorem 43.4.1 concerns the Poincaré series alone, so using it in the proof of Theorem 43.4.3 is not circular [Adr, Rem. 14.81].

What remains are the sizes. Write $\omega := \exp(-x^{\varepsilon/8})$ and $\sigma := x^{-C_H}$, with $C_H > 0$.

Problem 43.4.4. Let $\varepsilon > 0$ and $C_H > 0$. For large x: (i) $x^{C_H} \geq 20$, $11 \omega x^{C_H} \leq 1$ and $8n \leq b\omega^2$; (ii) if $\eta_0, \eta_1, \sqrt{e_0} \leq \omega$ and $\delta_T = 4n/b$, then $\eta_0, \eta_1, \sqrt{e_0} \leq \frac{\sigma}{2}$, $e_0 \leq \frac{1}{8}$, $\delta_T \leq \frac{1}{2}\omega^2 \leq \frac{1}{4}$, and $\big(5\sigma + 3\sqrt{\delta_T} + 4\sqrt{2e_0}\big) x^{C_H} \leq 6$.

The first half of (f) holds if $d_u^S \geq D^*/(C_0 \Gamma_0^2)$ for every u, the lower half of (2), and $|\mathcal{W}| \geq C_0 \Gamma_0^2 x^{C_H}$. Problems 43.4.2 and 43.4.4 with this remark are [Adr, Lemma 13.19]. The upper half of (2) shows that the words must be numerous; since a word is a string of $k - 3$ binary letters, this forces k to be large.

Problem 43.4.5. Assume $D^* \leq \sigma \sum_u d_u^S$, $d_u^S \leq C_0 D^* \Gamma_0^{-2}$ for every $u \in \mathcal{W}$, and $|\mathcal{W}| \leq 2^{k-3}$. Then $(k-3)\log 2 \geq 4 C_\Gamma m^5 n y + C_H \log x - \log C_0$. Hence $k \geq m^5 n$ for large x.

So (f) and (2) make k at least about $m^5 n$, which is about $x^{9/2}$ with the choices of [Adr, Prop. 5.15].

43.5. The record

Chapter 58 uses Theorem 43.3.1 with $H_0 := x^{C_H} b$. Call $0 \le \delta_R \le \frac{1}{2}$, $\eta \ge 0$, $0 \le \epsilon \le \frac{1}{4}$ and $(\eta + 4\sqrt{\epsilon})x^{C_H} \le 6$ the parameter bounds, and for $H_0 > 0$ consider the inequality

$$s - \log|\mathcal{W}| - \frac{1}{1-\delta_R}\Big(4u_F \log\Big(1 + \frac{H_0}{u_F}\Big) + (\eta + 4\sqrt{\epsilon})H_0\Big) \le \mu(R^\vee). \tag{3}$$

The condition (K3) of a chain record [Adr, Def. 18.9] asks for the parameter bounds and for (3) at one H_0 with $s - \mu_{\min}(R^\vee) \le H_0 \le x^{C_H} b$. The first part of (K4) asks $\mu_{\min}(R^\vee) = -\mu_{\max}(R)$ and $s + \mu_{\max}(R) \le x^{C_H} b$.

Theorem 43.3.1, with Theorems 43.1.1 and 43.4.1–43.4.3 and Problems 43.4.2 and 43.4.4, gives the following. The proof of (K3) in [Adr, Prop. 18.11(ii)–(iii)] takes $H_0 = x^{C_H} b$.

Theorem 43.5.1. *Suppose that, at the scale x, the data satisfy (a) and (b) of Theorem 43.3.1 and* (2)*, that $T_u \ge b$ and $d_u = d_u^S(1 - 1/(T_u - 1))^{2n}$ for every u, and that $C_0\Gamma_0^2 x^{C_H} \le |\mathcal{W}| \le 2^{k-3}$. Under Theorems 43.4.1, 43.4.2 and 43.4.3, for large x the parameter bounds and* (3) *hold for every $H_0 > 0$ with $H_0 \ge s - \mu_{\min}(R^\vee)$. If also $s - \mu_{\min}(R^\vee) \le x^{C_H} b$, taking $H_0 = x^{C_H} b$ gives (K3). In both assertions take $\eta = 5x^{-C_H} + 3\sqrt{4n/b}$ and $\epsilon = 2e_0$.*

The term $\log|\mathcal{W}|$ in (3) only weakens Theorem 43.3.1, and costs little.

Problem 43.5.1. If $\mathcal{W}$ is nonempty and $|\mathcal{W}| \le 2^{k-3}$, then $\log|\mathcal{W}| \le k$, and $k \le a = xe^{x^\varepsilon}$ for large x.

Problem 43.5.2. Let $0 \le u \le U$, $b, X > 0$ and $0 < H_0 \le Xb$. Then $u\log(1 + H_0/u) \le \log(1 + X)\,(U + b)$.

Problem 43.5.3. Assume (K3), $\mu_{\min}(R^\vee) = -\mu_{\max}(R)$, $s + \mu_{\max}(R) \le x^{C_H} b$, and $u_F \le U$ (note that $u_F \ge 0$, since s is a weighted mean of the $\mu(N_u) - h_u$). Then $s - e_3 \le \mu(R^\vee)$ with $e_3 = \log|\mathcal{W}| + 8\log(1 + x^{C_H})(U + b) + 12b$.

This is (A3) in the form of [Adr, Prop. 18.14]; compare [Adr, Cor. 13.18]. The bound $u_F \le U$ for the spread is [Adr, Lemma 18.13], which combines the records of the other inputs.

Deliverable. This chapter supplies the following part of a chain record [Adr, Def. 18.9]: condition (a), $|\mathcal{W}| \le 2^{k-3}$, compatible with the lower bound of Theorem 43.5.1 once $k \ge m^5 n$ with room to spare (Problem 43.4.5); condition (K3), by Theorem 43.5.1 with the height bound of (K4); and the first clause $\mu_{\min}(R^\vee) = -\mu_{\max}(R)$ of (K4), by Theorem 43.1.1. Theorems 43.1.1–43.3.1 are classical or proved in [Adr]. The record rests on Theorems 43.4.1–43.4.3, which in [Adr] rest on the hypotheses of [Adr, §14] listed after them, assumed there apart from rationality on the selected record [Adr, (14.15)]; and on (a) and (b) of Theorem 43.3.1, which come from the divisibility and the heights. The a priori range $s + \mu_{\max}(R) \le x^{C_H} b$ in (K4) is assumed in [Adr, Prop. 18.11(iii)]; we have not found a proof of it there, so in [Adr] it is part of [Adr, Hyp. 18.10], assumed there; see Section 58.7 of Chapter 58.

Hints

§43.1.

43.1.1: The filtration $0 \subset \mathcal{O}_K e_1 \subset L$ has semistable quotients of decreasing slopes; use the uniqueness in Theorem 7.1.1. The image $\varphi(L)$ is again an orthogonal sum of two lines.

§43.2.

43.2.1: Take $Y_1 = 1$ and $Y' = [B]_v$. Semistability makes $\mu_{\max}(N) = \mu(N)$.

43.2.2: Compare $\eta_0 + D/R$ with 5σ.

§43.3.

43.3.1: Compute the q-expansion of $\iota_0(F)$, and use Problems 11.1.1 and 11.3.1.

§43.4.

43.4.1: An entry of a matrix is bounded by its operator norm; use Theorem 11.3.1 and Problem 11.4.2.

43.4.2: Bernoulli's inequality.

43.4.3: K is positive semidefinite with $\ker K = \ker P$, and $K - 1$ is -1 on this kernel.

43.4.4: $x^{\varepsilon/8}$ eventually exceeds any multiple of $\log x$; and $3 + 4\sqrt{2} < 11$.

43.4.5: $\sum_u d_u^S \le |\mathcal{W}| \max_u d_u^S$.

§43.5.

43.5.1: $k \le 2x^{9/2+10\varepsilon}$.

43.5.2: Apply $\log z \le z - 1$ to $z = (1 + H_0/u)/(1 + X)$, and $1 - 1/z \le \log z$ to $z = 1 + X$.

43.5.3: Use the $H_0 \le x^{C_H} b$ supplied by (K3) in (3), and use Problem 43.5.2 with $X = x^{C_H}$.

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