HECKE OPERATORS AND THETA LIFTS

Prerequisites. Even unimodular lattices, E_8, theta series and the Hecke operators T_n on modular forms of level one [SeA, Ch. V and VII]. Quadratic forms over $\mathbb{F}_p$ and the number of their isotropic vectors (Problem 22.2.2). Gaussian binomial coefficients $\begin{bmatrix} a \\ b \end{bmatrix}_Q$, which count the b-dimensional subspaces of $\mathbb{F}_Q^a$ [Andr, Ch. 3]. From Chapter 21, *Masses and the count of forms*: Section 21.1, in particular Problems 21.1.1, 21.1.2 and 21.2.3. From Chapter 22, *The Siegel–Weil formula in the smallest case*: Problems 22.1.2 and 22.3.7. From Chapter 23, *Definite theta lifts and their norms*: Section 23.1. For Section 24.4, maximal orders in a definite quaternion algebra over $\mathbb{Q}$ [Gr], [Pi].

A theta lift is a sum over classes of lattices. A Hecke operator on a class set is also a sum over lattices, namely over the neighbours of a lattice. How do the two interact? Eichler found that a theta series summed over the neighbours of a lattice is a Hecke transform of the theta series of the lattice, up to an explicit multiple of the identity [Ei52], [Ei73]; see [An] for Siegel modular forms and [Ra] for the general theta correspondence. We prove this in degree one, and check it on class sets with two elements, where the Hecke operator is a 2×2 matrix.

24.1. Neighbours

How does one move from a lattice to the other lattices of its genus? Kneser's answer is to change it at one prime [Kn].

Let L be an even unimodular lattice of rank $n = 2k$, with $8 \mid n$, and let p be a prime. A *p-neighbour* of L is an even unimodular lattice $M \subset L \otimes \mathbb{Q}$ with $[L : L \cap M] = [M : L \cap M] = p$. Let $\mathcal{N}_p(L)$ be the set of p-neighbours.

Problem 24.1.1. Let p be odd. For $v \in L \setminus pL$ with $v \cdot v \equiv 0 \pmod{2p^2}$, let $L_v := \{x \in L : x \cdot v \equiv 0 \pmod{p}\}$. Then $M_v := L_v + \mathbb{Z}\, v/p$ is a p-neighbour of L, and M_v depends only on the line $\mathbb{F}_p \bar{v} \subset L/pL$. Every p-neighbour arises in this way, from exactly one isotropic line of L/pL.

Problem 24.1.2. The quadratic space L/pL over $\mathbb{F}_p$ is split, so that

$$|\mathcal{N}_p(L)| = \frac{(p^k - 1)(p^{k-1} + 1)}{p - 1}.$$

For $L = E_8$ and $p = 3, 5$ this is 1120 and 19656.

For $p = 2$ the same count holds, if v is chosen with $v \cdot v \equiv 0 \pmod 8$ [CL]; for E_8 it is 135. Since E_8 is the only even unimodular lattice of rank 8, all its neighbours are copies of E_8. For $n \geq 3$, Kneser proved that any two lattices of one genus are joined by a chain of p-neighbours [Kn].

Let Cl be a class set, with $\Gamma_L := \mathrm{O}(L)$. The *neighbour operator* on functions $\varphi \colon \mathrm{Cl} \to \mathbb{C}$ is $(T_p\varphi)(L) := \sum_{M \in \mathcal{N}_p(L)} \varphi(M)$, and its matrix $N_p(L, M) := \#\{M' \in \mathcal{N}_p(L) : M' \cong M\}$ is the *Brandt matrix* of the class set at p. Let $\langle \varphi, \psi \rangle := \sum_{L \in \mathrm{Cl}} \varphi(L)\overline{\psi(L)}/|\Gamma_L|$, as in Chapter 21.

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Problem 24.1.3. $N_p(L,M)/|\Gamma_L| = N_p(M,L)/|\Gamma_M|$. Hence T_p is self-adjoint for $\langle\,,\rangle$, and the constant function is an eigenvector with eigenvalue $|\mathcal{N}_p(L)|$.

In rank 16 the class set is $\{E_8 \perp E_8,\ D_{16}^+\}$, with $|\Gamma| = 2\cdot 696729600^2$ and $2^{15}\cdot 16!$ [CoS]. By a machine enumeration of the 32895 two-neighbours of $E_8 \perp E_8$, sorted by whether their roots span the lattice, 14670 of them are isometric to $E_8 \perp E_8$ and 18225 to D_{16}^+.

Problem 24.1.4. The Brandt matrix of the class set of rank 16 at $p = 2$ is $\begin{pmatrix} 14670 & 18225 \\ 12870 & 20025 \end{pmatrix}$, with eigenvalues 32895 and 1800.

The eigenvalue 1800 is $15\,\tau(2) + (2^4 + 2^5 + \cdots + 2^{10}) + 2^7$, where $\tau(2) = -24$ is the coefficient of Ramanujan's Δ. Chenevier and Lannes explain such identities by the Arthur parameters of the eigenvectors [CL]. In Section 24.2 we show that the theta series of degree one cannot see this eigenvector.

24.2. Eichler's commutation relation in degree one

For L as in Section 24.1, let $r_L(m) := \#\{x \in L : x\cdot x = 2m\}$, with $r_L(m) := 0$ unless $m \in \mathbb{Z}_{\geq 0}$, and let $\theta_L := \sum_m r_L(m)\, q^m$. It is a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ [SeA, Ch. VII], and T_{p^2} acts on its coefficients by

$$a(m) \mapsto a(p^2 m) + p^{k-1} a(m)\,[p \mid m] + p^{2k-2} a(m/p^2).$$

We compute $\sum_{M\in\mathcal{N}_p(L)} r_M(m)$ by counting, for each vector, the neighbours that contain it. Let p be odd.

Problem 24.2.1. Let $x \in L \setminus pL$ with $x\cdot x = 2m$. The number of $M \in \mathcal{N}_p(L)$ with $x \in M$ is $(p^{2k-2}-1)/(p-1)$ if $p \nmid m$, and $1 + p(p^{k-1}-1)(p^{k-2}+1)/(p-1)$ if $p \mid m$.

Problem 24.2.2. The vectors of $\bigcup_{M\in\mathcal{N}_p(L)} M \setminus L$ are the y/p with $y \in L\setminus pL$ and $y\cdot y \equiv 0 \pmod{2p^2}$, and each lies in exactly one neighbour.

Problems 24.2.1 and 24.2.2, together with $x \in pL \subset M$ for every M, give the following.

Theorem 24.2.1 (Eichler)**.** *For odd p,* $$\sum_{M\in\mathcal{N}_p(L)} \theta_M = T_{p^2}\theta_L + \frac{p^{2k-2}-p}{p-1}\,\theta_L.$$

Problem 24.2.3. Theorem 24.2.1 for $L = E_8$ and $p = 3$ reads $1120 = 757 + 363$. The same identity of constant terms holds for $p = 2$: $135 = 73 + 62$.

Let $\theta(\varphi) := \sum_{L\in\mathrm{Cl}} \varphi(L)\,\theta_L/|\Gamma_L|$ be the *theta lift* of a function on the class set. It is the analogue for these lattices of the lift of Chapter 23 to rank one in weight zero, with the vectors sorted by norm into a q-series instead of weighted by a Gaussian.

Problem 24.2.4. Let p be odd. Then $\theta(T_p\varphi) = \big(T_{p^2} + \frac{p^{2k-2}-p}{p-1}\big)\theta(\varphi)$. If $\theta(\varphi) \neq 0$ and $T_p\varphi = \mu\varphi$, and $\theta(\varphi)$ is an eigenform with $T_p\theta(\varphi) = \lambda\theta(\varphi)$, then $\mu = \lambda^2 - p^{k-1} + (p^{2k-2}-p)/(p-1)$.

So the theta lift intertwines T_p on the class set with a polynomial in T_p on modular forms; the explicit factor is the constant $(p^{2k-2}-p)/(p-1)$.

In rank 16, $\theta_{E_8\perp E_8} = \theta_{D_{16}^+}$ by Problem 22.1.2 of Chapter 22.

Problem 24.2.5. The eigenvector of Problem 24.1.4 with eigenvalue 1800 lies in the kernel of θ. For the constant function, the eigenvalue 32895 agrees with the formula of Problem 24.2.4 read at $p = 2$, where we have not proved it: $129^2 - 2^7 + (2^{14} - 2) = 32895$.

The two lattices are distinguished by their theta series of degree 4, and not before [CL]. A lift that kills an eigenvector says nothing about its eigenvalue; one needs a lift to a larger group, as in the Rallis inner product formula (Section 23.3).

24.3. Hermitian lattices of rank five

Let $\mathcal{O} = \mathbb{Z}[\mathrm{i}]$, let $p \equiv 1 \pmod 4$ be a prime, and let $p = \pi\bar\pi$ in $\mathcal{O}$. Let L be a unimodular Hermitian lattice of rank n, as in Chapter 21, and let $L^\sharp$ denote duals. Then $L/\pi L$ is an n-dimensional vector space over $\mathcal{O}/\pi = \mathbb{F}_p$. For a hyperplane $H \subset L/\pi L$ let L_H be its preimage in L, and

$$L'_H := L_H + \pi L_H^\sharp.$$

Problem 24.3.1. L'_H is unimodular and lies in the genus of L, with $L \cap L'_H = L_H$ of index p in both. Moreover $L'_H \subset \bar\pi^{-1}L$, and L is obtained from L'_H by the same construction with $\bar\pi$ in place of π.

We call L'_H a *π-neighbour* of L. There are $(p^n-1)/(p-1)$ of them, and $T_\pi\varphi(L) := \sum_H \varphi(L'_H)$ defines an operator on functions on Cl_n. By Problem 24.3.1 and the argument of Problem 24.1.3, $T_{\bar\pi}$ is the adjoint of T_π. Let $r_L(m) := \#\{x \in L : h(x,x) = m\}$ and $\theta_L := \sum_m r_L(m)\, q^m$, a modular form of weight n for $\Gamma_0(4)$ with character χ^n, where χ is the character modulo 4 [Iw, Ch. 10]. Since $\chi(p) = 1$, T_p acts on such forms by $a(m) \mapsto a(pm) + p^{n-1}a(m/p)$.

Problem 24.3.2. For $x \in L$, the number of H with $x \in L'_H$ is $(p^{n-1}-1)/(p-1)$ if $x \notin \pi L$, and $(p^n-1)/(p-1)$ if $x \in \pi L$. A vector $x \notin L$ lies in some L'_H exactly when $y := \bar\pi x \in L \setminus \bar\pi L$ and $p \mid h(y,y)$, and then in exactly one.

Problem 24.3.3. $\displaystyle\sum_H \theta_{L'_H} = T_p\theta_L + \frac{p^{n-1}-p}{p-1}\,\theta_L$. Hence, with $\theta(\varphi) := \sum_{L \in \mathrm{Cl}_n} \varphi(L)\theta_L/|\Gamma_L|$, one has $\theta(T_\pi\varphi) = \big(T_p + \frac{p^{n-1}-p}{p-1}\big)\theta(\varphi)$, and likewise for $\bar\pi$.

Here the factor is linear in T_p, because at a split prime the local group is $\mathrm{GL}_n(\mathbb{Q}_p)$ and a π-neighbour is an edge of type one in the sense of Section 24.5.

Now take $n = 5$, $p = 5$ and $\pi = 2 + \mathrm{i}$. By Problem 21.2.3, $\mathrm{Cl}_5 = \{I_5, E \perp I_1\}$, with $|\Gamma| = 122880$ and 184320, and $r(1) = 20$ and 4.

Problem 24.3.4. The matrix of T_π on Cl_5 is $\begin{pmatrix} 525 & 256 \\ 384 & 397 \end{pmatrix}$, with eigenvalues 781 and 141.

We checked this matrix by constructing all 781 neighbours of each lattice by machine. Let g be the cusp form called f in Chapter 22,

$$g = \frac{1}{4}\sum_{\alpha \in \mathcal{O}} \alpha^4 q^{|\alpha|^2} = \eta(\tau)^4\eta(2\tau)^2\eta(4\tau)^4 = q - 4q^2 + 16q^4 - 14q^5 - 64q^8 + 81q^9 + \cdots;$$

here η is Dedekind's function. The form g spans the cusp forms of weight 5 for $\Gamma_0(4)$ with character χ, and it has complex multiplication by $\mathbb{Q}(\mathrm{i})$. By Problem 22.3.7, $\theta_{I_5} - \theta_{E\perp I_1} = 16\,g$.

Problem 24.3.5. The eigenvector of T_π with eigenvalue 141 is $\varphi = (2, -3)$, and $\theta(\varphi) = g/3840$.

Problem 24.3.6. $a_5(g) = -14 = 2\,\mathrm{Re}\big((2+\mathrm{i})^4\big)$, and $141 = -14 + 155$, as Problem 24.3.3 predicts.

Here the lift is injective, so the matrix of T_π is that of $T_5 + 155$ on the span of θ_{I_5} and $\theta_{E \perp I_1}$. Neither theta series is an eigenform; the eigenforms are g and the lift of the constant function, with T_5-eigenvalues -14 and $626 = 781 - 155$.

24.4. Brandt matrices

The original setting of Eichler's relation is a quaternion algebra, where the theta series of the norm forms of maximal orders give the modular forms of weight two and prime level [Ei73], [Pi], [Gr]. We follow Gross [Gr, §§1–2] in the smallest case with two classes.

Let $B = (-1, -11)/\mathbb{Q}$, with $\mathrm{i}^2 = -1$, $j^2 = -11$ and $k = \mathrm{i}j$; it is ramified exactly at 11 and ∞. Let $\mathcal{O}_1 := \mathbb{Z} + \mathbb{Z}\mathrm{i} + \mathbb{Z}\frac{1+j}{2} + \mathbb{Z}\frac{\mathrm{i}+k}{2}$. There is one other class of maximal orders, represented by an order $\mathcal{O}_2$ with $\Theta_{\mathcal{O}_2} := \sum_{x \in \mathcal{O}_2} q^{N(x)} = 1 + 6q + 6q^3 + 24q^4 + 18q^5 + \cdots$. Let $w_i := |\mathcal{O}_i^\times|/2$. Gross defines the Brandt matrices $B(m)$, $m \geq 1$, whose entries $B_{ij}(m)$ count the elements of norm m in $\mathrm{Hom}(I_j, I_i)$, divided by $2w_j$, for left ideals I_i of a fixed maximal order with right orders $\mathcal{O}_i$; he shows that $B_{ii}(m) = r_{\mathcal{O}_i}(m)/2w_i$, that $w_j B_{ij}(m) = w_i B_{ji}(m)$, that $\sum_j B_{ij}(m) = \sigma(m)$ for $11 \nmid m$, and that $m \mapsto B(m)$ is the action of the Hecke operators T_m on $M_2(\Gamma_0(11))$ [Gr, §§1–2]. For a prime $\ell \neq 11$, $B_{ij}(\ell)$ counts the left ideals $J \subset I_i$ of the fixed order with $[I_i : J] = \ell^2$ and $J \cong I_j$; there are $\ell + 1$ such J in all, and they play the role of the ℓ-neighbours of Section 24.1.

Problem 24.4.1. $\mathcal{O}_1$ is an order, its norm form is $(a^2 + ac + 3c^2) + (b^2 + bd + 3d^2)$, and $\mathcal{O}_1^\times = \{\pm 1, \pm \mathrm{i}\}$. Thus $\Theta_{\mathcal{O}_1} = \big(\sum_{a,c} q^{a^2+ac+3c^2}\big)^2$.

Eichler's mass formula $\sum_i 1/w_i = (11-1)/12$ then gives $w_2 = 3$ [Gr].

Problem 24.4.2. $B(2) = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ and $B(3) = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$, with eigenvalues $3, -2$ and $4, -1$.

Problem 24.4.3. Let $f := q\prod_{n \geq 1}(1-q^n)^2(1-q^{11n})^2 = q - 2q^2 - q^3 + 2q^4 + q^5 + \cdots$. Then $\Theta_{\mathcal{O}_1} - \Theta_{\mathcal{O}_2} = -2f$, and $(1, -1)\,B(\ell) = a_\ell(f)\,(1, -1)$ for $\ell = 2, 3$.

The eigenvalues -2 and -1 are the traces of Frobenius of the elliptic curve $X_0(11)$ at 2 and 3.

24.5. A split place

In the chain of lifts of Adriazola's reconstruction [Adr, §10], the kernel of a lift at a split place is the indicator of $xL \subset L'$ for a matrix x, and the Hecke operators are sums over edges of lattices. The relation between them is a count of subspaces, of which Problems 24.3.2 and 24.3.3 are the case of rank one.

Let Q be a power of p, and let W be the unramified extension of $\mathbb{Z}_p$ with residue field $\mathbb{F}_Q$. For lattices $L'' \subset L$ in a $W[1/p]$-vector space, (L'', L) is an *edge of type* i if $pL \subset L'' \subset L$ and $\dim L/L'' = i$. Let $L \subset W[1/p]^N$ and $L' \subset W[1/p]^{N+g}$ be lattices, let x be an $(N+g) \times N$ matrix with $xL \subset L'$, and let ρ be the rank of x modulo p as a map $L/pL \to L'/pL'$.

Problem 24.5.1. The number of edges (L_1, L') of type i with $xL \subset L_1$ is $\begin{bmatrix} N+g-\rho \\ i \end{bmatrix}_Q$. The number of edges (L, A_1) of type i with $xA_1 \subset L'$ is $\begin{bmatrix} N-\rho \\ i \end{bmatrix}_Q$.

Problem 24.5.2. As functions of x (with $\mathbf{1}_{L,L'}(x) := [xL \subset L']$),

$$\sum_{(L_1,L') \text{ of type } i} \mathbf{1}_{L,L_1} = \sum_{l=0}^{i} Q^{(i-l)(g-l)} \begin{bmatrix} g \\ l \end{bmatrix}_Q \sum_{(L,A_1) \text{ of type } i-l} \mathbf{1}_{A_1,L'}.$$

For $N = 1$, $i = 1$ and $Q = p$ this is the first count of Problem 24.3.2.

Problem 24.5.3. For $0 \le l \le g$, $e_l(Q^{t/2}, Q^{t/2-1}, \ldots, Q^{-t/2}) = Q^{-l(g-l)/2} \begin{bmatrix} g \\ l \end{bmatrix}_Q$, where $t = g - 1$, the arguments step by Q^{-1}, and e_l is the elementary symmetric polynomial.

Summing Problem 24.5.2 against a kernel that depends on x only through $\mathbf{1}_{L,L'}(x)$ and on the lengths of the lattices through a power of p, and moving the sum over edges at the source to the other variable as in Problem 24.1.3, gives a relation of the form $E_{j+1}(X)\, K = K\, P(X)\, E_j(uX)$ between the generating series $E(X) = \sum_i \mathcal{E}_i X^i$ of normalized Hecke operators at the target and at the source. Here u is a unit and $P(X) = \prod_a (1 + aX)$ is scalar, with roots read off from Problem 24.5.3. This is the relation for one lift in [Adr, §10.4].

Problem 24.5.4. Suppose that $E_{j+1}(X)\, K_j = K_j\, P_{j+1}(X)\, E_j(X)$ modulo X^{r+1} for $0 \le j < m$, with $P_{j+1}(X) = \prod_{a \in x_{j+1}} (1 + aX)$, and where $E_j(X) = \sum_i \mathcal{E}_i^{(j)} X^i$, and let Y be the union of the families $x_1, \ldots, x_m$ and $\mathcal{K} := K_{m-1} \cdots K_0$. Then $\mathcal{K}\, \mathcal{E}_r^{(0)} = \sum_{l=0}^{r} (-1)^l h_l(Y)\, \mathcal{E}_{r-l}^{(m)}\, \mathcal{K}$, where h_l is the complete symmetric polynomial.

So a Hecke operator at the source of a chain of lifts moves to the target as a polynomial in the Hecke operators there, with coefficients $h_l(Y)$. Problem 24.2.4 is an instance with one lift. For divisibility along a chain, the valuations of these coefficients are what matter [Adr, §10.2].

Hints

§24.1.

24.1.1: v/p has integral products with L_v, and $(v/p) \cdot (v/p) \in 2\mathbb{Z}$. Compare discriminants: $[L : L_v] = p$. Conversely $M \cap L = L_v$ for any $v \in pM \setminus pL$. Changing v by pL_v or by a unit does not change M_v; to lift an isotropic $\bar{v}$ to $v \cdot v \equiv 0 \pmod{2p^2}$, adjust by pw with $w \cdot v \not\equiv 0$.

24.1.2: The discriminant of L/pL is $(-1)^k$ times a square. A split space of dimension $2k$ has $p^{2k-1} + p^k - p^{k-1}$ isotropic vectors, counting 0.

24.1.3: Count the Γ-orbits of pairs (L, M) with $M \in \mathcal{N}_p(L)$, weighted by the inverse order of the stabilizer of the pair, in two ways. The relation is symmetric because $L \in \mathcal{N}_p(M)$.

24.1.4: Row sums are 32895; use Problem 24.1.3 for the second row.

§24.2.

24.2.1: By Problem 24.1.1, $x \in M_v$ exactly when $\bar{x} \perp \bar{v}$. Count isotropic lines in $\bar{x}^\perp$: if $\bar{x}$ is anisotropic, $\bar{x}^\perp$ is non-degenerate of odd dimension $2k-1$, with p^{2k-2} isotropic vectors (split off hyperplanes and use Problem 22.2.2); if $\bar{x}$ is isotropic, $\bar{x}^\perp = \mathbb{F}_p \bar{x} \oplus W$ with W split of dimension $2k-2$.

24.2.2: Use the description of M_v in Problem 24.1.1.

24.2.3: $\sigma_3(9) = 757$ and $\sigma_3(4) = 73$.

24.2.4: Exchange the sums by Problem 24.1.3, then use Theorem 24.2.1. On an eigenform, $T_{p^2} = T_p^2 - p^{k-1}$.

24.2.5: $M_8(\mathrm{SL}_2(\mathbb{Z}))$ is spanned by E_4^2, and $T_2 E_4^2 = (1 + 2^7)E_4^2$. The eigenvector is orthogonal to the constants.

§24.3.

24.3.1: Work locally. Over $\mathbb{Z}_p$, $\mathcal{O} \otimes \mathbb{Z}_p = \mathbb{Z}_p \times \mathbb{Z}_p$, and a self-dual lattice is $\Lambda \oplus \Lambda^\vee$ for a $\mathbb{Z}_p$-lattice Λ.

24.3.2: Use the local description of L'_H from Problem 24.3.1, one component at a time.

24.3.3: Sum Problem 24.3.2 over the vectors of norm m, using $h(\bar\pi y, \bar\pi y) = p\,h(y,y)$ and $r_{\pi L}(m) = r_L(m/p)$.

24.3.4: Compare the coefficients of q in Problem 24.3.3 for $L = I_5$, using $r_{I_5}(5) = 8424$ and the values of $r(1)$. For the second row, do the same for $L = E \perp I_1$, with $r_{E\perp I_1}(5) = 8648$; or use Problem 24.1.3 together with complex conjugation, which exchanges π- and $\bar\pi$-neighbours and fixes both classes.

24.3.5: Use $122880 = 2 \cdot 61440$ and $184320 = 3 \cdot 61440$.

24.3.6: Compute $a_5(g)$ from the product. Then apply Problem 24.3.3 to φ and compare with Problem 24.3.5.

§24.4.

24.4.1: Write an element as $(a + c/2) + (b + d/2)\mathrm{i} + (c/2)j + (d/2)k$.

24.4.2: Find the diagonal entries from $r_{\mathcal{O}_1}(m)$ and from $\Theta_{\mathcal{O}_2}$, and the others from the row sums. Check $w_2 B_{12} = w_1 B_{21}$.

24.4.3: Both sides lie in $M_2(\Gamma_0(11))$, which has dimension 2; compare the coefficients of q^0, q^1, q^2.

§24.5.

24.5.1: $L_1 \supset pL'$ corresponds to a subspace of L'/pL' of codimension i containing the image of x. $A_1 \subset p^{-1}L$ corresponds to the subspace pA_1/pL of L/pL, which must lie in the kernel of x modulo p.

24.5.2: Problem 24.5.1 and the q-Vandermonde identity [Andr, Ch. 3]

$$\begin{bmatrix} a+b \\ i \end{bmatrix}_Q = \sum_l Q^{(i-l)(b-l)} \begin{bmatrix} b \\ l \end{bmatrix}_Q \begin{bmatrix} a \\ i-l \end{bmatrix}_Q .$$

24.5.3: Use the Gaussian binomial theorem

$$\prod_{s=0}^{g-1} (1 + Q^s z) = \sum_l Q^{l(l-1)/2} \begin{bmatrix} g \\ l \end{bmatrix}_Q z^l .$$

24.5.4: Work with power series in X modulo X^{r+1}; the scalars commute with the operators, and $\prod_{a\in Y}(1 + aX)^{-1} = \sum_l (-1)^l h_l(Y) X^l$.

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