CONGRUENCE MODULES

Prerequisites. Modules over a Dedekind domain and the elementary divisor theorem. Ideals in quadratic fields. Modular forms of level 1: the structure of M_k and S_k [SeA, Ch. VII, §3.2, Thm. 4], Eisenstein series and Δ [SeA, Ch. VII, §§4.2–4.3], Hecke operators and their action on q-expansions, and normalized eigenforms [SeA, Ch. VII, §§5.3–5.5]; Serre writes the weight as $2k$ and numbers the Bernoulli numbers differently. The Petersson product, for which the Hecke operators are self-adjoint [DS, §5.5]. From Chapter 29, *Denominators and the Stickelberger element*: the cost of a denominator (Problem 29.1.5) and Theorem 29.2.2.

In Chapter 29 a linear map that does not respect integral structures cost $\log[M + T(L) : M]$. Which map does the argument have to divide by? The period pairing sees a lifted form only through its components along Hecke eigenspaces, so the map is the projection of a lattice of integral forms onto a sum of eigenspaces [Adr, §17.1]. What does such a projection cost, and what controls the cost? The answer is a congruence module. We compute it for level 1, in weight 12, where it is $\mathbb{Z}/691\mathbb{Z}$, and in weight 24, where it detects a congruence modulo the prime 144169 between the two newforms.

31.1. Congruence modules

Let $\mathcal{O}$ be a Dedekind domain with fraction field K. A *lattice* in a finite-dimensional K-vector space V is a finitely generated $\mathcal{O}$-submodule that spans V. Let $V = V_1 \oplus V_2$, let π_1, π_2 be the two projections, and let $\Lambda \subset V$ be a lattice. The *congruence module* of V_1 in Λ is

$$\mathrm{C}(\Lambda; V_1) := \pi_1(\Lambda)/(\Lambda \cap V_1).$$

Problem 31.1.1. $\pi_1(\Lambda) = V_1 \cap (\Lambda + V_2)$, and

$$\mathrm{C}(\Lambda; V_1) \cong \Lambda/\big((\Lambda \cap V_1) \oplus (\Lambda \cap V_2)\big) \cong \mathrm{C}(\Lambda; V_2).$$

In particular $\mathrm{C}(\Lambda; V_1)$ is a finitely generated torsion $\mathcal{O}$-module, and it vanishes exactly when $\Lambda = (\Lambda \cap V_1) \oplus (\Lambda \cap V_2)$.

Problem 31.1.2. $(\Lambda + \pi_1(\Lambda))/\Lambda \cong \mathrm{C}(\Lambda; V_1)$. For $\mathcal{O} = \mathbb{Z}$, the least positive integer c with $c\,\pi_1(\Lambda) \subseteq \Lambda$ is the exponent of $\mathrm{C}(\Lambda; V_1)$.

So for $L = M = \Lambda$ and $T = \pi_1$ the index $[M + T(L) : M]$ of Problem 29.1.5 is $|\mathrm{C}(\Lambda; V_1)|$: the projection costs $\log|\mathrm{C}(\Lambda; V_1)|$.

Problem 31.1.3. Let $N \geq 1$, $V = \mathbb{Q}^2 = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2$, and $\Lambda = \{(a, b) \in \mathbb{Z}^2 : a \equiv b \pmod N\}$. Then $\mathrm{C}(\Lambda; \mathbb{Q}e_1) \cong \mathbb{Z}/N\mathbb{Z}$, whereas $\mathrm{C}(\mathbb{Z}^2; \mathbb{Q}e_1) = 0$.

The lattices of interest carry an action of a Hecke algebra. Let X be a finite set, and let $\mathbf{T} \subseteq \prod_{x \in X} \mathcal{O}$ be an $\mathcal{O}$-subalgebra, finitely generated as an $\mathcal{O}$-module, which separates the points of X. Write $t(x)$ for the coordinates of $t \in \mathbf{T}$. Let $\mathbf{T}$

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act on $V = \bigoplus_{x\in X} V_x$ by $t \cdot v = (t(x)v_x)_x$, and let Λ be a $\mathbf{T}$-stable lattice. The *congruence ideal* of x is

$$\mathfrak{c}_x := \{t(x) : t \in \mathbf{T},\ t(y) = 0 \text{ for all } y \neq x\} \subseteq \mathcal{O},$$

and we write $\mathrm{C}(\Lambda; V_x)$ for the congruence module with respect to $V = V_x \oplus \bigoplus_{y\neq x} V_y$.

Problem 31.1.4. $\mathfrak{c}_x$ annihilates $\mathrm{C}(\Lambda; V_x)$. If $\mathcal{O}$ is a discrete valuation ring with valuation v, then $\mathrm{len}_{\mathcal{O}} \mathrm{C}(\Lambda; V_x) \leq \dim V_x \cdot \min v(\mathfrak{c}_x)$.

Problem 31.1.5. If $\Lambda \cong \mathbf{T}$ as a $\mathbf{T}$-module, then $\mathrm{C}(\Lambda; V_x) \cong \mathcal{O}/\mathfrak{c}_x$. For $\mathbf{T} = \{(a, b) \in \mathbb{Z}^2 : a \equiv b \pmod N\}$ acting on itself, $\mathfrak{c}_x = N\mathbb{Z}$ at both points.

Thus the denominator of the projection onto an eigenspace is the congruence module; it is killed by the congruence ideal, and equals $\mathcal{O}/\mathfrak{c}_x$ when the lattice is free of rank one over $\mathbf{T}$.

A congruence module can also be read off from a pairing. Suppose that Λ carries a *perfect* pairing $\langle\,,\rangle\colon \Lambda \times \Lambda \to \mathcal{O}$, that is, an $\mathcal{O}$-bilinear form for which $\lambda \mapsto \langle\lambda, \cdot\rangle$ is an isomorphism $\Lambda \to \mathrm{Hom}_{\mathcal{O}}(\Lambda, \mathcal{O})$, and that $\langle V_1, V_2\rangle = \langle V_2, V_1\rangle = 0$. For a lattice $L \subset V_1$ let $L^\vee := \{v \in V_1 : \langle v, L\rangle \subseteq \mathcal{O}\}$.

Problem 31.1.6. $\pi_1(\Lambda) = (\Lambda \cap V_1)^\vee$. Hence $\mathrm{C}(\Lambda; V_1) \cong (\Lambda \cap V_1)^\vee/(\Lambda \cap V_1)$.

Problem 31.1.7. In the setting of Problem 31.1.6: (i) if $\Lambda \cap V_1 = \mathcal{O}f$, then $\mathrm{C}(\Lambda; V_1) \cong \mathcal{O}/\langle f, f\rangle\mathcal{O}$; (ii) if $\Lambda \cap V_1 = \mathcal{O}x \oplus \mathcal{O}y$ with $\langle x, x\rangle = \langle y, y\rangle = 0$, then $\mathrm{C}(\Lambda; V_1) \cong (\mathcal{O}/\langle x, y\rangle\mathcal{O}) \oplus (\mathcal{O}/\langle y, x\rangle\mathcal{O})$. For $\Lambda = \mathbb{Z}^2$ with the dot product and $V_1 = \mathbb{Q}(1, 1)$, $V_2 = \mathbb{Q}(1, -1)$, the congruence module is $\mathbb{Z}/2\mathbb{Z}$.

This is the mechanism behind Hida's identity in Section 31.4: a congruence module of an eigenform is the discriminant of the eigenform's piece of a self-dual lattice. Compare [DDT, §4.4, Lemma 4.17], where a power of the congruence ideal of the Hecke algebra is shown to lie in the ideal of this discriminant, with equality when the lattice is free over the Hecke algebra.

31.2. The Eisenstein line in weight 12

For even $k \geq 4$ let M_k and S_k be the spaces of modular forms and cusp forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$, identified with their q-expansions, and let $M_k(\mathbb{Z}) := M_k \cap \mathbb{Z}[[q]]$ and $S_k(\mathbb{Z}) := S_k \cap \mathbb{Z}[[q]]$. Let $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$ and

$$E_k := 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n, \qquad \Delta := q \prod_{n\geq 1} (1 - q^n)^{24} = \sum_{n\geq 1} \tau(n) q^n,$$

so that $E_4 = 1 + 240 \sum \sigma_3(n) q^n$, $E_6 = 1 - 504 \sum \sigma_5(n) q^n$, and $B_{12} = -691/2730$ [SeA, Ch. VII, §4]. We use the decomposition $M_k \otimes \mathbb{Q} = \mathbb{Q}E_k \oplus S_k(\mathbb{Q})$, in which both summands are stable under the Hecke operators.

Problem 31.2.1. $M_{12}(\mathbb{Z}) = \mathbb{Z}E_4^3 \oplus \mathbb{Z}\Delta$.

Problem 31.2.2. $E_4^3 = E_{12} + \frac{432000}{691}\Delta$. Hence $\pi_\Delta(M_{12}(\mathbb{Z})) = \frac{1}{691}\mathbb{Z}\Delta$, and $\mathrm{C}(M_{12}(\mathbb{Z}); \mathbb{Q}\Delta) \cong \mathbb{Z}/691\mathbb{Z}$.

The same computation for the Eisenstein line works in every weight.

Problem 31.2.3. Let $k \geq 4$ be even, and let N_k be the numerator of $B_k/2k$ in lowest terms. Then $\mathrm{C}(M_k(\mathbb{Z}); \mathbb{Q}E_k) \cong \mathbb{Z}/N_k\mathbb{Z}$. For $k = 4, 6, 8, 10, 14$ one has $N_k = \pm 1$, and $|N_{16}| = 3617$, $|N_{22}| = 77683 = 131{\cdot}593$, $|N_{24}| = 236364091 = 103{\cdot}2294797$.

Let $\mathbf{T}_{12}$ be the $\mathbb{Z}$-algebra of endomorphisms of $M_{12} \otimes \mathbb{Q}$ generated by the Hecke operators T_n, $n \geq 1$, normalized so that $T_n\Delta = \tau(n)\Delta$ and $T_n E_{12} = \sigma_{11}(n)E_{12}$ [SeA, Ch. VII, §5]. Map $\mathbf{T}_{12}$ to $\mathbb{Z} \times \mathbb{Z}$ by its eigenvalues on (Δ, E_{12}).

Problem 31.2.4. Each T_n preserves $M_{12}(\mathbb{Z})$, and $\mathbf{T}_{12} = \{(a,b) \in \mathbb{Z}^2 : a \equiv b \pmod{691}\}$. Thus the congruence ideal of Δ is $691\mathbb{Z}$. (Here $\tau(2) = -24$ and $\tau(3) = 252$.)

So the bound of Problem 31.1.4 is attained, although the lattice is not free.

Problem 31.2.5. $M_{12}(\mathbb{Z})$ is not free of rank one over $\mathbf{T}_{12}$.

The prime 691 is also the one in Theorem 29.2.2: since $691 \mid B_{12}$, the eigenspace $A^{(679)}$ of the 691-part of the class group of $\mathbb{Q}(\zeta_{691})$ is nonzero. That the same prime measures both is not an accident; Chapter 32 builds the class from the congruence.

31.3. Two newforms of weight 24

The space S_{24} has dimension 2 [SeA, Ch. VII, §3.2], and its two normalized eigenforms are not rational. Let $D := 144169$, a prime with $D \equiv 1 \pmod{24}$, let $K := \mathbb{Q}(\sqrt{D})$, $\mathcal{O} := \mathcal{O}_K$, and $\mathfrak{p} := \sqrt{D}\,\mathcal{O}$.

Problem 31.3.1. $S_{24}(\mathbb{Z}) = \mathbb{Z}g_1 \oplus \mathbb{Z}g_2$, where $g_1 = \Delta E_4^3 - 696\Delta^2 = q + 195660q^3 + \cdots$ and $g_2 = \Delta^2 = q^2 - 48q^3 + \cdots$.

On S_{24} one has $a_n(T_2 f) = a_{2n}(f) + 2^{23}a_{n/2}(f)$, with $a_{n/2} := 0$ for odd n; also $a_4(g_1) = 12080128$ and $a_4(g_2) = 1080$.

Problem 31.3.2. $T_2 g_1 = 20468736\, g_2$ and $T_2 g_2 = g_1 + 1080\, g_2$. The characteristic polynomial of T_2 on S_{24} is $x^2 - 1080x - 20468736$, with discriminant $24^2 D$ and roots $\lambda_\pm = 540 \pm 12\sqrt{D}$. The normalized eigenforms are $f_\pm = g_1 + \lambda_\pm g_2$.

Let $\mathbf{T}_{24}$ be the $\mathbb{Z}$-algebra generated by the T_n on S_{24}.

Problem 31.3.3. $\mathbf{T}_{24} = \mathbb{Z}[T_2] = \mathbb{Z} \oplus \mathbb{Z}T_2$, and $S_{24}(\mathbb{Z}) = \mathbf{T}_{24}\, g_2$ is free of rank one over $\mathbf{T}_{24}$.

Let $\Lambda := S_{24}(\mathbb{Z}) \otimes \mathcal{O}$ in $V := S_{24} \otimes K = Kf_+ \oplus Kf_-$, and let $\mathbf{T}_{24} \otimes \mathcal{O}$ act on it, through its eigenvalues, as a subalgebra of $\mathcal{O} \times \mathcal{O}$.

Problem 31.3.4. The congruence ideal of f_+ is $(\lambda_+ - \lambda_-)\mathcal{O} = 24\sqrt{D}\,\mathcal{O}$. Hence $\mathrm{C}(\Lambda; Kf_+) \cong \mathcal{O}/24\sqrt{D}\,\mathcal{O}$, and this can also be seen directly from Problem 31.3.2.

Problem 31.3.5. $a_n(f_+) \equiv a_n(f_-) \pmod{24\sqrt{D}\,\mathcal{O}}$ for all n. The primes 2 and 3 split in K, and $\mathrm{C}(\Lambda; Kf_+)$ has length 1 at $\mathfrak{p}$, length 3 at each prime above 2, and length 1 at each prime above 3.

The congruence modulo $\mathfrak{p}$, the ramified prime of the Hecke field, is the interesting one. The primes 2 and 3 are smaller than the weight, outside the range $p > k$ of Hida's theorem below (Theorem 31.4.1).

31.4. Hida's identity

What, in terms of f alone, decides whether $\mathrm{C}(\Lambda; Kf)$ is nonzero at $\mathfrak{p}$? Let $k \geq 12$ be even, let $f = \sum a(n)q^n \in S_k$ be a normalized eigenform, let $K \subset \mathbb{C}$ be a number field containing the coefficients of all normalized eigenforms in S_k, $\mathcal{O} := \mathcal{O}_K$, and

$\Lambda := S_k(\mathbb{Z}) \otimes \mathcal{O}$. Let $Y \subset S_k \otimes K$ be the span of the other normalized eigenforms, and write $\mathrm{C}(f) := \mathrm{C}(\Lambda; Kf)$ for the decomposition $S_k \otimes K = Kf \oplus Y$. Let

$$\langle g, h \rangle := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} g(z)\,\overline{h(z)}\,y^k\,\frac{dx\,dy}{y^2}$$

be the Petersson product.

Problem 31.4.1. Y is the orthogonal complement of f, and the projection to Kf is $g \mapsto \frac{\langle g,f\rangle}{\langle f,f\rangle} f$. Hence $\mathrm{C}(f) \cong J_f/\mathcal{O}$, where J_f is the $\mathcal{O}$-module generated by the numbers $\langle g, f\rangle/\langle f, f\rangle$ with $g \in S_k(\mathbb{Z})$. For $f = f_+$ of Section 31.3, $J_f = (24\sqrt{D})^{-1}\mathcal{O}$.

So the denominator is a division by $\langle f, f\rangle$, and Hida's identity says how far that division fails to be integral. For p prime write $a(p) = \alpha_p + \beta_p$ with $\alpha_p\beta_p = p^{k-1}$, and let

$$L(\mathrm{Sym}^2 f, s) := \prod_p \bigl(1 - \alpha_p^2 p^{-s}\bigr)^{-1}\bigl(1 - p^{k-1-s}\bigr)^{-1}\bigl(1 - \beta_p^2 p^{-s}\bigr)^{-1} \qquad (\mathrm{Re}\, s > k).$$

In the unitary normalization, the *adjoint L-function* is $L(s, \mathrm{Ad}\, f) := L(\mathrm{Sym}^2 f, s + k - 1)$; its value at $s = 1$ is $L(\mathrm{Sym}^2 f, k)$.

Problem 31.4.2. If $a_m := (\alpha^{m+1} - \beta^{m+1})/(\alpha - \beta)$ and $w := \alpha\beta$, then

$$\sum_{m\ge 0} a_m^2 X^m = \frac{1 + wX}{(1 - \alpha^2 X)(1 - wX)(1 - \beta^2 X)}.$$

Hence $\sum_{n\ge1} a(n)^2 n^{-s} = \zeta(s - k + 1)\, L(\mathrm{Sym}^2 f, s)/\zeta(2s - 2k + 2)$.

Rankin's method [Ra] computes the residue of this Dirichlet series at $s = k$; in terms of $L(\mathrm{Sym}^2 f, s)$, which is holomorphic there [Sh],

$$\langle f, f\rangle = \frac{(k-1)!}{2^{2k-1}\pi^{k+1}}\, L(\mathrm{Sym}^2 f, k) \tag{1}$$

[Za, p. 109, (3)–(5)]. For $f = \Delta$ both sides equal $1.0353620568\ldots \times 10^{-6}$, the value of $\|\Delta\|^2$ found from Petersson's formula in Section 11.2 of Chapter 11.

The transcendental part of $\langle f, f\rangle$ is measured by periods. Let

$$r_f(X) := \int_0^{i\infty} f(z)(X - z)^{k-2}\,dz,$$

and let r_f^+ and r_f^- be its even and odd parts in X. By Manin's theorem there are $\omega_f^\pm \in \mathbb{C}^\times$ with $r_f^\pm \in \omega_f^\pm K[X]$ [Mn]. For $f = \Delta$ one has [KZ]

$$r_\Delta^+ = \omega^+\Bigl(\tfrac{36}{691}(X^{10} - 1) + X^2 - 3X^4 + 3X^6 - X^8\Bigr),$$
$$r_\Delta^- = \omega^-\bigl(4X^9 - 25X^7 + 42X^5 - 25X^3 + 4X\bigr).$$

Problem 31.4.3. The two polynomials in parentheses, and $X^{10} - 1$, satisfy $P(X) + X^{10}P(-1/X) = 0$.

The polynomial $X^{10} - 1$ is a coboundary in the Eichler–Shimura cohomology, and the prime in front of it is the 691 of Problem 31.2.2. For a cusp form one discards this direction. Fix a prime $\mathfrak{p}$ of $\mathcal{O}$, and let $\Omega_f^\pm \in \mathbb{C}^\times$ be such that the coefficients of X^n in $r_f^\pm$ with $0 < n < k - 2$ lie in $\Omega_f^\pm\mathcal{O}_\mathfrak{p}$ and generate it, where

$\mathcal{O}_\mathfrak{p}$ is the localization of $\mathcal{O}$ at $\mathfrak{p}$. For $\mathfrak{p}$ above a prime $p > k$ these are Hida's periods, defined by generators of the f-eigenlines in the parabolic cohomology of $\mathrm{SL}_2(\mathbb{Z})$ with coefficients in polynomials of degree at most $k-2$ over $\mathcal{O}_\mathfrak{p}$: the period polynomial represents the class of f there, modulo the coboundary $X^{k-2}-1$. The following theorem is due to Hida [Hi1, Hi2], with a contribution of Ribet [Ri]; see [Gh, Thm. 1]. We use it as a black box.

Theorem 31.4.1 (Hida)**.** *The number $\eta_f := \langle f, f\rangle/(\mathrm{i}\,\Omega_f^+\Omega_f^-)$ lies in K. If $\mathfrak{p}$ lies above a prime $p > k$, then $\mathrm{C}(f)\otimes\mathcal{O}_\mathfrak{p} \neq 0$ if and only if $\eta_f \in \mathfrak{p}\mathcal{O}_\mathfrak{p}$.*

By (1), η_f is, up to a $\mathfrak{p}$-adic unit, the algebraic part $L(1,\mathrm{Ad}\,f)/(\pi^{k+1}\mathrm{i}\,\Omega_f^+\Omega_f^-)$ of the adjoint L-value. In outline, the proof applies Problem 31.1.7(ii) to the parabolic cohomology lattice, self-dual for $p > k$, whose f-part is spanned by isotropic $\pm$ lines; the main difficulty is to compare this cohomological congruence module with $\mathrm{C}(f)$ [Gh, §3]; in weight 2 see [DDT, §4.4]. Two checks with PARI/GP [PARI]: with the coefficient of X^2 in r_Δ^+ as Ω^+ and ω^- as Ω^-, one has $\eta_\Delta = \pm 2^{-10}$, with no prime $p > 12$, as it must be since $S_{12} = \mathbb{C}\Delta$; had one kept the coefficient $36/691$ of $X^{10}-1$, one would have found 691. For $f = f_+$ of Section 31.3, take for Ω^+ and Ω^- the coefficients of X^2 in r_f^+ and of X in r_f^-, which are admissible at every prime above $p > 24$. Then the norm of η_f is D times a rational number whose numerator and denominator have no prime factor above 24, so $v_\mathfrak{p}(\eta_f) = 1$, matching Problem 31.3.5.

The denominator in the argument. In the reconstruction [Adr, §17.1], at a twisted place the lattice Λ of integral lifted forms is split into the eigencomponents V_{good} seen by the period pairing and the others V_{bad}, and the defect is the length of $V_{\mathrm{good}}\cap(\Lambda+V_{\mathrm{bad}})/(\Lambda\cap V_{\mathrm{good}})$. By Problems 31.1.1 and 31.1.2, this is the congruence module of V_{good}: the denominator of the projection that the period pairing forces. By Problem 31.1.4 it is bounded by congruence ideals [Adr, §17.5]. In the setting of Theorem 31.4.1 such congruences are detected by adjoint L-values, which the argument does not control. It uses instead that a congruence with a reducible eigensystem produces a Galois cohomology class, as the congruence ideal $691\mathbb{Z}$ of Δ in Section 31.2 produces the nonzero group $A^{(679)}$. Chapter 30 computed the groups of this kind for Tate twists over $\mathbb{Q}$, and Chapter 32 bounds congruence ideals by them in this way.

Exercise. For $k = 16, 18, 20, 22, 26$ the space S_k is spanned by one normalized eigenform f_k, and $a_\ell(f_k) \equiv 1 + \ell^{k-1} \pmod{N_k}$ for every prime ℓ.

Hints

§31.1.

31.1.1: Consider $\Lambda \to \pi_1(\Lambda)/(\Lambda\cap V_1)$, $\lambda \mapsto \pi_1(\lambda)$, and compute its kernel.
31.1.2: The map $\pi_1(\Lambda) \to (\Lambda + \pi_1(\Lambda))/\Lambda$.
31.1.3: $\pi_1(\Lambda) = \mathbb{Z}e_1$.
31.1.4: Apply to Λ an element of $\mathbf{T}$ that vanishes off x. For the length, $\pi_x(\Lambda)$ is free of rank $\dim V_x$.
31.1.5: Identify V_x with K by $t \mapsto t(x)$; then $\pi_x(\mathbf{T}) = \mathcal{O}$.
31.1.6: $\Lambda\cap V_1$ is a direct summand of Λ, so an $\mathcal{O}$-valued functional on it extends to Λ; then use that the pairing is perfect.
31.1.7: Compute the dual basis.

§31.2.

31.2.1: Subtract a multiple of E_4^3 to kill the constant term; $S_{12} = \mathbb{C}\Delta$.

31.2.2: Compare the coefficients of 1 and q. Then use $\gcd(432000, 691) = 1$.

31.2.3: The projection to $\mathbb{Q}E_k$ is $g \mapsto a_0(g)E_k$, and some $E_4^a E_6^b$ lies in $M_k(\mathbb{Z})$ with constant term 1.

31.2.4: For the first claim use the formula for $a_n(T_m f)$ in [SeA, Ch. VII, §5.3]. For the second, Problem 31.2.2 gives one inclusion; for the other use T_2, T_3 and $\gcd(2073, 176896) = 691$.

31.2.5: For $g = aE_4^3 + b\Delta$, compute the index of $\mathbf{T}_{12}\, g$ in $M_{12}(\mathbb{Z})$ using $(1, 1)$ and $(691, 0)$; note $432000 \equiv 125 \pmod{691}$.

§31.3.

31.3.1: As in Problem 31.2.1, with $S_{24} = \Delta M_{12}$.

31.3.2: A normalized eigenform is determined by $a_1 = 1$ and $a_2 =$ its T_2-eigenvalue [SeA, Ch. VII, §5.4].

31.3.3: $\mathbf{T}_{24} \otimes \mathbb{Q} = \mathbb{Q}[T_2]$, and the matrix of T_2 in the basis g_1, g_2 has an entry 1.

31.3.4: Write $t = a + bT_2$ with $a, b \in \mathcal{O}$, and apply Problem 31.1.5. For the direct check, write g_1, g_2 in terms of $f_\pm$.

31.3.5: Write $f_+ - f_-$ in the basis g_1, g_2. For the splitting, reduce D modulo 8 and 3.

§31.4.

31.4.1: The Hecke operators are self-adjoint, and the eigenforms have distinct eigenvalues. $S_k(\mathbb{Z})$ has a basis $g_1, \ldots, g_d$ with $a_i(g_j) = \delta_{ij}$ for $i, j \le d$, built from the forms $\Delta^j E_4^a E_6^b$ as in Problems 31.2.1 and 31.3.1, so $\Lambda = (S_k \otimes K) \cap \mathcal{O}[[q]]$; hence $\Lambda \cap Kf = \mathcal{O}f$ because $a_1(f) = 1$. For the example use Problem 31.3.4.

31.4.2: Expand a_m^2 as a combination of α^{2m}, w^m and β^{2m}, and sum three geometric series; for $\alpha = \beta$, read a_m as $\sum_{i+j=m} \alpha^i \beta^j$.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

[DDT] H. Darmon, F. Diamond and R. Taylor, *Fermat's Last Theorem*, in: Current Developments in Mathematics, 1995, International Press, 1994, 1–154.

[DS] F. Diamond and J. Shurman, *A First Course in Modular Forms*, Grad. Texts in Math. 228, Springer, 2005.

[Gh] E. Ghate, *An introduction to congruences between modular forms*, in: Current Trends in Number Theory (Allahabad, 2000), Hindustan Book Agency, New Delhi, 2002, 39–58.

[Hi1] H. Hida, *Congruences of cusp forms and special values of their zeta functions*, Invent. Math. **63** (1981), 225–261.

[Hi2] H. Hida, *On congruence divisors of cusp forms as factors of the special values of their zeta functions*, Invent. Math. **64** (1981), 221–262.

[KZ] W. Kohnen and D. Zagier, *Modular forms with rational periods*, in: Modular Forms (Durham, 1983), R. A. Rankin (ed.), Ellis Horwood, Chichester, 1984, 197–249.

[Mn] Yu. I. Manin, *Periods of parabolic forms and p-adic Hecke series*, Math. USSR-Sb. **21** (1973), 371–393.

[PARI] The PARI Group, *PARI/GP*, version 2.17.1, Univ. Bordeaux, `https://pari.math.u-bordeaux.fr/`.

[Ra] R. A. Rankin, *Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions. I, II*, Proc. Cambridge Philos. Soc. **35** (1939), 351–356 and 357–372.

[Ri] K. A. Ribet, *Mod p Hecke operators and congruences between modular forms*, Invent. Math. **71** (1983), 193–205.

[SeA] J.-P. Serre, *A Course in Arithmetic*, Grad. Texts in Math. 7, Springer, 1973.

[Sh] G. Shimura, *On the holomorphy of certain Dirichlet series*, Proc. London Math. Soc. (3) **31** (1975), 79–98.

[Za] D. Zagier, *Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields*, in: Modular Functions of One Variable VI, Lecture Notes in Math. 627, Springer, 1977, 105–169.