THE ANALYTIC FRAME: WINDOWS, TRANSLATES AND AUXILIARY CHARACTERS

Prerequisites. From Chapter 1: G, G_χ, c_χ and G^ψ (equations (1), (3) and Section 1.4), and Theorems 1.2.3 and 1.5.1. From Chapter 2: Problem 2.1.7 and Theorem 2.2.2. From Chapter 3: Theorems 3.3.1 and 3.4.1. From Chapter 4: all of it, in particular Γ, Φ, the characters $\theta_{\xi,j,l}$, $P_{t,w}$, M_w, $R_{q,d}$, R_ξ, $\mathcal{K}(x)$, and Theorems 4.2.1 and 4.4.1. From Chapter 5: Theorem 5.2.1 and Section 5.3. The zero-free region, the exceptional zero and the prime number theorem for characters [Dav, Chs. 14 and 20]; the conductor–discriminant formula [Wa, Ch. 3].

The earlier chapters of Part I proved each analytic step in its simplest setting: one window theorem for ζ, one field E with $H = (\mathbb{Z}/4n\mathbb{Z})^\times/\{\pm1\}$, one auxiliary prime at a time. In what generality does the proof use them? This chapter states the analytic part of the argument as Adriazola proves it [Adr, §§2–6], as cited theorems, each with its simplest incarnation in the earlier chapters and its status. The problems specialize the statements, check their exponents, and deduce the forms used later. The analytic deductions in [Adr] use published theorems. Theorem 36.4.3(ii) also assumes the Selmer bound on windows [Adr, Hyp. 17.11]. The final implication, Theorem 36.5.1, takes Theorem A as input. The proof of Theorem A in [Adr, §18.6] uses chain records [Adr, Hyp. 18.10]. The Lean formalization [Lean] proves the final theorem; the status of each input is recorded in Section 58.7 of Chapter 58.

Throughout, ψ is a primitive Dirichlet character of conductor M and order o, $N_0 := 8M$, and $0 < \varepsilon < \frac{1}{52}$. We write G_χ for the complex function (3); it is the function G^*_χ of [Adr, (5.1)], whose real part [Adr] calls G_χ. We keep the notation of Chapter 4, with u for the translates: $\Gamma = \mathrm{Gal}(E/\mathbb{Q})$, $\Phi = \mathrm{Gal}(E/K)$ with $K = \mathbb{Q}(\sqrt{-d})$, $n = |H|$, $A_0 \subset H$ of index o, and $G_\theta = G_{\chi_\theta}$. The element $\iota_d \in \Gamma$ acts by -1 on $\sqrt{-d}$ and trivially on $\sqrt{-q}$ and on H; it generates the inertia group at d.

36.1. Windows

Chapter 1 excluded zeros by an upper bound on windows $[Y, Y^{3/2}]$, via Kaczorowski–Pintz and a separate argument for a real zero. Adriazola proves one theorem that does both.

Theorem 36.1.1 ([Adr, Thm. 2.4])**.** *Let $F\colon [1,\infty) \to \mathbb{R}$ be measurable with $|F(x)| \le Bx$, and suppose $A(s) := \int_1^\infty F(x)x^{-s-1}\,dx$ extends to a meromorphic function on $\mathbb{C}$. Let $0 \le \theta < 1$. Suppose that every pole p of A with $\mathrm{Re}\, p > \theta$ is simple with $\mathrm{Re}\, p < 1$, and that the residue at every real pole $p > \theta$ is positive. If there are $\lambda > 1$ and C such that $F(x) \le Cx^\theta$ on $[Y, Y^\lambda]$ for arbitrarily large Y, then A has no pole in $\mathrm{Re}\, s > \theta$.*

Simplest incarnation: Theorem 1.2.3. *Status*: proved in [Adr]; a quantitative form of Landau's theorem, with Taylor coefficients at a real point as in Section 1.5. For

Date: November 25, 2026.

$G_+ := G + \sum_i w_i \operatorname{Re} G_{\chi_i}$ with $w_i > 0$ and nontrivial χ_i, it gives [Adr, Cor. 2.5]: a bound $G_+(x) \le Cx^\theta$ on such windows implies $\zeta(s) \ne 0$ and $L(s,\chi_i) \ne 0$ for $\operatorname{Re} s > \theta$.

Problem 36.1.1. (i) Theorem 36.1.1 applies to $F = G$ with $\lambda = \frac{3}{2}$, $C = 1$, and gives Theorem 1.2.3; the residue at a real zero $\beta \in (\theta, 1)$ of ζ of multiplicity m is $m/(\beta(1-\beta)) > 0$. (ii) The corollary with $\chi_j = \psi_j$ and $w_j = 1/(3o+1)$ for $1 \le j < o$ is Theorem 1.5.1.

36.2. The excess and the average

Let E be the field of Theorem 36.4.1 below, and identify Γ with $H \times \{\pm 1\}^2$ as there. Let $c' := ([1], -1, -1)$; it is the complex conjugation c when $\psi(-1) = 1$. Define the cells, Λ_d, $P_{t,w}$ and M_w as in Chapter 4, with c' in place of c. For $a > 0$, $t \in H$ and $w\colon \Phi \to (0,\infty)$, Adriazola's *excess* is

$$\operatorname{De}(a,t,w) := \frac{1}{2n} \sum_{g \notin \Phi} \sum_{\theta} \overline{\theta(g)}\, w(tgc')\, G_\theta\Big(\frac{a}{w(tgc')}\Big).$$

Theorem 36.2.1 (excess identity, [Adr, Prop. 3.3 and Cor. 3.4])**.** *For all $a > 0$, $t \in H$ and $w\colon \Phi \to (0,\infty)$, $P_{t,w}(a) = M_w(a) + \operatorname{De}(a,t,w)$. Here q, d are as in Theorem 36.3.3, so that $d \equiv 7 \pmod 8$ and $d \equiv -1$ modulo q and modulo every odd prime dividing M; then the primes dividing $2qM$ split in K and contribute nothing to $P_{t,w}$.*

Simplest incarnation: Theorem 4.2.1. *Status*: proved in [Adr]; an identity between finite sums.

The average removes the characters that are not trivial on A_0 and keeps the $4o$ characters $\theta_{\xi,j,l}$, $\xi \in \langle\psi\rangle$ (Problem 4.5.1). Let $T_2 \subset A_0$, $\delta := |T_2|/n$, and call a cell thick for $u \in A_0$ if it lies in row 1 and in a column of uT_2, as in (2). Let $T \subset H$ have N_T elements, and let $\kappa, \nu \ge 0$ with $|\sum_{t\in T} \xi(t) - \kappa N_T| \le \nu N_T$ for $\xi \in \langle\psi\rangle \smallsetminus 1$; put $\kappa_1 := 1$ and $\kappa_\xi := \kappa$ otherwise.

Theorem 36.2.2 ([Adr, Lemma 4.1 and Thm. 4.2])**.** *Let $x \ge 3$, $b, b', X' > 0$ with $b'X' \le xb$, $0 \le \Delta \le b$, $\vartheta \ge 0$, and $m \ge 1$. Let $|a_t/b - x| \le 1$ for $t \in T$, and for $u \in A_0$, $1 \le j \le m$ let $w_{u,j}\colon \Phi \to (0,\infty)$ satisfy $b \le w_{u,j} \le b + \Delta$ on the thick cells for u and $w_{u,j} \le 2b'$ on the others. Suppose $|G_\theta(y)| \le \vartheta y$ for all θ and $y \ge X'/3$, and put $\mathcal{M} := \log(x+1) + C_d + \max|c_{\chi_\theta}|$ with C_d absolute, the maximum over the $4o$ characters $\theta_{\xi,j,l}$ with $\xi \in \langle\psi\rangle$ (as in $\mathcal{K}(x)$ of Problem 4.4.4). If the mean over $u \in A_0$ of $\sum_{t,j} \operatorname{Re} \operatorname{De}(a_t, t, w_{u,j})$ is at most* Err*, then*

$$N_T m b\, \frac{\delta}{2}\Big(\sum_{\xi\in\langle\psi\rangle} \kappa_\xi \operatorname{Re} G_\xi(x) - \nu \sum_{\xi\in\langle\psi\rangle} |G_\xi(x)| - {\sum}' |G_\theta(x)| - 4o\,\mathcal{M}\Big) \le \mathrm{Err} + N_T m\big(4n\vartheta(x+1)b + 12n(x+1)\Delta\big),$$

where $\sum'$ runs over the $3o$ characters $\theta_{\xi,j,l}$, $\xi \in \langle\psi\rangle$, $(j,l) \ne (0,0)$.

Simplest incarnation: Theorem 4.4.1. *Status*: proved in [Adr]. Here ϑ, Δ, $\mathcal{M}$ are the numbers ω, η', M of [Adr]. Adriazola then divides by $N_T m b \delta/2$: if the sizes satisfy [Adr, (4.6), (4.7)] and $\mathrm{Err} \le x^{C\varepsilon}\mathcal{B}$, where

$$\mathcal{B} := N_T a\Big(1 + \frac{m^2}{k}\Big) + m\Big(2nmb\, x^\varepsilon + \frac{\delta n\, k\, b\, x^{2\varepsilon}}{n^2}\Big),$$

and if $4n(x+1)\vartheta \le 1/x$, $12n(x+1)\Delta \le b/x$ and $4x^{\varepsilon-1} + \nu\sum_\xi |G_\xi(x)| + 4o\mathcal{M} \le \frac{3}{13}x^{1/2+(C+3)\varepsilon}$, then $\sum_\xi \kappa_\xi \operatorname{Re} G_\xi(x) \le x^{1/2+(C+3)\varepsilon} + \sum'|G_\theta(x)|$ [Adr, Prop. 4.3 and Cor. 4.4]. Status: proved in [Adr].

Problem 36.2.1. (i) For $\psi = 1$, $\mathrm{De}(a,t,w) = \sum_{\rho\in\Phi} w(\rho) R_{t,\rho}(a/w(\rho))$; so Theorem 36.2.1 is Theorem 4.2.1. (ii) Take $\psi = 1$, so that $A_0 = H$, and $m = 1$, $T = \{t\}$, thick weights b. The mean over $u \in H$ of the thick part of $\mathrm{De}(a,t,w_u)$ is $\frac{\delta b}{2}\sum_{(j,l)} \theta_{1,j,l}(\iota_d)\, G_{\theta_{1,j,l}}(a/b) = \frac{\delta b}{2} R_{q,d}(a/b)$, which is Problem 4.4.3.

The set T is chosen with prescribed shares on the cosets of A_0.

Theorem 36.2.3 ([Adr, Lemma 5.18])**.** *If for every $\gamma \in H$ the number of $t \in T$ in γA_0 differs from $\lambda_\gamma N_T$ by at most $\delta' N_T$, where $\lambda_\gamma = 4/(3o+1)$ for $\gamma \in A_0$ and $3/(3o+1)$ otherwise, then $|\sum_{t\in T}\xi(t) - N_T/(3o+1)| \le o\,\delta' N_T$ for $\xi \in \langle\psi\rangle \smallsetminus 1$.*

Simplest incarnation: Problem 4.5.3, with $\delta' = 0$. *Status*: proved in [Adr]; elementary.

Problem 36.2.2. With $\delta' = x^{-1+\varepsilon}$, Theorem 36.2.3 gives the hypothesis of Theorem 36.2.2 with $\kappa = 1/(3o+1)$ and $\nu = o\,x^{-1+\varepsilon}$. If $|G_\xi(y)| \le B_\psi\, y$ for $\xi \in \langle\psi\rangle$, then $\nu\sum_\xi |G_\xi(x)| \le o^2 B_\psi\, x^\varepsilon$.

So the ν term costs x^ε, far below $x^{1/2}$. What remains are the $3o$ values $|G_\theta(x)|$ in $\sum'$, at the scale x, and the thin-scale bound, at scales $y \ge X'/3$.

36.3. The auxiliary characters

The characters in $\sum'$ have conductors divisible by q or d, larger than any power of x; no zero-free region for a fixed L-function bounds them at one point. The primes q, d are therefore chosen, as in Chapters 2 and 3, so that the bound holds on a whole window. The classical inputs, with absolute constants, are listed in [Adr, §5.1]: Chebyshev's bounds; the prime number theorem for characters with the exceptional zero [Dav, Ch. 20]; Page's theorem [Pa]; Montgomery's density theorem [Mon, Thm. 12.2]; and the prime number theorem in progressions in Linnik's range, which [Adr, (C7)] derives from Gallagher's theorem [Gal, Thm. 7] with the exceptional zero kept in the main term. All are classical.

Theorem 36.3.1 (thin scales, [Adr, Prop. 5.3])**.** *There are absolute $0 < c_1 \le \frac12$ and C_1 such that the following holds. Let χ be primitive modulo $N \ge 3$, $\lambda \ge \max(1, \log N)$ and $0 < c_* \le \frac12$, and suppose χ is not real or $L(\beta,\chi) \ne 0$ for $1 - c_* < \beta < 1$. Then for $y \ge 2$ and $t_0 := \log y$,*

$$\frac{|G_\chi(y)|}{y} \le \frac{2e^{-c_* t_0}}{c_*} + C_1\lambda^4\Big(\frac{2\lambda}{c_1}e^{-c_1 t_0/(2\lambda)} + \frac{8}{c_1^2}\Big(1 + \frac{c_1\sqrt{t_0}}{2}\Big)e^{-c_1\sqrt{t_0}/2}\Big).$$

The same decay holds for G.

Simplest incarnation: the thin-scale bound of Section 4.4, which Chapter 4 assumed. *Status*: classical; proved in [Adr] from the prime number theorem for characters.

Theorem 36.3.2 (first prime, [Adr, Thm. 5.10])**.** *Let $J \ge 1$ and $0 < \tau \le \frac12$. For Y large in terms of $N_0, J, \varepsilon, \tau$, put $z := Y^{\varepsilon/5}$ and let $\mathcal{P}$ be the set of primes $q \equiv 3 \pmod 8$ in $(e^{z/2}, e^z]$. For all $q \in \mathcal{P}$ outside a set of at most $Y^{-\tau}|\mathcal{P}|$ elements, every primitive ξ of conductor dividing N_0 and $\chi := \xi\chi_q$ satisfy: (i) $L(s,\chi) \ne 0$*

for $\operatorname{Re} s > 1 - \frac{1}{400}$, $|\Im s| \le N_0 e^z$*; (ii)* $|c_\chi| \le 401\, c_0\, J \log Y$*, with* c_0 *the Chebyshev constant; (iii)* $|G_\chi(y)| \le y^{1/2+\tau}$ *for every* $y \in [Y, Y^J]$.

Simplest incarnation: Theorem 3.3.1, and Theorem 2.2.2. *Status*: proved in [Adr] from the classical inputs above; Gallagher's density theorem is not needed for q.

Theorem 36.3.3 (second prime, [Adr, Thm. 5.12 and Cor. 5.13])**.** *Let* J, τ, Y, z *be as in Theorem 36.3.2, let* $q \equiv 3 \pmod 8$ *be a prime in* $(e^{z/2}, e^z]$*, let* $z \log^2 Y \le z' \le J^2 z \log^2 Y$*, and let* $\mathcal{P}'$ *be the set of primes* $d \in (e^{z'/2}, e^{z'}]$ *with* $d \equiv 7 \pmod{16}$ *and* $d \equiv -1$ *modulo* q *and modulo every odd prime* $p \le z$*. For all* $d \in \mathcal{P}'$ *outside a set of at most* $Y^{-\tau}|\mathcal{P}'|$ *elements, conclusions (i)–(iii) of Theorem 36.3.2 hold for* $\chi := \xi \chi_q^a \chi_d$*,* $a \in \{0,1\}$*, with* $N_0 e^{z+z'}$ *in (i). Moreover, if (i) holds for* $\xi\chi_q$ *and for* $\xi\chi_q^a\chi_d$*, then every real primitive* χ *whose conductor is divisible by* q *or* d *and divides* $2^j M q d$ *has* $L(\beta, \chi) \ne 0$ *for* $1 - \frac{1}{400} < \beta < 1$.

Simplest incarnation: Section 3.4 and Theorem 3.4.1. *Status*: proved in [Adr] from the classical inputs, with Linnik's range through Gallagher's theorem. [Adr] keeps the exceptional zero and Siegel's theorem, so its constants are not effective; with Theorem 9.3.1 of Chapter 9 the exceptional zero does not occur, Siegel's theorem is not needed, and this source of ineffectivity disappears. The congruences on d fix $\left(\frac{s}{d}\right)$ for every s whose prime factors are at most z; this is why $\log d$ must exceed log of the modulus $16q\prod_{p\le z} p$, of order z, by the factor $\log^2 Y$.

Problem 36.3.1. (i) For $M = 1$, $J = \frac{3}{2}$ and $\tau = \varepsilon$, the case $\xi = 1$ of Theorem 36.3.2(iii) is the bound of Theorem 3.3.1 for $A = \frac{3}{2}$, with $q \equiv 3 \pmod 8$ and $\log q \in (z/2, z]$ in place of $\log q$ of size $(\log Y)^3$. (ii) If the conclusions hold at a scale Y_0 with exponent J, and $Y_0 \le Y$, $Y^2 \le Y_0^J$, then they hold on the window $[Y, Y^{3/2}]$; with $z' = z \log^2 Y_0$ there, $\log q \le Y_0^{\varepsilon/5}$ and $\log d \le Y_0^{\varepsilon/5} \log^2 Y_0$.

Problem 36.3.2. Take $N_0 = 8M$. Each of the $3o$ characters in $\sum'$ is $\xi\chi_q^j\chi_d^l$ with ξ primitive inducing a power of ψ and $(j,l) \ne (0,0)$, and Theorems 36.3.2(iii) and 36.3.3(iii) bound all of them: $|G_\theta(x)| \le x^{1/2+\varepsilon}$, so $\sum'|G_\theta(x)| \le 3o\, x^{1/2+\varepsilon}$. For $\psi = 1$ these are $\chi_q, \chi_d, \chi_q\chi_d$; for $\psi = \chi_8$ there are six.

Problem 36.3.3. Let $\log N \le \lambda := A\, x^{\varepsilon/5} \log^2 x$ with A fixed, and $t_0 \ge x^{\varepsilon/3} - \log 3$. For x large in terms of A, c_*, ε, the right side of Theorem 36.3.1 is at most $\exp(-x^{\varepsilon/8})$. The exponents that matter are $\frac{1}{3} - \frac{1}{5} = \frac{2}{15} > \frac{1}{8}$ and $\frac{1}{6} > \frac{1}{8}$.

By Theorem 36.3.3, the real characters of conductor divisible by q or d have no zero in $(1 - \frac{1}{400}, 1)$, and the finitely many others depend on M alone; so c_* depends on M only. With $\log N \le \lambda$ (Problem 36.4.3), Problem 36.3.3 gives the thin-scale bound with $\vartheta = \exp(-x^{\varepsilon/8})$ for all $4n$ characters θ of Γ [Adr, proof of Thm. 6.3].

36.4. The host field and the data

Theorem 36.4.1 (host field, [Adr, Prop. 5.14])**.** *Let* $q \ne d$ *be primes* $\equiv 3 \pmod 4$ *not dividing* M*,* $e \ge 0$*,* $N_1 := \operatorname{lcm}(2^{e+2}, M)$*,* $U_1 := \{a \in (\mathbb{Z}/N_1)^\times : a \equiv \pm 1 \bmod 2^{e+2},\ \psi(a) = 1\}$*,* $H := (\mathbb{Z}/N_1)^\times/U_1$*, and let* $E \subset \mathbb{Q}(\zeta_{N_1 qd})$ *be the field fixed by the classes that lie in* U_1 *modulo* N_1 *and are squares modulo* q *and* d*. Then* E *is a CM field containing* $\sqrt{-q}, \sqrt{-d}$*, with group* $H \times \{\pm 1\}^2$*;* $n := |H| = c\, 2^e$ *with* $c \mid o$*; the image* A_0 *of* $\ker\psi$ *is cyclic of index* o*; and the characters of* H *trivial on* A_0 *are the powers of* ψ.

Simplest incarnation: Problem 4.1.1, the case $\psi = 1$, where $H = (\mathbb{Z}/4n\mathbb{Z})^\times/\{\pm 1\}$. *Status*: classical (class field theory of $\mathbb{Q}(\zeta_N)$).

Let x be the scale, and put

$$b := e^{x^\varepsilon}, \qquad a := xb, \qquad X' := \exp(x^{\varepsilon/3}), \qquad b' := a/X', \qquad \delta_1 := x^{-\varepsilon/8}.$$

Theorem 36.4.2 ([Adr, Props. 5.15–5.17, 5.20 and 18.7, Cor. 18.8])**.** *For x large in terms of ε and ψ, and n as in Theorem 36.4.1 with $x^{2+8\varepsilon} \le n \le 2x^{2+8\varepsilon}$: (i) there are even $m \in [\sqrt{x\delta_1}, 2\sqrt{x\delta_1}]$ and $k \in [x^{9/2+10\varepsilon}, 2x^{9/2+10\varepsilon}]$, and with $|T| \ge \frac{1}{2}\delta_1 n$, $\delta \ge x^{-\varepsilon}$ they satisfy the size conditions* [Adr, (4.6), (4.7)] *of the division by $N_T mb\delta/2$; (ii) there are nodes $w_{u,j}$ with $b \le w_{u,j} \le b + \Delta$ at the thick cells, $1 \le w_{u,j} \le 2b'$ elsewhere, where $\Delta := W + mh_0$, W is the number of roots of unity in E and $h_0 := 2\lceil x^{402}\rceil$; and then $12n(x+1)x\Delta \le b$ and $4n(x+1)\exp(-x^{\varepsilon/8}) \le 1/x$; (iii) for $L \ge 2$ and $n_0 \ge 1$ there is a set T_2' of integers s with $4Ls < n_0$, meeting no line in three points as in Problem 5.2.4, and $|T_2'| \ge (n_0/L)\exp(-36\sqrt{\log n_0 \log L})$; (iv) there is $T \subset H$ with $\frac{1}{2}\delta_1 n \le |T| \le 2\delta_1 n$, satisfying Theorem 36.2.3 with $\delta' = x^{-1+\varepsilon}$, and with few tops whose Frobenius orbits are not good* [Adr, Def. 5.19].

Simplest incarnations: (iii) is Theorem 5.2.1, (iv) is Section 5.3. *Status*: proved in [Adr]; (iii) is Behrend's construction [Beh], (iv) a random choice.

Problem 36.4.1. Let $n_0 = n/o$ with $x^{2+8\varepsilon} \le n \le 2x^{2+8\varepsilon}$, and $2 \le L \le C\log x$. For x large in terms of ε, o, C, the bound of Theorem 36.4.2(iii) is at least $x^{-\varepsilon}n$; so $\delta \ge x^{-\varepsilon}$ for $T_2 := \{a_0^s : s \in T_2'\}$, a_0 a generator of A_0; the map $s \mapsto a_0^s$ is injective on T_2' because $0 \le s < n_0 = |A_0|$.

Problem 36.4.2. Under Theorem 36.4.2(ii), with $\vartheta = \exp(-x^{\varepsilon/8})$, $X := X'/3$ and $\delta \ge x^{-\varepsilon}$: (i) $a_t/w_{u,j}(\rho) \ge X$ at every thin cell, if $|a_t/b - x| \le 1$ and $x \ge 3$; (ii) the last two terms of Theorem 4.4.1 are at most $3x^{\varepsilon-1}$, using $\sum_{r\le u}\Lambda(r) \le 1.04\,u$ [RS].

Problem 36.4.3. The root discriminant $\mathrm{rd}_E := |\mathrm{disc}\,E|^{1/[E:\mathbb{Q}]}$ satisfies $\log \mathrm{rd}_E \le \log(8Mn) + \log q + \log d$. If $\log q \le Y_0^{\varepsilon/5}$, $\log d \le Y_0^{\varepsilon/5}\log^2 Y_0$ and $Y_0 \le x$, then $\ell_a := \log(a\ \mathrm{rd}_E) \le 2x^\varepsilon$ for large x (the number Λ_a of [Adr, Lemma 18.15(iii)]), and every χ_θ has log conductor at most $4x^{\varepsilon/5}\log^2 x$.

Admissible data at the scale x for the window $[Y, Y^{3/2}]$ with base Y_0 [Adr, Def. 18.1] consist of: q, d as in Theorems 36.3.2 and 36.3.3 at the scale Y_0, with exponent J and $z' = z\log^2 Y_0$; e with $x^{2+8\varepsilon} \le c\,2^e \le 2x^{2+8\varepsilon}$ and the field of Theorem 36.4.1; m, k, T_2, T as in Theorem 36.4.2; two level primes g_0, g_1 of size $(kb)^8$ split in K, from Thorner–Zaman [TZ] with Stark's bound [Sta]; three primes $\equiv 1$ modulo a conductor, from Linnik's theorem [Lin]; and condition (D6): the Tate–Shafarevich groups $\mathrm{Sha}_\ell(E,i)$, $1 \le i < x^{400}$, have total length $\sum_\ell \log\ell \cdot \mathrm{length} \le \exp(\frac{1}{2}x^\varepsilon)$.

Theorem 36.4.3 ([Adr, Prop. 6.2 and Prop. 18.4])**.** *(i) For Y large and every $x \in [Y, Y^{3/2}]$, the data other than (D6) exist with $Y_0 = Y$. (ii) Assume the Selmer bound on windows* [Adr, Hyp. 17.11]*. There is $J \ge 2$ such that for arbitrarily large Y there is $Y_0 \le Y$ with $Y^2 \le Y_0^J$ for which admissible data with base Y_0 exist at every $x \in [Y, Y^{3/2}]$.*

Simplest incarnation: the example $q = 11$, $d = 1319$, $n = 4$ of Chapter 4. *Status*: (i) is proved in [Adr] from published theorems; (ii) is proved in [Adr] from the Selmer bound on windows; in [Adr] this is [Adr, Hyp. 17.11], assumed there. The formalization proves the final theorem; see Section 58.7 of Chapter 58. Condition

(D6) is the only reason for the base Y_0: Theorem 36.3.2 must be applied at Y_0, and Problem 36.3.1(ii) carries it to the window.

36.5. The last step

Theorem A of [Adr] asserts: there is an absolute C such that for every ψ and small ε there is J with the following property, for arbitrarily large Y and some $Y_0 \le Y$ with $Y^2 \le Y_0^J$. At every $x \in [Y, Y^{3/2}]$ there are admissible data with base Y_0, weights $a \le a_t \le a + a/(kx^{13})$ and nodes as in Theorem 36.4.2(ii), with $\sum_{t,j} \operatorname{Re} \mathrm{De}(a_t, t, w_{u,j}) \le x^{C\varepsilon}\mathcal{B}$ for every $u \in A_0$. One can take $C = 5$. [Adr, §18.6] derives it from [Adr, Hyp. 17.11] and the chain records [Adr, Hyp. 18.10], both assumed there. The formalization proves the final theorem, constructing chain records on a family of data that it selects (Section 58.7 of Chapter 58). In [Adr, Hyp. 18.10] the data and weights are chosen at each scale of the window, and then, for each $u \in A_0$, nodes and a chain record for them.

Theorem 36.5.1 ([Adr, Thm. 6.3])**.** *Theorem A implies that $L(s, \psi) \neq 0$ for $\operatorname{Re} s > \frac{1}{2}$.*

Simplest incarnation: Theorems 4.4.1 and 1.5.1. *Status*: proved in [Adr] from the theorems of this chapter.

Problem 36.5.1. Assume Theorem A. On its windows, for large Y, $G^{\psi}(x) \le x^{1/2+(C+4)\varepsilon}$. The only input from Part II is the bound for $\sum \operatorname{Re} \mathrm{De}$; all other hypotheses of Theorem 36.2.2 and of the division by $N_T mb\delta/2$ are supplied by Theorems 36.2.3–36.4.3 and Problems 36.2.2–36.4.3.

Since C does not depend on ε, Theorem 36.1.1 with $\theta = \frac{1}{2} + (C+4)\varepsilon$ and $\varepsilon \to 0$ gives Theorem 36.5.1.
The example $\psi = \chi_8$. Take $q = 11$, $d = 1319$, $n = 4$, $H = \{[1], [3], [5], [7]\}$ and $\psi := \xi^2$ as in Section 4.5, so that $\chi_\psi = \chi_8$, $o = 2$, $A_0 = \{[1], [7]\}$. No subset of H has the shares $\frac{4}{7}$, $\frac{3}{7}$ on the two cosets, so let T be the multiset $\{[1], [1], [7], [7], [3], [3], [5]\}$, with sums and means over T counted with multiplicity; in the argument n is large and a set suffices.

Problem 36.5.2. The mean of the thick part of (1) over $u \in A_0$ and $t \in T$ is $\frac{\delta b}{2} R^{\psi}(a/b)$, with $R^{\psi} := R_{q,d} + \frac{1}{7} R_{\xi^2}$. Its untwisted part is $G + \frac{1}{7} G_{\chi_8} = G^{\psi}$.

Problem 36.5.3. $R^{\psi} - G^{\psi} = G_{\chi_{11}} - G_{\chi_{1319}} - G_{\eta} + \frac{1}{7}(G_{\chi_8\chi_{11}} - G_{\chi_8\chi_{1319}} - G_{\chi_8\eta})$, with primitive characters of conductors 11, 1319, 14509, 88, 10552, 116072. If these six satisfy $|G_{\chi}(x)| \le x^{1/2+\varepsilon}$, then $G^{\psi}(x) \le R^{\psi}(x) + \frac{24}{7} x^{1/2+\varepsilon}$.

These six are the characters of Problem 36.3.2 for $\psi = \chi_8$; here q and d are far too small, and the example only names them.

Problem 36.5.4. $L(\sigma, \chi_8) > 0$ for $0 < \sigma < 1$.

The argument does not use this: for a general ψ a real zero of $L(s, \psi)$ is excluded only by the positive residue in Theorem 36.1.1.
Deliverable. This chapter proves none of the conditions (K1)–(K8) of a chain record [Adr, Def. 18.9]; it supplies what they are stated about and what follows from them. It gives the admissible data, the weights and the nodes $w_{u,j}$ (Theorems 36.4.1–36.4.3), with the numbers b, b', a, $\mathcal{B}$ and $\ell_a \le 2x^{\varepsilon}$ that enter (K1)–(K8); the sum S of (K1), which is the left side of Theorem 36.2.1 summed over t and j; and the

nodes in the first inequality of (K7). Conversely, given chain records at every scale of a window, [Adr, Props. 18.14 and 18.16] give $\sum \operatorname{Re} \mathrm{De} \le x^{5\varepsilon}\mathcal{B}$ for records with the exponents $e_D = 2$, $e_L = 0$ of [Adr, Hyp. 18.10], and this chapter turns that into the Riemann hypothesis for $L(s,\psi)$: Theorem 36.4.2(ii) with Problem 36.4.2(i) gives the normalization condition (V2), and Problem 36.4.2(ii) gives (V3), of the final chapter, *Assembly at one scale*; Problems 36.3.2 and 36.3.3 give its window conditions (W1) and (W2); and Problem 36.4.1 with Theorem 36.4.2(iv) gives $\delta \ge x^{-\varepsilon}$ and $N_T \ge \frac{1}{2}x^{-\varepsilon/8}n$. The only input here without a published proof is (D6): in [Adr] it rests on [Adr, Hyp. 17.11], assumed there; see Section 58.7 of Chapter 58.

Hints

§36.1.

36.1.1: (i) $|G(x)| \le Bx$ by Theorem 1.1.1, since $|x^\rho| \le x$; then Problem 1.1.4, and $\rho(\rho-1) < 0$ for real $\rho \in (0,1)$. (ii) Problem 1.4.3.

§36.2.

36.2.1: (i) Put $\rho := tgc$. (ii) Orthogonality over $u \in H$ leaves the characters trivial on H; the thick cells for u are g with tgc in row 1, column in uT_2.

36.2.2: $|\langle\psi\rangle| = o$.

§36.3.

36.3.1: (ii) $Y^{3/2} \le Y^2 \le Y_0^J$.

36.3.2: Problem 4.5.1; χ_q and χ_d are induced by Legendre symbols since $q \equiv d \equiv 3 \pmod 4$.

36.3.3: Every prefactor is a power of x.

§36.4.

36.4.1: $\sqrt{\log n \, \log\log x} = o(\log x)$.

36.4.2: (i) $b' = xb/X'$. (ii) Use $\Delta \le b/(12nx(x+1))$ and $4n(x+1)\vartheta \le 1/x$.

36.4.3: Every character of E has conductor dividing $N_1 qd$, and $2^e \le n$.

§36.5.

36.5.1: Divide by $N_T mb\delta/2$; bound $\mathcal{M}$ by Theorems 36.3.2(ii) and 36.3.3(ii) for $(j,l) \ne (0,0)$, while the c_{ψ_j} depend on ψ alone, and $\sum'$ by Problem 36.3.2.

36.5.2: Problems 4.5.2 and 4.5.3.

36.5.3: Problem 4.5.1, and the triangle inequality.

36.5.4: Group the terms in blocks of eight, and use that $t \mapsto t^{-\sigma}$ is convex.

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