EISENSTEIN CONGRUENCES AND SELMER CLASSES

Prerequisites. Modular forms of level 1, Eisenstein series, Δ and the Hecke operators [SeA, Ch. VII]; Serre writes the weight as $2k$ and numbers the Bernoulli numbers differently. Complete discrete valuation rings and Hensel's lemma. The Chebotarev density theorem. From Chapter 29, *Denominators and the Stickelberger element*: the eigenspaces $A^{(i)}$ of Section 29.2 and Theorem 29.2.2. From Chapter 30, *Tate–Shafarevich groups of Tate twists*, with $\ell = p$: the groups $G_{\mathbb{Q},p}$, G_p, $\mathrm{Sha}_p(\mathbb{Q}, e)$, Problems 30.1.4 and 30.2.1, Theorem 30.1.1, and the Euler characteristic formula stated before Problem 30.1.8. From Chapter 31, *Congruence modules*: Problem 31.2.2. Galois representations attached to modular forms are stated in Section 32.3 with references.

Chapter 30 computed $\mathrm{Sha}_p(\mathbb{Q}, e)$ for odd e from the theorem of Mazur and Wiles, which was used as a black box. How does one produce classes in Sha, and how does one bound them? Ribet's answer to the first question [Ri] is that a cusp form congruent to an Eisenstein series gives a Galois representation that is reducible modulo p, and a good choice of lattice turns it into an unramified class. Read in reverse, a class in Sha is a tangent vector at the reducible "Eisenstein point". This direction is the one that bounds Sha. We work with level 1, and the example to keep in mind is $k = 12$, $p = 691$. Throughout, p is an odd prime, χ is the p-adic cyclotomic character and ω is its reduction modulo p.

32.1. The congruence

For even $k \geq 4$ let $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$, and normalize the Eisenstein series as

$$E_k := 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n)\, q^n,$$

so that E_k has rational coefficients and constant term 1 [SeA, Ch. VII, §4]. Let $\Delta = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum \tau(n) q^n$. Since $B_{12} = -691/2730$, the coefficients of E_{12} have 691 in the denominator. By Problem 31.2.2 of Chapter 31, $691\,(E_4^3 - E_{12}) = 432000\,\Delta$.

Problem 32.1.1. $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ for all $n \geq 1$.

The same happens whenever p divides B_k/k.

Problem 32.1.2. Let $k \geq 4$ be even, and let p divide the numerator of B_k/k. There is a cusp form g of weight k and level 1 with coefficients in $\mathbb{Z}_{(p)}$ such that $a_n(g) \equiv \sigma_{k-1}(n) \pmod{p}$ for all $n \geq 1$.

Problem 32.1.3. With g as in Problem 32.1.2, for every prime ℓ and every $n \geq 1$,

$$a_n(T_\ell\, g) \equiv (1 + \ell^{k-1})\, a_n(g) \pmod{p}.$$

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The cusp forms with coefficients in $\mathbb{Z}_{(p)}$ form a lattice stable under the Hecke operators. By the lifting lemma of Deligne and Serre [DeS, Lemme 6.11], Problem 32.1.3 gives a normalized eigenform f of weight k and level 1, and a prime $\lambda \mid p$ of the field generated by its coefficients, such that

$$a_\ell(f) \equiv 1 + \ell^{k-1} \pmod{\lambda} \qquad \text{for every prime } \ell. \tag{1}$$

In particular $a_p(f) \equiv 1$: f is *ordinary* at λ. For $k = 12$ we may take $f = \Delta$, and $\tau(691) \equiv 1 \pmod{691}$.

32.2. Lattices

A congruence between f and an Eisenstein series says that a Galois representation is reducible modulo λ. Which reduction one sees depends on a lattice. Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer π, valuation v with $v(\pi) = 1$, and residue field $\mathbb{F}$. Let G be a profinite group and $\rho\colon G \to \mathrm{GL}(V)$ a continuous irreducible representation on a 2-dimensional K-space. A *lattice* is a free $\mathcal{O}$-submodule of rank 2 that spans V. Suppose that for one, hence (by the Brauer–Nesbitt theorem [CR, (30.16)]) every, G-stable lattice L, the semisimplification of $L/\pi L$ is $\chi_1 \oplus \chi_2$ with characters $\chi_1 \neq \chi_2\colon G \to \mathbb{F}^\times$.

Problem 32.2.1. G-stable lattices exist.

Problem 32.2.2. Choose $\sigma_0 \in G$ with $\chi_1(\sigma_0) \neq \chi_2(\sigma_0)$. Then $\rho(\sigma_0)$ has eigenvalues $\lambda_1, \lambda_2 \in \mathcal{O}$ with $\lambda_i \equiv \chi_i(\sigma_0) \pmod{\pi}$, and every G-stable lattice L has a basis e_1, e_2 with $\rho(\sigma_0)e_i = \lambda_i e_i$.

Fix such L, e_1, e_2, write $\rho(\sigma) = \left(\begin{smallmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{smallmatrix}\right)$ in this basis, and put $m := \min_\sigma v(b(\sigma))$, $n := \min_\sigma v(c(\sigma))$. By irreducibility b and c are not identically zero, so m, n are finite. For $r \in \mathbb{Z}$ let $L_r := \pi^r \mathcal{O} e_1 \oplus \mathcal{O} e_2$.

Problem 32.2.3. L_r is G-stable if and only if $-n \le r \le m$, and every G-stable lattice is homothetic to exactly one L_r: the stable lattices form a segment of length $m + n$ in the tree of $\mathrm{PGL}_2(K)$.

Problem 32.2.4. $m+n \ge 1$. On $L_m/\pi L_m$ the group G acts by $\left(\begin{smallmatrix} \chi_1 & * \\ 0 & \chi_2 \end{smallmatrix}\right)$, a nonsplit extension; on $L_{-n}/\pi L_{-n}$ it acts by a nonsplit $\left(\begin{smallmatrix} \chi_1 & 0 \\ * & \chi_2 \end{smallmatrix}\right)$, with sub χ_2 and quotient χ_1; and for $-n < r < m$ it acts by $\chi_1 \oplus \chi_2$.

Problems 32.2.1–32.2.4 prove the following. It is due to Ribet [Ri, Prop. 2.1].

Theorem 32.2.1 (Ribet's lemma)**.** *There is a G-stable lattice on whose reduction G acts by a nonsplit extension* $\left(\begin{smallmatrix} \chi_1 & * \\ 0 & \chi_2 \end{smallmatrix}\right)$.

32.3. The unramified class

We state three inputs. Let f be a normalized eigenform of weight k and level 1, let $\lambda \mid p$ be a prime of the field of its coefficients, and let K be the completion at λ.

- (D) There is a continuous $\rho_f\colon G_{\mathbb{Q},p} \to \mathrm{GL}_2(K)$ with $\operatorname{tr}\rho_f(\mathrm{Frob}_\ell) = a_\ell(f)$ and $\det\rho_f = \chi^{k-1}$, for every prime $\ell \neq p$ [De].
- (I) ρ_f is irreducible [Ri2, §2].
- (O) If $a_p(f)$ is a λ-adic unit, then $\rho_f|_{G_p}$ stabilizes a line on which the inertia group I_p acts by χ^{k-1}, and G_p acts on the quotient by an unramified character [Wi, Thm. 2.1.4].

Chapter 34, *Black boxes: Galois representations and local–global compatibility*, states Galois representations of this kind as black boxes, over CM fields. Its Section 34.4 shows in the simplest case, the eigenform f of weight 2 and level 11, where ρ_f comes from: it is the rational p-adic Tate module of $J_0(11)$, and the Eichler–Shimura relation gives the analogue of (D).

From now on k is even with $4 \le k \le p-3$, p divides B_k (equivalently, since $k < p$, the numerator of B_k/k), and f, λ are as in (1).

Problem 32.3.1. The semisimplification of ρ_f modulo λ is $1 \oplus \omega^{k-1}$, and $1 \ne \omega^{k-1}$ on G_p, and even on I_p.

Apply Theorem 32.2.1 with $\chi_1 = 1$ and $\chi_2 = \omega^{k-1}$, and let $\bar\rho = \left(\begin{smallmatrix} 1 & \bar b \\ 0 & \omega^{k-1} \end{smallmatrix}\right)$ be the resulting reduction.

Problem 32.3.2. $\kappa(\sigma) := \bar b(\sigma)\,\omega^{1-k}(\sigma)$ is a cocycle on $G_{\mathbb{Q},p}$ with values in $\mathbb{F}(\omega^{1-k})$, and its class is not zero.

Problem 32.3.3. The class of κ vanishes on G_p.

Since $H^1(G_{\mathbb{Q},p}, \mathbb{F}(\omega^{1-k})) = H^1(G_{\mathbb{Q},p}, \mathbb{Z}/p(1-k)) \otimes_{\mathbb{F}_p} \mathbb{F}$, and similarly for G_p, Problems 32.3.1–32.3.3, Problem 30.1.4 and Theorem 30.1.1 prove the following.

Theorem 32.3.1 (Ribet)**.** *Let k be even with $4 \le k \le p-3$, and suppose that $p \mid B_k$. Then $\mathrm{Sha}_p(\mathbb{Q}, k-1) \ne 0$, and $A^{(p-k)} \ne 0$.*

This is the converse to Herbrand's theorem [Ri]; see also [Wa, Thm. 6.18]. For $k = 12$ and $p = 691$ the input is Δ, and the conclusion is that $\mathbb{Q}(\zeta_{691})$ has an unramified extension of degree 691 on which $\mathrm{Gal}(\mathbb{Q}(\zeta_{691})/\mathbb{Q})$ acts through $\omega^{-11} = \omega^{679}$. The proof gives Sha $\ne 0$ and no more. Congruences with Eisenstein series are measured by the Eisenstein ideal of the Hecke algebra, which Mazur introduced in weight 2 and prime level [Ma]. Mazur and Wiles refined Ribet's method to show that the length of $\mathrm{Sha}_p(\mathbb{Q}, k-1)$ is at least $v_p(B_k/k)$; the analytic class number formula gives the reverse inequality [MW].

32.4. Tangent vectors at the Eisenstein point

Ribet goes from a modular form to a class. Alpöge's argument needs the reverse: a bound for Sha from the modular side. What is a class of Sha from the point of view of Galois representations? Let $\rho_0 := 1 \oplus \omega^{k-1} \colon G_{\mathbb{Q},p} \to \mathrm{GL}_2(\mathbb{F}_p)$, the reduction of the representation $1 \oplus \chi^{k-1}$ of the Eisenstein series E_k; we call it the *Eisenstein point*. Let $\mathbb{F}_p[\varepsilon]$ be the ring with $\varepsilon^2 = 0$. A *lift* of ρ_0 is a continuous homomorphism $\rho_\varepsilon \colon G_{\mathbb{Q},p} \to \mathrm{GL}_2(\mathbb{F}_p[\varepsilon])$ that reduces to ρ_0 modulo ε. Let $\mathrm{ad}\,\rho_0$ be the space $M_2(\mathbb{F}_p)$ with σ acting by conjugation by $\rho_0(\sigma)$. See [Mz, §1] for the general theory.

Problem 32.4.1. Writing $\rho_\varepsilon = (1 + \varepsilon M)\rho_0$ identifies the lifts of ρ_0, up to conjugation by matrices $1 + \varepsilon X$, with $H^1(G_{\mathbb{Q},p}, \mathrm{ad}\,\rho_0)$.

The torus $T = \{\mathrm{diag}(1,t)\}$ commutes with ρ_0, and acts on lifts by conjugation.

Problem 32.4.2. $\mathrm{ad}\,\rho_0 = \mathbb{F}_p E_{11} \oplus \mathbb{F}_p E_{22} \oplus \mathbb{F}_p(\omega^{1-k}) E_{12} \oplus \mathbb{F}_p(\omega^{k-1}) E_{21}$, where E_{ij} are the matrix units, and T acts on these four summands, and on their H^1, with weights 0, 0, -1, $+1$.

A cocycle κ with nonzero class in $H^1(G_{\mathbb{Q},p}, \mathbb{F}_p(\omega^{1-k}))$ gives the nonsplit $\rho_\kappa = \left(\begin{smallmatrix} 1 & \kappa\omega^{k-1} \\ 0 & \omega^{k-1} \end{smallmatrix}\right)$. For $t \in \mathbb{F}_p^\times$, conjugating by $\mathrm{diag}(t,1)$ replaces κ by $t\kappa$, so all the $\rho_{t\kappa}$ are

isomorphic; at $t = 0$ one reaches ρ_0, and to first order at $t = 0$ the family is the lift $(1 + \varepsilon\kappa E_{12})\rho_0$, of weight -1. This is the sense in which a class is a tangent vector at the Eisenstein point.

Which lifts are allowed? By (O), ρ_f has a line on which I_p acts by χ^{k-1} with unramified quotient. Call a lift ρ_ε *ordinary* if $\mathbb{F}_p[\varepsilon]^2$ has a G_p-stable direct summand L of rank 1, reducing to $\mathbb{F}_p e_2$, on which I_p acts by ω^{k-1}, with I_p acting trivially on the quotient. Let $t_{\mathrm{ord}} \subseteq H^1(G_{\mathbb{Q},p}, \operatorname{ad}\rho_0)$ be the set of classes of ordinary lifts, and $t_{\mathrm{ord}}^{(j)}$ its part of weight j.

Problem 32.4.3. The lift with $M = \left(\begin{smallmatrix} \alpha_1 & \beta \\ \gamma & \alpha_2 \end{smallmatrix}\right)$ is ordinary if and only if α_1 and α_2 vanish on I_p and the class of β vanishes in $H^1(G_p, \mathbb{F}_p(\omega^{1-k}))$. Hence $t_{\mathrm{ord}}^{(0)} = 0$, and $t_{\mathrm{ord}}^{(+1)} = H^1(G_{\mathbb{Q},p}, \mathbb{F}_p(\omega^{k-1}))$.

With Problem 30.1.4 and Theorem 30.1.1, Problems 32.4.1–32.4.3 prove the following.

Theorem 32.4.1. $t_{\mathrm{ord}} = t_{\mathrm{ord}}^{(-1)} \oplus t_{\mathrm{ord}}^{(+1)}$, *and* $t_{\mathrm{ord}}^{(-1)} \cong \mathrm{Sha}_p(\mathbb{Q}, k-1)[p] \cong \mathrm{Hom}(A^{(p-k)}, \mathbb{F}_p)$.

Problem 32.4.4. $\dim t_{\mathrm{ord}}^{(+1)} \geq 1$.

The Hecke algebra sees a representation only through its traces.

Problem 32.4.5. Every ordinary lift of ρ_0 to $\mathbb{F}_p[\varepsilon]$ has trace $1 + \omega^{k-1}$.

The traces see β and γ only through their products. Return to ρ_f with the basis e_1, e_2 of Problem 32.2.2, with $\chi_1 = 1$ and $\chi_2 = \omega^{k-1}$, and let $I \subseteq \mathcal{O}$ be the ideal generated by the products $b(\sigma)c(\tau)$, $\sigma, \tau \in G_{\mathbb{Q},p}$. Following Bellaïche and Chenevier [BC, Ch. 1], we call it the *reducibility ideal*.

Problem 32.4.6. $I = \pi^{m+n}\mathcal{O}$, and a and d are characters of $G_{\mathbb{Q},p}$ modulo I.

Problem 32.4.7. $a \equiv 1$ and $d \equiv \chi^{k-1}$ modulo I. Hence $v(a_\ell(f) - 1 - \ell^{k-1}) \geq m+n$ for every prime $\ell \neq p$.

Problem 32.4.8. For $f = \Delta$ and $p = 691$, $m + n = 1$: up to homothety ρ_Δ has exactly two stable lattices. Moreover $v_{691}(\tau(\ell) - 1 - \ell^{11}) = 1$ for every prime $\ell \leq 37$.

Thus the trace differences $a_\ell(f) - 1 - \ell^{k-1}$ bound how far f lies from the Eisenstein point. To bound Sha one bounds the weight -1 tangent space of an ordinary deformation ring through ρ_0 by the trace differences at its points. This requires knowing that all the points are modular, an automorphy theorem for residually reducible representations. For GL_2 over $\mathbb{Q}$, such theorems are due to Skinner and Wiles [SW]. For $k = 12$ and $p = 691$ the two sides agree: $\mathrm{Sha}_{691}(\mathbb{Q}, 11)$ has length 1 by Theorem 30.1.2, and $v_{691}(\tau(2) - 1 - 2^{11}) = 1$. In the reconstruction [Adr, §17.4] a class of Sha gives, as here, a tangent vector of weight -1 at an Eisenstein point of rank three, for a unitary group in three variables (the setting of [BC]), and the weight -1 cotangent module is bounded by trace differences. The automorphy statement needed there is (G3) of [Adr, §17.4]; its status is recorded in Section 33.4 of Chapter 33.

32.5. From the CM field to characters

The fields of the argument are abelian CM fields E, not $\mathbb{Q}$. A class over E becomes a class over $\mathbb{Q}$ twisted by a character, as in Problem 30.2.1, now with E ramified anywhere.

Let E be an abelian number field with $\Gamma := \mathrm{Gal}(E/\mathbb{Q})$ of order prime to p. Let $\mathcal{O} := \mathbb{Z}_p[\mu_{|\Gamma|}]$, let $W := W_e \otimes_{\mathbb{Z}_p} \mathcal{O}$ with $W_e = \mathbb{Q}_p/\mathbb{Z}_p(-e)$ as in Chapter 30, and let $G_E \subseteq G_{\mathbb{Q}}$ be the absolute Galois groups. Let $\mathrm{Sha}_p(E,e)_{\mathcal{O}}$ be the group of classes in $H^1(G_E, W)$ that vanish on the inertia group I_w of every place $w \nmid p$ and on the decomposition group D_w of every $w \mid p$. Inflation–restriction shows that $\mathrm{Sha}_p(E,e)_{\mathcal{O}} = \mathrm{Sha}_p(E,e) \otimes_{\mathbb{Z}_p} \mathcal{O}$, with $\mathrm{Sha}_p(E,e)$ as in Chapter 30. For a character $\psi\colon \Gamma \to \mathcal{O}^\times$, let $\mathrm{Sha}_p(\mathbb{Q}, e; \psi)_{\mathcal{O}}$ be defined in the same way from $H^1(G_{\mathbb{Q}}, W \otimes \psi^{-1})$. Let Γ act on $\mathcal{O}[\Gamma]$ by left multiplication, so that $G_{\mathbb{Q}}$ acts on it through Γ, and let $\gamma \in \Gamma$ act on $H^1(G_E, W)$ by $(\gamma c)(h) := \tilde{\gamma}\, c(\tilde{\gamma}^{-1} h \tilde{\gamma})$, for any lift $\tilde{\gamma} \in G_{\mathbb{Q}}$ of γ.

Problem 32.5.1. $H^1(G_E, W) \cong H^1(G_{\mathbb{Q}}, \mathcal{O}[\Gamma] \otimes_{\mathcal{O}} W)$, and $\mathcal{O}[\Gamma] \otimes_{\mathcal{O}} W \cong \bigoplus_\psi W \otimes \psi$ as $G_{\mathbb{Q}}$-modules, the sum over all characters ψ of Γ.

Problem 32.5.2. Let v be a prime of $\mathbb{Q}$, and $D = D_v$ or I_v. A class of $H^1(G_E, W)$ vanishes on D_w, respectively on I_w, for every place $w \mid v$ of E if and only if its image under Problem 32.5.1 vanishes on D.

Problem 32.5.3. $\mathrm{Sha}_p(E,e)_{\mathcal{O}} \cong \bigoplus_\psi \mathrm{Sha}_p(\mathbb{Q}, e; \psi)_{\mathcal{O}}$. The summand $\mathrm{Sha}_p(\mathbb{Q}, e; \psi)_{\mathcal{O}}$, which comes from the summand $W \otimes \psi^{-1}$ of Problem 32.5.1, is the part on which Γ acts through ψ.

So a class in the ψ-part of $\mathrm{Sha}_p(E,e)_{\mathcal{O}}$ that is killed by p is the image of a class in $H^1(G_{\mathbb{Q}}, \mathbb{F}(\omega^{-e}\bar{\psi}^{-1}))$, with $\mathbb{F}$ the residue field of $\mathcal{O}$, as in Problem 30.1.4. As in §32.4, this is a tangent vector of weight -1 at the Eisenstein point $1 \oplus \bar{\psi}\,\omega^e$. On the minus part each summand is bounded by a generalized Bernoulli number, by the main conjecture for abelian fields [MW]; over $\mathbb{Q}$ this is Theorem 30.1.2, and for $E \subseteq \mathbb{Q}(\zeta_p)$ the classes killed by p are given by Theorem 29.2.2 through Problem 30.2.2. On the plus part the bound must come from the trace differences at automorphic points, as in Problem 32.4.7.

Hints

§32.1.

32.1.1: Reduce the identity modulo 691, noting $432000 + 65520 = 720 \cdot 691$.
32.1.2: The forms $E_4^a E_6^b$.
32.1.3: $a_n(T_\ell g) = a_{\ell n}(g) + \ell^{k-1} a_{n/\ell}(g)$, and σ_{k-1} satisfies the same relation with eigenvalue $\sigma_{k-1}(\ell)$.

§32.2.

32.2.1: The stabilizer of any lattice is open, hence of finite index.
32.2.2: Hensel's lemma for the characteristic polynomial. The projector $(\rho(\sigma_0) - \lambda_2)/(\lambda_1 - \lambda_2)$ has entries in $\mathcal{O}$.
32.2.3: By Problem 32.2.2, a stable lattice is the sum of its intersections with Ke_1 and Ke_2. Write ρ in the basis $\pi^r e_1, e_2$.
32.2.4: The only lines stable under $\bar{\rho}(\sigma_0)$ are $\mathbb{F}e_1$ and $\mathbb{F}e_2$. A representation with semisimplification $\chi_1 \oplus \chi_2$ has a stable line.

§32.3.

32.3.1: Chebotarev, Brauer–Nesbitt, and (1). ω maps I_p onto $\mathbb{F}_p^\times$.

32.3.2: Multiply out $\bar{\rho}(\sigma\tau)$. A coboundary can be removed by conjugating with a unipotent upper triangular matrix.

32.3.3: Reduce the line of (O) inside the lattice of Theorem 32.2.1, and compare it with $\mathbb{F}e_1$ under I_p.

§32.4.

32.4.1: Expand $\rho_\varepsilon(\sigma\tau) = \rho_\varepsilon(\sigma)\rho_\varepsilon(\tau)$.

32.4.2: Conjugate each E_{ij} by $\rho_0(\sigma)$ and by $\mathrm{diag}(1,t)$.

32.4.3: A rank-one summand reducing to $\mathbb{F}_p e_2$ is spanned by $e_2 + \varepsilon x e_1$. For the weight 0 part, $\mathbb{Q}$ has no nontrivial extension unramified at every prime.

32.4.4: The Euler characteristic formula of Chapter 30.

32.4.5: Problem 32.4.3.

32.4.6: The upper left entry of $\rho(\sigma\tau)$.

32.4.7: In the basis of L_m, ρ is upper triangular modulo I. Compare the line $(\mathcal{O}/I)e_1$ with the reduction of the line of (O) under I_p. A character of $G_{\mathbb{Q},p}$ unramified at p is trivial. Then use the determinant.

32.4.8: Problems 32.2.4 and 32.4.7 at $\ell = 2$, where $\tau(2) = -24$. The last statement is a finite computation with the product for Δ.

§32.5.

32.5.1: Shapiro's lemma. The elements $|\Gamma|^{-1}\sum_\gamma \psi(\gamma)^{-1}\gamma$ are orthogonal idempotents.

32.5.2: Mackey's formula. The double cosets $D_v\backslash G_\mathbb{Q}/G_E$ correspond to the places of E above v; for I_v each place w occurs $f(w/v)$ times, its residue degree, through conjugates of I_w.

32.5.3: The decomposition of Problem 32.5.1 is $G_\mathbb{Q}$-stable, hence stable under every D. Under Problem 32.5.1, γ acts on $\mathcal{O}[\Gamma]\otimes_\mathcal{O} W$ by right multiplication by γ^{-1} on $\mathcal{O}[\Gamma]$.

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