CHOW FORMS AND AN EFFECTIVE HILBERT–SAMUEL BOUND

Prerequisites. Jensen's formula; Gauss's lemma for polynomials over a factorial ring; binary forms and their discriminants; Bézout's theorem for plane curves. Mahler measures [BG, §1.6] and Chow forms [GKZ, Chs. 3–4] are recalled in the text. Hadamard's inequality $|\det(v_1,\dots,v_r)| \le \prod_j \|v_j\|$ for vectors of a Hermitian space of dimension r (for integer matrices, Problem 6.1.3). From Chapter 6, *The slope method*: Section 6.3 (degrees; over $\mathbb{Q}$, $\deg L = -\log \operatorname{covol} L$). From Chapter 26, *Covolumes of lattices of sections*: the lattice E_D and Problems 26.1.1 and 26.1.5, and Theorem 26.1.1, of Section 26.1, and Theorem 26.2.1 and formula (1) of Section 26.2.

In Chapter 26 the covolume of $E_D = H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(D))$ was computed exactly because the monomials are orthogonal. On a curve $C \subset \mathbb{P}^2$ no orthogonal basis of sections is known, and the arithmetic Hilbert–Samuel theorem gives $\deg H^0(C, \mathcal{O}(D))$ only up to $o(D^2)$, with an error that is not explicit. How can one bound this degree from below with every constant explicit? The reconstruction [Adr, proof dependency map, item HSAt] describes the covolume bound of the argument as a Chow-form determinant argument in the style of Nesterenko and Philippon [NP], with Mahler heights, Hadamard's inequality and a monomial count. This chapter treats the ingredients: Mahler measures, Chow forms as heights, the exact covolume for points on $\mathbb{P}^1$, and a determinant of sections of $\mathbb{P}^1$, where one sees what Hadamard's inequality loses.

27.1. Mahler measures

For a nonzero $P \in \mathbb{C}[x_1,\dots,x_N]$ let

$$m(P) := \int_{[0,1]^N} \log\bigl|P(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_N})\bigr|\, d\theta, \qquad M(P) := e^{m(P)},$$

the (logarithmic) Mahler measure [BG, §1.6]. For G homogeneous on $\mathbb{C}^N$, or multihomogeneous on $\mathbb{C}^{N_1}\times\cdots\times\mathbb{C}^{N_k}$, let $m_S(G)$ be the mean of $\log|G|$ over the product of the unit spheres S^{2N_j-1}, each with its invariant probability measure. Write $|\cdot|_2$ for the Euclidean norm and $H_k := \sum_{j=1}^{k} 1/j$, with $H_0 = 0$.

Problem 27.1.1. $M(a\prod_i (x-\alpha_i)) = |a|\prod_i \max(1,|\alpha_i|)$, and $M(PQ) = M(P)M(Q)$. Hence if a binary form factors as $F = \prod_i(\beta_i x - \alpha_i y)$, then $M(F(x,1)) = \prod_i \max(|\alpha_i|,|\beta_i|)$.

Problem 27.1.2. Order the monomials of $P \neq 0$ lexicographically with x_N most significant: compare the exponents of x_N first, then those of x_{N-1}, and so on. Let a be the coefficient of the largest monomial of P. Then $m(P) \ge \log|a|$. In particular $M(P) \ge 1$ for every nonzero $P \in \mathbb{Z}[x_1,\dots,x_N]$.

Problem 27.1.3. Let z be uniform on the unit sphere of $\mathbb{C}^N$, $N \ge 2$. Then $|z_1|^2$ has density $(N-1)(1-t)^{N-2}$ on $[0,1]$, and $\mathbb{E}\log|z_1|^2 = -H_{N-1}$.

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The measure m_S on spheres was introduced by Philippon [Ph1]. Lelong [Le] proved that $0 \le m(G) - m_S(G) \le \frac{1}{2} H_{N-1}\, k$ for every $G \in \mathbb{C}[u_1, \dots, u_N]$ homogeneous of degree k; see [KPS, (1.2)]. We need the lower bound in the following form.

Problem 27.1.4. If $G \in \mathbb{Z}[u_1, \dots, u_N]$ is nonzero and homogeneous of degree k, then $m_S(G) \ge -\frac{1}{2} H_{N-1}\, k$.

Problem 27.1.5. For $p \in \mathbb{C}^N \setminus 0$, $m_S(u_1 p_1 + \cdots + u_N p_N) = \log|p|_2 - \frac{1}{2} H_{N-1}$.

27.2. Points on the projective line

The smallest case is a hypersurface in $\mathbb{P}^1$, a finite set of points, where the covolume is exact. Let $F \in \mathbb{Z}[x, y]$ be a primitive binary form of degree $\delta \ge 1$ with $\operatorname{disc} F \ne 0$, and factor $F = \prod_{i=1}^{\delta} (\omega_{i1} x - \omega_{i0} y)$ with $\omega_i \in \mathbb{C}^2$. The ω_i are determined up to $\omega_i \mapsto t_i \omega_i$ with $\prod t_i = 1$, and the points $X = \{(\omega_{i0} : \omega_{i1})\}$ are the zeros of F. The polynomial

$$f_X(u_0, u_1) := \prod_i (u_0 \omega_{i0} + u_1 \omega_{i1}) = F(u_1, -u_0)$$

vanishes exactly when the hyperplane $u_0 x + u_1 y = 0$ meets X; it is the *Chow form* of X.

Problem 27.2.1. We have

$$m_S(f_X) = \sum_i \log|\omega_i|_2 - \tfrac{1}{2}\delta, \qquad m(F) = \sum_i \log \max(|\omega_{i0}|, |\omega_{i1}|),$$

so that $m(F) \le m_S(f_X) + \frac{1}{2}\delta \le m(F) + \frac{1}{2}\delta \log 2$. For $\delta = 1$ and $F = qx - py$, $m(F)$ is the Weil height of $(p : q)$.

For $D \ge \delta - 1$ let $\Lambda_D := \mathbb{Z}[x, y]_D / F\, \mathbb{Z}[x, y]_{D-\delta}$, the lattice of sections of $\mathcal{O}(D)$ on X, with

$$\|g\|^2 := \sum_{i=1}^{\delta} \frac{|g(\omega_i)|^2}{|\omega_i|_2^{2D}}.$$

Problem 27.2.2. $F\, \mathbb{Z}[x, y]_{D-\delta} = \mathbb{Z}[x, y]_D \cap F\, \mathbb{Q}[x, y]_{D-\delta}$, so Λ_D is free of rank δ, and $\|\cdot\|$ is a Euclidean norm on $\Lambda_D \otimes \mathbb{R}$. For linearly independent $g_1, \dots, g_\delta \in \Lambda_D$,

$$|\det(g_j(\omega_i))| \big/ \textstyle\prod_i |\omega_i|_2^D$$

is the covolume of the lattice they span.

Let a be the coefficient of x^δ in F, and suppose $a \ne 0$. Let $V := \prod_{i<i'} (\omega_{i'0}\omega_{i1} - \omega_{i0}\omega_{i'1})$.

Problem 27.2.3. The images of $x^j y^{D-j}$, $0 \le j < \delta$, span a sublattice of index $|a|^{D-\delta+1}$ in Λ_D.

Problem 27.2.4. $\det\bigl(\omega_{i0}^j \omega_{i1}^{D-j}\bigr)_{1 \le i \le \delta,\, 0 \le j < \delta} = \pm a^{D-\delta+1} V$, and $V^2 = \operatorname{disc} F$, where $\operatorname{disc} F = a^{2\delta-2} \prod_{i<i'} (\alpha_i - \alpha_{i'})^2$ for $F = a \prod_i (x - \alpha_i y)$.

So for a basis g_j of Λ_D the determinant $\det(g_j(\omega_i))$ is $\pm V$: the index of Problem 27.2.3 cancels the power of a in Problem 27.2.4. When $a = 0$ one first applies a substitution in $\mathrm{GL}_2(\mathbb{Z})$, which changes $\det(g_j(\omega_i))$ and V at most by a sign. With Problems 27.2.1 and 27.2.2 this proves the following.

Theorem 27.2.1. *For $D \geq \delta - 1$,*

$$\deg \Lambda_D = D \sum_i \log|\omega_i|_2 - \tfrac{1}{2}\log|\operatorname{disc} F| = D\big(m_S(f_X) + \tfrac{1}{2}\delta\big) - \tfrac{1}{2}\log|\operatorname{disc} F|.$$

Problem 27.2.5. For $F = x^2 + y^2$, $\deg \Lambda_D = (D-1)\log 2$. For $F = x^2 - 2y^2$, $\deg \Lambda_D = D\log 3 - \frac{3}{2}\log 2$; for $D = 1$ this is $0.0588\ldots$

The slope grows linearly in D with coefficient $\frac{1}{\delta}\big(m_S(f_X) + \frac{1}{2}\delta\big)$, a Mahler measure of the Chow form plus a constant. For a curve there is no such closed form.

27.3. Chow forms

Let $X \subset \mathbb{P}^n_{\mathbb{Q}}$ be irreducible of dimension d and degree δ. The $(d+1)$-tuples of hyperplanes $u^{(0)}, \ldots, u^{(d)}$ with a common zero on $X(\mathbb{C})$ form a hypersurface defined over $\mathbb{Q}$ and irreducible over $\mathbb{Q}$. Its equation $f_X \in \mathbb{Z}[u^{(0)}, \ldots, u^{(d)}]$, taken primitive, is irreducible over $\mathbb{Q}$ and multihomogeneous of degree δ in each block; it is the Chow form of X, unique up to sign [GKZ, Ch. 3, §2 and Ch. 4, §1]. For a hypersurface $X = V(F)$, f_X is F evaluated at the vector of signed $n \times n$ minors of $(u^{(0)}, \ldots, u^{(n-1)})$, the common zero of the n hyperplanes. Philippon used Mahler measures of f_X as heights [Ph1]. He also proved that the Fubini–Study height of Bost, Gillet and Soulé [BGS], the arithmetic self-intersection of $\bar{\mathcal{O}}(1)$ on X, is

$$h_{\mathrm{FS}}(X) = m_S(f_X) + \tfrac{1}{2}(d+1)\,\delta\, H_n \tag{27.1}$$

[Ph3]. By Problem 27.2.1, Theorem 27.2.1 reads $\deg \Lambda_D = D h_{\mathrm{FS}}(X) - \frac{1}{2}\log|\operatorname{disc} F|$.

From now on $C = V(F) \subset \mathbb{P}^2$, with $F \in \mathbb{Z}[x_0, x_1, x_2]$ primitive and irreducible of degree δ. Lines are $s(x) = s_0x_0 + s_1x_1 + s_2x_2$, and $s \times l$ is the cross product.

Problem 27.3.1. $f_C(s, l) = F(s \times l)$. If the line $s = 0$ meets C in δ distinct points $P_1, \ldots, P_\delta \in \mathbb{C}^3$, then $f_C(s, l) = c(s)\prod_j l(P_j)$ with $c(s) \neq 0$.

The hypothesis holds for s outside a proper Zariski-closed set: through a point $Q \notin C$ only finitely many lines are tangent to C or pass through a singular point, by Bézout applied to C and the polar curve $\sum_k Q_k\, \partial F/\partial x_k = 0$.

Problem 27.3.2. For the line $x_2 = 0$ in $\mathbb{P}^2$, $f = s_0l_1 - s_1l_0$ and $m_S(f) = -1$ on $S^5 \times S^5$. For $\mathbb{P}^1$ itself, with $f = s_0l_1 - s_1l_0$ on $S^3 \times S^3$, $m_S(f) = -\frac{1}{2}$. In both cases (27.1) gives $\frac{1}{2}$, the value of formula (1) of Chapter 26 for $N = 1$.

Now let the line move, so that the model changes while the curve over $\mathbb{C}$ stays a projective line. For an integer $N \geq 0$ let $C_N \subset \mathbb{P}^2$ be the line $x_2 = Nx_0$; thus C_0 is the line of Problem 27.3.2. Let $H^0(C_N, \mathcal{O}(T))$ be the lattice of restrictions of $\mathbb{Z}[x_0, x_1, x_2]_T$ to C_N, with the norm

$$\|g\|^2 := \int_{C_N(\mathbb{C})} \frac{|g(z)|^2}{|z|_2^{2T}}\, d\mu_N(z),$$

where μ_N is the probability measure of the metric induced by the Fubini–Study metric of $\mathbb{P}^2$. For $N = 0$ this is the lattice E_T of Chapter 26.

Problem 27.3.3. The restrictions of $x_0^i x_1^{T-i}$, $0 \leq i \leq T$, form an orthogonal $\mathbb{Z}$-basis of $H^0(C_N, \mathcal{O}(T))$, and

$$\deg H^0(C_N, \mathcal{O}(T)) - \deg H^0(C_0, \mathcal{O}(T)) = \tfrac{1}{4}T(T+1)\log(1+N^2).$$

Problem 27.3.4. $f_{C_N}(s,l) = (s\times l)_2 - N(s\times l)_0$, which is $\det(p,s,l)$ for $p = (-N,0,1)$, and $h_{\mathrm{FS}}(C_N) - h_{\mathrm{FS}}(C_0) = \frac{1}{2}\log(1+N^2)$.

By Theorem 26.2.1 with $d=1$, the coefficient of T^2 in $\deg H^0(C_N,\mathcal{O}(T))$ is $\frac{1}{2}h_{\mathrm{FS}}(C_N)$. The change of height in Problem 27.3.4 therefore predicts a change $\frac{1}{4}\log(1+N^2)$ in that coefficient, which is the coefficient of T^2 in Problem 27.3.3. The height determines the leading term exactly; the term $\frac{1}{4}T\log(1+N^2)$ is invisible to it.

27.4. A determinant of sections of the projective line

What does Hadamard's inequality lose when the sections are not orthogonal? Take $X=\mathbb{P}^1$, so that $\Lambda_D = E_D$ with the norm of Chapter 26. For $s,l\in\mathbb{C}^2$ write also $s=s_0x+s_1y$ and $l=l_0x+l_1y$, and consider the sections $s^a l^{D-a}$, $0\le a\le D$.

Problem 27.4.1. In the monomial basis, $\det\bigl(s^a l^{D-a}\bigr)_{0\le a\le D} = \pm(s_0l_1 - s_1l_0)^{D(D+1)/2}$.

Problem 27.4.2. If s and l are orthonormal, the columns are orthogonal in $E_D\otimes\mathbb{C}$, Hadamard's inequality is an equality, and it gives Problem 26.1.1 again.

Problem 27.4.3. Bounding the norm of each section $s^a l^{D-a}$ by 1 for unit vectors s,l, and averaging over s,l in the unit spheres, gives $\deg E_D \ge -\frac{1}{4}D(D+1)$. For $D=10$ this is -27.5, against $\deg E_{10} = 32.1969\ldots$

So at an orthonormal pair nothing is lost, and the main term $\frac{1}{4}D^2$ of Theorem 26.1.1 comes from the small norms of the columns, about $\binom{D}{a}^{-1/2}$. Averaging over s and l loses $\frac{1}{2}D^2$, because it spreads the pair over pairs whose sections are far from orthogonal. For a curve $C\subset\mathbb{P}^2$ no orthogonal family of this kind is known, and the argument described in [Adr, proof dependency map, item HSAt] uses Chow forms and a determinant estimate. The argument needs a lower bound of this kind, with explicit constants, on varieties of growing dimension, where Section 26.4 recorded that such bounds lose a factor exponential in the dimension.

Hints

Section 27.1.

- **27.1.1:** Jensen's formula for $x-\alpha$. For $\beta_i=0$ the factor is a constant.
- **27.1.2:** Induction on N, using Problem 27.1.1 in the last variable for fixed values of the others.
- **27.1.3:** Write $z=g/|g|_2$ with g a standard complex Gaussian vector; $|g_k|^2$ are independent exponential variables.
- **27.1.4:** The measure on the sphere is invariant under the torus. For fixed moduli $r_k=|u_k|$ apply Problem 27.1.2 to $G(r_1x_1,\ldots,r_Nx_N)$, then average over the moduli with Problem 27.1.3.
- **27.1.5:** Unitary invariance moves $\bar p$ to a multiple of the first basis vector.

Section 27.2.

- **27.2.1:** Problems 27.1.5 and 27.1.1; $m(F)=m(F(x,1))$, since F is homogeneous.
- **27.2.2:** Gauss's lemma. Evaluation at the ω_i, divided by $|\omega_i|_2^D$, is an isometry of $\Lambda_D\otimes\mathbb{C}$ onto $\mathbb{C}^\delta$.

27.2.3: Order the monomials by the power of x and add the columns $Fx^k y^{D-\delta-k}$.

27.2.4: Factor $\omega_{i1}^{D-\delta+1}$ out of row i. Write $\omega_i = t_i(\alpha_i, 1)$ with $\prod t_i = a$.

27.2.5: Theorem 27.2.1.

Section 27.3.

27.3.1: For fixed s, $l \mapsto s \times l$ maps the lines onto the points of the line $s = 0$.

27.3.2: Fix s and apply Problem 27.1.5 in l; then average $\log(|s_0|^2 + |s_1|^2)$ with Problem 27.1.3.

27.3.3: Substitute $x_2 = Nx_0$ for the basis. For the norm, put $u := \sqrt{1+N^2}\, x_0$ and $v := x_1$. On C_N, $|z|_2^2 = |u|^2 + |v|^2$, so $(u : v)$ identifies $C_N(\mathbb{C})$, with its metric, with $\mathbb{P}^1(\mathbb{C})$.

27.3.4: Problem 27.3.1, then a matrix in $\mathrm{SU}(3)$ taking $p/|p|_2$ to $(0, 0, 1)$, applied to s and l; then Problem 27.3.2 and (27.1).

Section 27.4.

27.4.1: The matrix is Sym^D of a 2×2 matrix; check diagonal matrices first.

27.4.2: The Fubini–Study norm is invariant under $\mathrm{U}(2)$, which moves (s, l) to (y, x).

27.4.3: Hadamard's inequality in $E_D \otimes \mathbb{C}$, with $|s(z)| \leq |z|_2$; then Problems 27.4.1, 27.3.2 and 26.1.5.

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