HECKE NONDEGENERACY ON THE TREE

Prerequisites. Modules over a discrete valuation ring and the elementary divisor theorem [La, Ch. III, §7]. Unramified extensions of $\mathbb{Q}_p$ and Teichmüller representatives. The tree of GL_2 over a local field [SeT, Ch. II, §1]. Legendre's formula $v_p(m!) = (m - s_p(m))/(p-1)$, where $s_p(m)$ is the sum of the base-p digits of m. From Chapter 6, *The slope method*: Problems 6.4.4 and 6.5.3, and Theorem 6.5.1. The language of dual Weyl modules [Ja, II.2, II.4] is used only in remarks.

In Chapter 6 a gain at a prime was recorded by passing to a sublattice, and Theorem 6.5.1 charges for it the length of the quotient. When the sublattice is spanned by images of a Hecke operator, how long is that quotient? We answer the question for GL_2, weight k, and the operator that sums over the neighbours of a vertex of the tree. The answer is in terms of binomial coefficients, and Kummer's theorem bounds it. This is the simplest case of the loss in the reconstruction of Adriazola [Adr, §12].

Why Hecke operators? In the argument the maps of a chain are theta lifts, and the sublattices to which Problem 6.5.3 is applied are spanned by images of Hecke operators. Hecke operators are attached to theta lifts only in Chapter 24, *Hecke operators and theta lifts*: there a Hecke operator on a class set is a sum over the neighbours of a lattice, as the operator here is a sum over the neighbours of a vertex, and the theta lift carries it to a Hecke operator on modular forms. This chapter uses nothing from Chapter 24.

Throughout, K is the unramified extension of $\mathbb{Q}_p$ of degree f, with ring of integers W, residue field $\mathbb{F}_Q$ where $Q = p^f$, and valuation v normalized by $v(p) = 1$. We fix an integer $k \geq 1$.

19.1. The tree and the integral forms

A *lattice* is a free W-submodule of rank 2 of K^2. For a lattice L, its *sources* are the lattices L'' with $pL \subset L'' \subset L$ and $L/L'' \cong \mathbb{F}_Q$. The vertices of the tree are the classes of lattices up to homothety, and the neighbours of the class of L are the classes of its sources [SeT, Ch. II, §1].

Problem 19.1.1. The sources of L correspond to the lines of $L/pL \cong \mathbb{F}_Q^2$. There are $Q+1$ of them, and their classes are pairwise distinct.

Let V be the space of homogeneous polynomial functions of degree k on K^2, on which $\mathrm{GL}_2(K)$ acts by $(gF)(u) = F(g^{-1}u)$. For a lattice L with dual $L^\vee = \mathrm{Hom}_W(L, W)$, put

$$H(L) := \mathrm{Sym}^k(L^\vee) \subset V.$$

If x, y is the basis of $L^\vee$ dual to a basis of L, then $H(L) = \bigoplus_{j=0}^{k} W x^{k-j} y^j$. Over $\mathbb{Q}_p$, $H(\mathbb{Z}_p^2)$ is the standard integral form of the dual Weyl module $\nabla(\lambda)$ with $\lambda = (0, -k)$ for this action; its reduction modulo p is $\nabla(\lambda)$ over $\mathbb{F}_p$, because $H^1(\mathbb{P}^1, \mathcal{O}(k)) = 0$, the case of GL_2 in Kempf's vanishing theorem [Ja, II.2, II.4]. For GL_n the analogous

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integral bases are the standard bideterminants of De Concini, Eisenbud and Procesi [DEP]; see also [Gr]. In rank 2 they are the monomials.

Problem 19.1.2. $H(gL) = gH(L)$ for $g \in \mathrm{GL}_2(K)$, and $H(cL) = c^{-k}H(L)$ for $c \in K^\times$. If $L'' \subset L$, then $H(L) \subset H(L'')$.

Problem 19.1.3. Let L'' be a source of L. There is a basis x, y of $L^\vee$ such that $L''^\vee = Wx + Wp^{-1}y$. Hence the elementary divisors of $H(L)$ in $H(L'')$ are $1, p, \dots, p^k$, we have $p^k H(L'') \subset H(L)$, and $\mathrm{length}_W H(L'')/H(L) = k(k+1)/2$.

A *form* is a function F from lattices to V. It is *integral* if $F(L) \in H(L)$ for every L. The Hecke operator and its normalization are

$$(TF)(L) := \sum_{L''} F(L''), \qquad Y := p^k T,$$

where the sum runs over the $Q+1$ sources of L. Let

$$S(L) := \sum_{L''} p^k H(L'') \subset H(L).$$

Problem 19.1.4. Y maps integral forms to integral forms, and $\{(YF)(L) : F \text{ integral}\} = S(L)$.

Problem 19.1.5. $S(L) = \sum_M \mathrm{Sym}^k(M)$, the sum over the $Q+1$ lattices M with $pL^\vee \subset M \subset L^\vee$ and $L^\vee/M \cong \mathbb{F}_Q$.

Problem 19.1.6. For $Q = 2$ and $k = 2$, $S(L) = Wx^2 + 2Wxy + Wy^2$. For $k = 1$ and every Q, $S(L) = H(L)$.

The *loss* at L is $\mathrm{length}_W H(L)/S(L)$. By Problem 6.5.3, recording a gain of the operator Y as a change of lattice at L lowers the degree by $\log Q$ times the loss. It does not depend on L, by Problem 19.1.2. By Problem 19.1.3 a single source costs $k(k+1)/2$. The question is how much the $Q+1$ sources save together.

19.2. Elementary divisors

Let $S \subset H$ be a submodule of full rank n of a free W-module. By the elementary divisor theorem [La, Ch. III, §7] there is a basis $h_1, \dots, h_n$ of H with $S = \bigoplus_i p^{e_i} W h_i$, and the exponents $e_1 \le \cdots \le e_n$ are unique. Then $\mathrm{length}_W H/S = e_1 + \cdots + e_n$. For an integer $D \ge 0$ and a real $\rho \in [0,1]$, we say that S is *nondegenerate up to p^D on a share ρ* when $\#\{i : e_i \le D\} \ge \rho n$.

Problem 19.2.1. Let A be a matrix whose rows are the coordinates, in a basis of H, of a generating set of S. Then $e_1 + \cdots + e_l$ is the least valuation of an $l \times l$ minor of A. In particular the e_i do not change under a finite unramified extension of W.

Problem 19.2.2. $\#\{i : e_i \le D\}$ is the largest N for which there are $s_1, \dots, s_N \in S$ and $\varphi_1, \dots, \varphi_N \in \mathrm{Hom}_W(H, W)$ with $\varphi_a(s_b) = p^D \delta_{ab}$.

The vectors s_b are *witnesses* and the functionals φ_a are *tests*. In practice one finds a test for each witness and then assembles them. The next problem does the assembly.

Problem 19.2.3. Let Λ be a W-module and $\varphi_1, \dots, \varphi_M \colon \Lambda \to W$ linear maps. Suppose that for every $c \in W^M$ with some c_i a unit there is $G \in \Lambda$ with $v\big(\sum_i c_i \varphi_i(G)\big) \le D$. Then the image of $G \mapsto (\varphi_i(G))_i$ contains $p^D W^M$.

Problem 19.2.4. Let A, A' be $n \times n$ matrices over W with $A \equiv A' \pmod{p^T}$ for an integer $T \geq 1$, with exponents e_i and e_i' (where $e_i = \infty$ for the zero elementary divisors). Then $\min(e_i, T) = \min(e_i', T)$ for every i.

Problem 19.2.4 lets one compute exponents below T modulo p^T. The manuscript [Adr, §12] uses Problems 19.2.3 and 19.2.4 to reduce nondegeneracy to tests on single edges.

19.3. Kummer's theorem

The exponents will be valuations of binomial coefficients.

Problem 19.3.1. For $0 \leq j \leq n$, $v_p\binom{n}{j} = \big(s_p(j) + s_p(n-j) - s_p(n)\big)/(p-1)$, and this is the number of carries in the addition of j and $n-j$ in base p.

This is due to Kummer [Ku]; see [Gra] for a survey.

Problem 19.3.2. For $1 \leq j \leq n$, $v_p\binom{n}{j} \leq \lfloor \log_p n \rfloor - v_p(j)$. In particular $p^{v_p\binom{n}{j}} \leq n$.

Problem 19.3.3. Every $\binom{Q-1}{j}$ with $0 \leq j \leq Q-1$ is prime to p. For $0 < j < p^m$, $v_p\binom{p^m}{j} = m - v_p(j)$, so Problem 19.3.2 is sharp.

19.4. The Fock form

For $F = \sum_m F_m x^m$ and $G = \sum_m G_m x^m$ in $K[x_1, \ldots, x_n]$, with multi-indices m and $m! = \prod_s m_s!$, let

$$\langle F, G \rangle := \sum_m F_m G_m \, m!, \qquad \text{that is,} \qquad \langle x^m, x^{m'} \rangle = m! \, \delta_{mm'}.$$

In one variable over $\mathbb{C}$, with complex conjugation on G, this is the Gaussian inner product of Problem 6.4.4; it is the inner product of the Fock model [Ba].

Problem 19.4.1. $\langle x_s F, G \rangle = \langle F, \partial_s G \rangle$ for every s. Hence $\langle F, G \rangle = \big(F(\partial)G\big)(0)$, where $F(\partial)$ is F with each x_s replaced by $\partial/\partial x_s$.

Problem 19.4.2. For $F \in K[x, y]$ homogeneous of degree k and $a, b \in K$, $\langle (ax + by)^k, F \rangle = k! \, F(a, b)$.

Let $H = \bigoplus_{j=0}^{k} W x^{k-j} y^j$, and let $H^\sharp := \{G \in K[x, y]$ homogeneous of degree k : $\langle G, F \rangle \in k! \, W$ for all $F \in H\}$.

Problem 19.4.3. $H^\sharp = \bigoplus_{j=0}^{k} \binom{k}{j} W x^{k-j} y^j$, and $p^{\lfloor \log_p k \rfloor} H \subset H^\sharp \subset H$.

So $\langle \cdot, \cdot \rangle / k!$ is a perfect pairing between $H^\sharp$ and H. Measured against H, it is nondegenerate up to $p^{\lfloor \log_p k \rfloor}$ on all of the rank. This is the mechanism: Kummer's theorem turns a factorial normalization into a logarithmic loss.

19.5. The span at a vertex

Fix L and a basis x, y of $L^\vee$. The lattices M of Problem 19.1.5 are $M_a := W(x + ay) + pL^\vee$ for $a \in W$, which depends only on a modulo p, and $M_\infty := Wy + pL^\vee$.

Problem 19.5.1. Suppose $Q > k$. The W-span of $\{\lambda^k : \lambda \in L^\vee\}$ is $H(L)^\sharp$. For $Q = 2$ and $k = 3$ the span is a sublattice of colength 1 in $H(L)^\sharp = H(L)$.

Problem 19.5.2. $\mathrm{Sym}^k(M_a)$ is spanned by the $(x+ay)^{k-i}(py)^i$ with $0 \le i \le k$. If $Q > k$, then for each i the W-span of $\{(x+ay)^{k-i}y^i : a \in W\}$ is $\bigoplus_{j=i}^{k} \binom{k-i}{j-i} W x^{k-j}y^j$.

Problems 19.1.5 and 19.5.2, together with $\mathrm{Sym}^k(M_\infty) = \bigoplus_j p^{k-j} W x^{k-j}y^j$, prove the following. Then Problem 19.3.2 gives the corollary.

Theorem 19.5.1. *Suppose $Q > k$. Then $S(L) = \bigoplus_{j=0}^{k} p^{\varepsilon_j} W x^{k-j}y^j$, where the exponent ε_j of the monomial $x^{k-j}y^j$ is*

$$\varepsilon_j = \min\Big(k-j,\ \min_{0\le i\le j}\Big(i + v_p\binom{k-i}{j-i}\Big)\Big).$$

In particular $\varepsilon_j \le \min\big(j,\, k-j,\, v_p\binom{k}{j}\big) \le \lfloor \log_p k \rfloor$. So $p^{\lfloor \log_p k\rfloor} H(L) \subset S(L)$, and the loss is at most $\sum_j v_p\binom{k}{j}$.

When $Q > k$, the normalized Hecke operator is therefore nondegenerate up to $p^{\lfloor \log_p k\rfloor}$ on all of the rank. The bound $\varepsilon_j \le v_p\binom{k}{j}$ uses only the terms with $i = 0$ in Problem 19.5.2, that is, the kth powers of Problem 19.5.1. The terms with $i \ge 1$ lower the exponents further.

Problem 19.5.3. $\varepsilon_j = \varepsilon_{k-j}$. For $k = Q-1$ the loss is 0. For $k = p < Q$ the loss is $p-1$. For $p = 3$, $k = 6$ and $Q \ge 9$ the ε_j, in the order $j = 0, \dots, k$, are $0,1,1,0,1,1,0$. For $p = 2$, $k = 8$ and $Q \ge 16$ they are $0,1,1,1,1,1,1,1,0$, so the loss is 7, although $\sum_j v_2\binom{8}{j} = 17$.

What if there are few edges?

Problem 19.5.4. If $Q \le k$, then at most $Q+1$ of the exponents of $S(L)$ in $H(L)$ vanish, and the loss is at least $k - Q$.

In fact, by Problem 19.1.3 each source supplies at most m generators of $S(L)$ modulo $p^m H(L)$, so at most $(Q+1)m$ of the exponents are less than m. Hence the largest exponent is at least $(k+1)/(Q+1) - 1$, and for fixed Q the loss grows quadratically in k. For $Q = 2$ and $k = 15$, a Smith form computation gives the exponents $0,0,0,1,1,1,2,2,2,3,3,3,4,4,4,5$, a loss of 35. For $Q = 3$ and $k = 15$ it gives $0,0,0,0,1,1,1,1,2,2,2,2,4,4,4,4$. We do not know a closed formula for the exponents when $Q \le k$.

A vertex must therefore have more edges than the dimension of the representation, or at least comparably many, before the operator Y loses only a logarithm. In the manuscript [Adr, §12] the residue field is $\mathbb{F}_Q$ with $Q = p^{n'}$ and n' large. There the lattices have rank $d \ge 2$, a source L'' of L has $pL \subset L''$ and $L/L'' \cong \mathbb{F}_Q^r$ for a fixed $1 \le r < d$, and a lattice has at least $Q^{r(d-r)}$ sources. Each source has to carry only a share $Q^{-r(d-r)}$ of the rank, and the tests are pairings with a twisted form of the Fock form of Section 19.4. With the gain of Chapters 17 and 18 and this loss, the divisibility side is in place; the next part, from Chapter 20 on, turns to the sizes: periods, norms of theta lifts and covolumes.

Hints

§19.1.

19.1.1: $L'' \mapsto L''/pL$. For the last statement, compare the indices in L of two homothetic sources.

19.1.2: $(gL)^\vee = (g^{-1})^\mathsf{T} L^\vee$ under the pairing. An inclusion of lattices reverses on duals.

19.1.3: Choose a basis u_1, u_2 of L with $L'' = Wu_1 + Wpu_2$, by the elementary divisor theorem.

19.1.4: For the image, take F supported on the sources of L.

19.1.5: $p^k \operatorname{Sym}^k(L''^\vee) = \operatorname{Sym}^k(pL''^\vee)$, and $L'' \mapsto pL''^\vee$ is a bijection onto the lattices M.

19.1.6: List the three lattices M and expand.

§19.2.

19.2.1: Multiplying A on either side by an invertible matrix does not change the ideal generated by its $l \times l$ minors.

19.2.2: For $\geq$, use the dual basis of (h_i). For $\leq$, the matrix $(\varphi_a(s_b))$ factors through S.

19.2.3: The image is a submodule of W^M; compare it with $p^D W^M$.

19.2.4: Compare the cokernels of A and A' after tensoring with $W/p^T W$.

§19.3.

19.3.1: Legendre's formula, and each carry lowers the digit sum by $p-1$.

19.3.2: The digits of j below position $v_p(j)$ vanish, so no carry occurs there. A carry out of the top digit of n is impossible.

19.3.3: Problem 19.3.1.

§19.4.

19.4.1: Check it on monomials. Then induct on the degree of F.

19.4.2: Problem 19.4.1, or expand both sides.

19.4.3: The monomials are orthogonal. For the inclusions, Problem 19.3.2.

§19.5.

19.5.1: The Teichmüller representatives are distinct modulo p, so their Vandermonde determinant is a unit. For $Q = 2$, look for a cubic form that takes values in $2\mathbb{Z}_2$ on $\mathbb{Z}_2^2$.

19.5.2: The same Vandermonde argument, applied to the vectors $\left(\binom{k-i}{j-i} a^{j-i}\right)_j$.

19.5.3: The symmetry $x \leftrightarrow y$ of $H(L)$ permutes the lattices M. Then Problem 19.3.3.

19.5.4: Reduce modulo p. Only the kth powers of $Q+1$ linear forms survive.

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