SIEGEL–WEIL IN THE SMALLEST CASE

Prerequisites. Modular forms for $\mathrm{SL}_2(\mathbb{Z})$, theta series of even unimodular lattices, and the mass formula, as in [SeA, Ch. V, §2.3 and Ch. VII, §§2–6]. Quadratic forms over finite fields [SeA, Ch. IV, §1.7] and Hensel's lemma [SeA, Ch. II, §2.2]. For §22.3: the space of modular forms of weight n on $\Gamma_0(4)$ with character χ^n, its dimension and its basis in terms of θ [Ko, Ch. IV, §1], and its Eisenstein series [DS, §4.5]. From Chapter 21, *Masses and the count of forms*: the class sets Cl_n, the groups Γ_L, the masses M_n, Problems 21.1.2, 21.1.4 and 21.2.3, and the forms of §21.3 with Problem 21.3.2.

A class set is a finite set of lattices, and an individual lattice in it can be complicated. The average over the class set, each lattice weighted by the inverse of the order of its automorphism group, is simple: this is Siegel's theorem, and in Weil's form it says that the mass-weighted average of theta series is an Eisenstein series. We check it where it can be checked by hand, then on the class set of Chapter 21 in rank 5, and then say why the Gram matrices of the theta lifts in the argument are controlled by it.

22.1. Even unimodular lattices

Let L be an even unimodular positive definite lattice of rank m, with inner product $x \cdot x$, and let $r_L(j)$ be the number of $x \in L$ with $x \cdot x = 2j$. Then $8 \mid m$, and $\theta_L(\tau) := \sum_{j \geq 0} r_L(j) q^j$, $q = e^{2\pi \mathrm{i} \tau}$, is a modular form of weight $k := m/2$ for $\mathrm{SL}_2(\mathbb{Z})$ [SeA, Ch. VII, Theorem 8]. Let $\mathcal{M}_k$ be the space of such forms and

$$E_k := 1 - \frac{2k}{B_k} \sum_{j \geq 1} \sigma_{k-1}(j) q^j, \qquad \Delta := q \prod_{j \geq 1} (1 - q^j)^{24} = \sum_{j \geq 1} \tau(j) q^j,$$

so that $E_4 = 1 + 240 \sum \sigma_3(j) q^j$, $E_8 = 1 + 480 \sum \sigma_7(j) q^j$, $E_{12} = 1 + \frac{65520}{691} \sum \sigma_{11}(j) q^j$. (Serre writes E_2, E_4, E_6 for these, indexing by half the weight.) The spaces $\mathcal{M}_4$ and $\mathcal{M}_8$ are spanned by E_4 and E_8, and $\mathcal{M}_{12}$ by E_{12} and Δ [SeA, Ch. VII, §3.2]. We write E_8 also for the root lattice; the context decides. Let $D_{16}^+ := D_{16} \cup (D_{16} + (\frac{1}{2}, \ldots, \frac{1}{2}))$, where $D_{16} = \{x \in \mathbb{Z}^{16} : \sum x_i \text{ even}\}$ [CoS, Ch. 4, §7].

Problem 22.1.1. $r_{E_8}(j) = 240\, \sigma_3(j)$ for $j \geq 1$. In particular E_8 has 240 roots and 2160 vectors with $x \cdot x = 4$.

Problem 22.1.2. $\theta_{E_8 \oplus E_8} = \theta_{D_{16}^+} = E_8$. Both lattices have 480 roots and 61920 vectors with $x \cdot x = 4$, and $\sigma_7(j) = \sigma_3(j) + 120 \sum_{i=1}^{j-1} \sigma_3(i) \sigma_3(j-i)$.

Problem 22.1.3. $E_8 \oplus E_8$ and D_{16}^+ are not isometric.

So the theta series does not determine the lattice. Milnor observed that the tori $\mathbb{R}^{16}/(E_8 \oplus E_8)$ and $\mathbb{R}^{16}/D_{16}^+$ therefore have the same Laplace spectrum without being isometric [Mi].

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Problem 22.1.4. If Λ is an even unimodular lattice of rank 24 without roots (the Leech lattice [CoS, Ch. 4, §11]), then $\theta_\Lambda = E_{12} - \frac{65520}{691}\Delta$, and Λ has 196560 vectors with $x \cdot x = 4$.

In rank 24 the theta series is $E_{12} + c_L\Delta$ with $c_L = r_L(1) - \frac{65520}{691}$ [SeA, Ch. VII, §6.6], and c_L varies with L. Let C_m be the set of isometry classes of even unimodular lattices of rank m, let $g_L := |O(L)|$, and let $\mu_m := \sum_{L\in C_m} g_L^{-1}$ be the mass. Siegel proved [Si] that

$$\sum_{L\in C_m} \frac{\theta_L}{g_L} = \mu_m\, E_{m/2}, \tag{1}$$

see [SeA, Ch. VII, §6.5, Remark], and the Minkowski–Siegel mass formula [SeA, Ch. V, §2.3]

$$\mu_m = \frac{|B_k|}{2k} \prod_{j=1}^{k-1} \frac{|B_{2j}|}{4j}, \qquad m = 2k.$$

One has $C_8 = \{E_8\}$ with $g_{E_8} = |W(E_8)| = 696729600$, and $C_{16} = \{E_8 \oplus E_8, D_{16}^+\}$ with $g = 2\,|W(E_8)|^2$ and $g = 2^{15}\,16!$ respectively [SeA, Ch. V, §2.3].

Problem 22.1.5. The mass formula holds for $m = 8$ and $m = 16$. In the average (1) for $m = 16$, $E_8 \oplus E_8$ has weight $286/691$ and D_{16}^+ has weight $405/691$.

Problems 22.1.1 and 22.1.2 prove (1) for $m = 8$ and 16: there every theta series is the Eisenstein series, because $\dim \mathcal{M}_{m/2} = 1$. In rank 24 the statement has content.

Problem 22.1.6. For $m = 24$, (1) is equivalent to $\sum_{L\in C_{24}} c_L/g_L = 0$. Hence the mass-weighted mean number of roots of an even unimodular lattice of rank 24 is $65520/691$; in particular $\mu_{24}^{-1}\sum_L \theta_L/g_L$ is not the theta series of a lattice.

22.2. Local densities

Why should an average of theta series be an Eisenstein series? Its coefficients are averages of representation numbers, and Siegel's theorem says that these are products of local quantities. Let $Q(x) := x \cdot x/2$ on L. For a prime p and $j \geq 1$ the *local density* is

$$\alpha_p(j) := \lim_{r\to\infty} p^{r(1-m)}\, \#\{x \in L/p^r L : Q(x) \equiv j \pmod{p^r}\},$$

and the *archimedean density* is $\alpha_\infty(j) := \frac{d}{dt}\,\mathrm{vol}\{x \in L\otimes\mathbb{R} : Q(x) \leq t\}\big|_{t=j}$, the volume being normalized so that L has covolume 1. In this language Siegel's theorem reads

$$\frac{1}{\mu_m} \sum_{L\in C_m} \frac{r_L(j)}{g_L} = \alpha_\infty(j) \prod_p \alpha_p(j) \qquad (j \geq 1), \tag{2}$$

and the densities depend only on $L\otimes\mathbb{Z}_p$ and $L\otimes\mathbb{R}$, which are the same for every $L \in C_m$. For every prime p, the reduction L/pL with the form Q mod p is isometric to the orthogonal sum H^k of $k = m/2$ hyperbolic planes, $Q = x_1y_1 + \cdots + x_ky_k$ [Ca, Ch. 8].

Problem 22.2.1. $\alpha_\infty(j) = (2\pi)^k j^{k-1}/\Gamma(k)$.

Problem 22.2.2. The number of $x \in \mathbb{F}_p^{2k}$ with $x_1y_1 + \cdots + x_ky_k = j$ is $p^{2k-1} - p^{k-1}$ for $j \neq 0$, and $p^{2k-1} + p^k - p^{k-1}$ for $j = 0$.

Problem 22.2.3. If $p \nmid j$, then $\alpha_p(j) = 1 - p^{-k}$.

Problem 22.2.4. If $p \| j$, then $\alpha_p(j) = (1 - p^{-k})(1 + p^{1-k})$.

The same argument, splitting off the solutions with $x \equiv 0 \pmod p$, gives $\alpha_p(j) = (1-p^{-k})(1+p^{1-k}) + p^{2-2k}\alpha_p(j/p^2)$ when $p^2 \mid j$, the last term coming from $x = px'$ with $Q(x') \equiv j/p^2$. By induction on $v = v_p(j)$, $\alpha_p(j) = (1 - p^{-k}) \sum_{i=0}^{v} p^{i(1-k)}$.

Problem 22.2.5. $\alpha_\infty(j) \prod_p \alpha_p(j) = -\frac{2k}{B_k}\, \sigma_{k-1}(j)$.

Thus the right side of (2) is the jth coefficient of E_k, and (2) is (1) coefficient by coefficient. With Problems 22.1.1 and 22.1.2 this proves the following.

Theorem 22.2.1. *For $m = 8$ and $m = 16$, and every $j \geq 1$, the mass-weighted mean of $r_L(j)$ over C_m is $\alpha_\infty(j) \prod_p \alpha_p(j)$.*

Weil reformulated (1) as an identity on adelic groups: the integral of the theta function over $O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A})$ taken with the invariant measure of total mass one, is an Eisenstein series, under hypotheses that include the absolute convergence of the Eisenstein series [We65, Theorem 5]. For a positive definite space the quotient is compact, and the integral is the average over the genus. This is the *Siegel–Weil formula in the convergent range*.

22.3. The Hermitian class set over $\mathbb{Z}[\mathrm{i}]$

Return to the class sets Cl_n of Hermitian lattices over $\mathcal{O} = \mathbb{Z}[\mathrm{i}]$ from Chapter 21. For $L \in \mathrm{Cl}_n$ let $\theta_L(\tau) := \sum_{x \in L} q^{h(x,x)}$, and let $\theta := \sum_{j \in \mathbb{Z}} q^{j^2}$. Let χ be the nontrivial character modulo 4, and $\mathcal{M}_n(4, \chi^n)$ the space of modular forms of weight n on $\Gamma_0(4)$ with character χ^n. It has dimension $1 + \lfloor n/2 \rfloor$ and basis $\theta^{2n-4i} F^i$, $0 \leq i \leq n/2$, where $F := \sum_{j \text{ odd}} \sigma_1(j) q^j$ [Ko, Ch. IV, §1]; so an element is determined by its coefficients of $q^0, \ldots, q^{\lfloor n/2 \rfloor}$. Each θ_L lies in $\mathcal{M}_n(4, \chi^n)$.

For Hermitian lattices Siegel's theorem is due to Braun [Br], and it is a case of Weil's theorem [We65, Theorem 5]; for the unitary formulation see [Li, Theorem 2.3.1]. It reads, for $n \geq 3$,

$$\sum_{L \in \mathrm{Cl}_n} \frac{\theta_L}{|\Gamma_L|} = M_n\, E^{(n)}, \qquad E^{(n)} = 1 + \sum_{j \geq 1} \alpha_\infty(j) \prod_p \alpha_p(j)\, q^j, \tag{3}$$

with the densities of I_n defined as in §22.2 with $Q(x) := h(x,x)$, and $E^{(n)}$ an Eisenstein series in $\mathcal{M}_n(4, \chi^n)$. The condition $n \geq 3$ is where Eisenstein series of weight n converge absolutely.

Problem 22.3.1. $\sum_{(c,d) \neq (0,0)} |c\tau + d|^{-s}$ converges if and only if $s > 2$.

Problem 22.3.2. $\theta_{I_n} = \theta^{2n}$, and $\theta_{E \perp I_1}(\tau) = E_4(2\tau)\, \theta(\tau)^2$, with E as in Problem 21.2.3.

For $n = 3$ and 4, $\mathrm{Cl}_n = \{I_n\}$ (Problem 21.1.4), so (3) says that θ^{2n} is itself an Eisenstein series. This is Jacobi's formula for sums of $2n$ squares; for $n = 4$ and odd j it reads $r_8(j) = 16\, \sigma_3(j)$. Here are its local factors.

Problem 22.3.3. For I_4 and odd j, $\alpha_\infty(j) = \pi^4 j^3/6$ and $\alpha_\infty(j) \prod_{p \text{ odd}} \alpha_p(j) = 16\, \sigma_3(j)$.

Problem 22.3.4. For odd j, the number of $x \in (\mathbb{Z}/8\mathbb{Z})^8$ with $x_1^2 + \cdots + x_8^2 \equiv j \pmod 8$ is 8^7.

Solutions modulo 8 with some x_i odd lift uniformly to $\mathbb{Z}/2^r\mathbb{Z}$, so Problem 22.3.4 gives $\alpha_2(j) = 1$, and the product of all the local densities is $16\,\sigma_3(j) = r_8(j)$.

In rank 5 the class set has two elements, I_5 and $E \perp I_1$ (Problem 21.2.3), and $\dim \mathcal{M}_5(4,\chi) = 3$, one more than the number of Eisenstein series. Let $e_4(j) := \sum_{d\mid j,\, d \text{ odd}} \chi(d)\, d^4$ and $e_4'(j) := \sum_{d\mid j,\, j/d \text{ odd}} \chi(j/d)\, d^4$. The series $G_5 := 1 + \frac{4}{5}\sum_{j\ge 1}(e_4(j) + 16e_4'(j))q^j$ is a combination of the two Eisenstein series of $\mathcal{M}_5(4,\chi)$ [DS, §4.5].

Problem 22.3.5. In Cl_5, the weights $|\Gamma_L|^{-1}/M_5$ are $3/5$ for I_5 and $2/5$ for $E \perp I_1$.

Problem 22.3.6. $\frac{3}{5}\,\theta^{10} + \frac{2}{5}\,E_4(2\tau)\theta^2 = G_5$. The coefficients of q in θ^{10}, $E_4(2\tau)\theta^2$ and G_5 are 20, 4 and 68/5.

So in rank 5 neither theta series is an Eisenstein series, but their mass-weighted average is, as (3) predicts. What is left over is a cusp form. Let $f := \frac{1}{4}\sum_{\alpha\in\mathcal{O}} \alpha^4 q^{|\alpha|^2} = q - 4q^2 + 16q^4 - 14q^5 + \cdots$, the theta series of $\mathcal{O}$ with the harmonic polynomial α^4; it is a cusp form in $\mathcal{M}_5(4,\chi)$ [Iw, Ch. 10].

Problem 22.3.7. $\theta^{10} - E_4(2\tau)\theta^2 = 16f$. Hence $\theta^{10} = G_5 + \frac{32}{5}f$ and $E_4(2\tau)\theta^2 = G_5 - \frac{48}{5}f$.

The first identity of the last line is Liouville's formula $r_{10}(j) = \frac{4}{5}(e_4(j) + 16e_4'(j)) + \frac{8}{5}\sum_{|\alpha|^2=j} \alpha^4$. The cusp parts $\frac{32}{5}f$ and $-\frac{48}{5}f$ cancel in the average with weights $3/5$ and $2/5$, and only there.

22.4. Gram rows and nonvanishing

The theta lifts of the argument go from a class set Cl_n to a much larger class set $\mathrm{Cl}_{n'}$, of which only the mass is known (Chapter 21, §21.2). Their Gram matrices are sums over $\mathrm{Cl}_{n'}$. We look at the smallest case, $n = 1$.

For $a \ge 0$ and $L' \in \mathrm{Cl}_{n'}$ put

$$v_a(L') := \sum_{x\in L'} x^a\, e^{-\pi h(x,x)} \in \mathrm{Sym}^a(L'_{\mathbb{C}}),$$

with x^a as in §21.3. Up to the factor $|\Gamma_{I_1}| = 4$, this is the theta lift of the next chapter, *Definite theta lifts and their norms*, applied to the form on $\mathrm{Cl}_1 = \{I_1\}$ whose value is the basis vector of $\mathrm{Sym}^a(\mathbb{C})$. For $x, y \in L'$ let $\mathrm{Gram}(x,y)$ be the matrix with entries $h(x,x)$, $h(x,y)$, $h(y,x)$, $h(y,y)$, and for a 2×2 Hermitian matrix T over $\mathcal{O}$ put

$$R_{n'}(T) := \sum_{L'\in \mathrm{Cl}_{n'}} \frac{1}{|\Gamma_{L'}|}\, \#\{(x,y) \in L'^2 : \mathrm{Gram}(x,y) = T\}.$$

Problem 22.4.1. v_a is a form of weight (a) on $\mathrm{Cl}_{n'}$, and $v_a = 0$ unless $4 \mid a$.

Problem 22.4.2. $\|v_a\|^2 = \sum_T R_{n'}(T)\, T_{12}^a\, e^{-\pi\,\mathrm{tr}\,T}$, the sum running over positive semi-definite T.

So the norm of the lift, and in the same way every entry of its Gram matrix, depends on $\mathrm{Cl}_{n'}$ only through the averaged representation numbers $R_{n'}(T)$. The Siegel–Weil formula of degree 2 evaluates them: for $n' > 4$ and T positive definite,

$R_{n'}(T) = M_{n'}\,\alpha_\infty(T)\prod_p \alpha_p(T)$, where $\alpha_v(T)$ are the local densities of representations of T by $I_{n'}$, defined as in §22.2 with pairs of vectors in place of vectors [We65, Theorem 5]. Here $n' > 4$ is the convergent range for pairs, as $n \geq 3$ was for single vectors. In particular $R_{n'}(T) > 0$ as soon as T is represented by $I_{n'}$ over every $\mathbb{Z}_p$: a statement about the average that need not hold for any single lattice.

Problem 22.4.3. Let 1_2 be the identity matrix. Then I_5 contains 320 pairs (x, y) with $\mathrm{Gram}(x, y) = 1_2$, and $E \perp I_1$ contains none. Hence $R_5(1_2) = 1/384$.

For $a = 0$ one can take the lift one step further and pair it with the constant form.

Problem 22.4.4. For $n' \geq 3$, $\langle v_0, 1\rangle = M_{n'}\,E^{(n')}(\mathrm{i}/2)$. For $n' = 5$ this is $G_5(\mathrm{i}/2)/73728$, and $G_5(\mathrm{i}/2) = 2.06195\ldots$.

The general statement is the Rallis inner product formula, which we state without proof. Let φ be a form on Cl_n of weight λ that is an eigenvector of the Hecke operators at almost all primes, and let $\Theta\varphi$ be its theta lift to $\mathrm{Cl}_{n'}$ with $n' > 4n$. Then $\|\Theta\varphi\|^2/\|\varphi\|^2$ is the product of finitely many local factors and of a partial L-function of φ at a point of its half-plane of absolute convergence. The proof unfolds $\|\Theta\varphi\|^2$ into an integral of $\varphi \otimes \bar{\varphi}$ against the genus average of the doubled theta series, the analogue of Problem 22.4.2, and evaluates that average by the Siegel–Weil formula [Ra]; for unitary groups in the stable range see [JSL]. Since an Euler product does not vanish where it converges absolutely, the L-value is nonzero, and the lift is nonzero once the local factors are.

In Adriazola's reconstruction of the argument the convergent-range Siegel–Weil formula is the one external analytic theorem behind the Gram-fibre identities [Adr, §14.4]; the near orthogonality that these identities feed into rests there also on counting and pre-trace statements, [Adr, Hyps. 14.61 and 14.71–14.77], assumed there (see Chapter 11, and the ledger in Section 58.7 of Chapter 58). Further, the norms of the lifts enter through the Rallis inner product formula in the convergent range [Adr, Prop. 15.52(i)]; [Adr, §15.13] specifies the forms on a selected realization, and the formula for the lifts of the chain is assumed there. In both places the class sets are far too large to handle one lattice at a time, and only the average, through its local densities, is used.

Hints

§22.1.

22.1.1: $\dim \mathcal{M}_4 = 1$.

22.1.2: $\dim \mathcal{M}_8 = 1$, and $\theta_{E_8\oplus E_8} = \theta_{E_8}^2$. For the roots of D_{16}^+, compute $x \cdot x$ on the non-trivial coset.

22.1.3: Is each lattice spanned by its roots?

22.1.4: Write $\theta_\Lambda = E_{12} + c\Delta$ and compare coefficients of q; then use $\tau(2) = -24$.

22.1.5: $B_2, B_4, \ldots, B_{16}$; for $m = 16$ one needs $B_8 = -1/30$ and $B_{14} = 7/6$.

22.1.6: Subtract $\mu_{24}E_{12}$ from both sides of (1), and compare coefficients of q.

§22.2.

22.2.1: Since L is unimodular, the volume is that of a Euclidean ball of radius $\sqrt{2t}$ in dimension $2k$.

22.2.2: Induction on k: count first the solutions with $x_k = 0$, then those with $x_k \neq 0$.

22.2.3: Every solution modulo p of $Q(x) \equiv j$ has $x \not\equiv 0$, and the gradient of Q is nonzero there, for every p, because L is unimodular; apply Hensel's lemma.

22.2.4: Separate the solutions with $x \not\equiv 0 \pmod p$, which lift as in Problem 22.2.3, from those with $x \equiv 0$, for which $Q(x) \equiv 0 \pmod{p^2}$.

22.2.5: The Euler product of $\zeta(k)$, Euler's formula for $\zeta(k)$ in terms of B_k, and $\sigma_{k-1}(j) = j^{k-1}\sigma_{1-k}(j)$.

§22.3.

22.3.1: Count the pairs with $N \le |c\tau + d| < N + 1$.

22.3.2: $h(x, x)$ is additive on orthogonal sums, and $h(x, x) = x \cdot x$ is even on E.

22.3.3: As in Problem 22.2.1, but $\{Q \le t\}$ is now the ball of radius $\sqrt{t}$. The counting argument of Problems 22.2.3–22.2.4 and the remark after them uses only that $Q \bmod p$ is isometric to H^4, which holds for $x_1^2 + \cdots + x_8^2$ and odd p. Then $\zeta(4) = \pi^4/90$.

22.3.4: Write the count as $\frac{1}{8}\sum_{t \bmod 8} e(-tj/8)\, g(t)^8$ with the Gauss sums $g(t) := \sum_{x \bmod 8} e(tx^2/8)$.

22.3.5: Problem 21.1.2 and $|\Gamma_E| = 46080$.

22.3.6: Both sides lie in $\mathcal{M}_5(4, \chi)$; compare the coefficients of q^0, q^1, q^2.

22.3.7: As in Problem 22.3.6; then combine with it.

§22.4.

22.4.1: Substitute $x \mapsto \gamma x$ for $\gamma \in \Gamma_{L'}$, and $x \mapsto \mathrm{i}x$.

22.4.2: Problem 21.3.2: $\langle x^a, y^a \rangle = h(x, y)^a$. Then group the pairs by their Gram matrix.

22.4.3: Problem 21.2.3: the vectors of $E \perp I_1$ with $h(x, x) = 1$ are ue, $u \in \mu_4$.

22.4.4: What is $v_0(L')$ as a value of $\theta_{L'}$?

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