BLACK BOXES: GALOIS REPRESENTATIONS AND LOCAL–GLOBAL COMPATIBILITY

Prerequisites. Local and global class field theory, including ray class groups and the Chebotarev density theorem [Ne, Ch. V–VI, §VII.13]. Elliptic curves at the level of [Si1, Ch. III, V]; the curve $y^2 = x^3 - x$ as in Chapter 14, *Complex multiplication and reduction at an inert prime*. Modular forms of weight 2 and Hecke operators [DS, Ch. 1–6]. Parabolic induction and the Steinberg representation of GL_2 over a p-adic field [BH, Ch. 3]. The words crystalline, potentially semistable and Hodge–Tate weight are used only in statements. Everything below that is not posed as a problem is a published theorem, stated with a reference, and is a black box for this chapter.

34.1. Weil–Deligne representations and local Langlands maps

What does a Galois representation remember at a prime $p \neq \ell$, and what does an automorphic representation give to compare it with?

Let $K/\mathbb{Q}_p$ be finite, with uniformizer ϖ_K and residue field of order q. Let W_K be its Weil group, I_K the inertia subgroup, and $\mathrm{Frob} \in W_K$ a geometric Frobenius. Let $\mathrm{Art}_K \colon K^\times \to W_K^{ab}$ be the reciprocity map, normalized so that ϖ_K goes to a geometric Frobenius, and let $|w| := |\mathrm{Art}_K^{-1}(w)|$, so that $|\mathrm{Frob}| = q^{-1}$. A *Weil–Deligne representation* is a pair (r, N) of a representation $r \colon W_K \to \mathrm{GL}(V)$ on a finite-dimensional space over $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$, trivial on an open subgroup of I_K, and an endomorphism N with $r(w) N r(w)^{-1} = |w| N$. It is *Frobenius-semisimple* if $r(\mathrm{Frob})$ is semisimple. Let $\mathrm{Sp}(2)$ be the pair with $V = \mathbb{C}^2$ (or $\overline{\mathbb{Q}}_\ell^2$), $r(w) = \mathrm{diag}(1, |w|)$ and $Ne_1 = e_2$, $Ne_2 = 0$.

Problem 34.1.1. In a Weil–Deligne representation, N is nilpotent.

Problem 34.1.2. A two-dimensional Weil–Deligne representation with $N \neq 0$ is isomorphic to $\mathrm{Sp}(2) \otimes \chi$ for exactly one character χ of W_K.

Let $\ell \neq p$. By Grothendieck's monodromy theorem [Ta, 4.2.1–4.2.2], every continuous $\rho \colon W_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ determines a Weil–Deligne representation $\mathrm{WD}(\rho) = (r, N)$, characterized by $\rho(\mathrm{Frob}^m \sigma) = r(\mathrm{Frob}^m \sigma) \exp(t_\ell(\sigma) N)$ for $m \in \mathbb{Z}$ and $\sigma \in I_K$, where $t_\ell \colon I_K \to \mathbb{Z}_\ell$ is a surjection onto the ℓ-part of tame inertia. We write $\mathrm{WD}(\rho)^{F\text{-ss}}$ for its Frobenius-semisimplification.

Problem 34.1.3. If $\rho(I_K)$ is finite, then $N = 0$. If $\rho(I_K)$ is a nontrivial group of unipotent matrices, then $N \neq 0$.

For characters $\chi_1, \dots, \chi_n$ of $K^\times$, let $i(\chi_1, \dots, \chi_n)$ be the normalized parabolic induction from the Borel subgroup of $\mathrm{GL}_n(K)$. Let St be the Steinberg representation of $\mathrm{GL}_2(K)$, the irreducible subquotient of $i(|\cdot|^{1/2}, |\cdot|^{-1/2})$ that is not a character.

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Theorem 34.1.1 (Harris–Taylor, Henniart)**.** *For each $n \geq 1$ there is a bijection rec_K from irreducible admissible representations of $\mathrm{GL}_n(K)$ to Frobenius-semisimple n-dimensional Weil–Deligne representations of W_K such that $\mathrm{rec}_K(\chi) = \chi \circ \mathrm{Art}_K^{-1}$ for $n = 1$, $\mathrm{rec}_K(\pi \otimes \chi \circ \det) = \mathrm{rec}_K(\pi) \otimes \mathrm{rec}_K(\chi)$, $\det \mathrm{rec}_K(\pi) = \mathrm{rec}_K(\omega_\pi)$ for the central character ω_π, $\mathrm{rec}_K(\pi^\vee) = \mathrm{rec}_K(\pi)^\vee$, and L- and ϵ-factors of pairs are preserved. These properties determine rec_K.*

See [HT], [He] and, for another proof, [Sch]. If $i(\chi_1, \dots, \chi_n)$ is irreducible, then rec_K sends it to $\bigoplus_i \chi_i \circ \mathrm{Art}_K^{-1}$ with $N = 0$; for every irreducible subquotient π of $i(\chi_1, \dots, \chi_n)$, the semisimplification of the W_K-representation underlying $\mathrm{rec}_K(\pi)$ is $\bigoplus_i \chi_i \circ \mathrm{Art}_K^{-1}$; and $\mathrm{rec}_K(\mathrm{St} \otimes \chi) = \mathrm{Sp}(2) \otimes \chi|\cdot|^{-1/2} \circ \mathrm{Art}_K^{-1}$ [Ku]. Only these properties are used below.

34.2. Galois representations over CM fields

Let F be a CM field with totally real subfield F^+ and complex conjugation c, let ℓ be a prime and $\iota\colon \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. Let ϵ_ℓ be the ℓ-adic cyclotomic character. A cuspidal automorphic representation π of $\mathrm{GL}_a(\mathbb{A}_F)$ is *polarized* if $\pi^c \cong \pi^\vee \otimes (\chi \circ N_{F/F^+} \circ \det)$ for a character χ of $\mathbb{A}_{F^+}^\times/(F^+)^\times$ satisfying a sign condition at the real places, and *regular algebraic* if π_∞ has the infinitesimal character of an irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_a$; see [BLGGT, §2.1].

Theorem 34.2.1. *Let π be a regular algebraic polarized cuspidal automorphic representation of $\mathrm{GL}_a(\mathbb{A}_F)$. There is a continuous semisimple $r = r_{\ell,\iota}(\pi)\colon G_F \to \mathrm{GL}_a(\overline{\mathbb{Q}}_\ell)$ such that:* (i) *$r^c \cong r^\vee \otimes \epsilon_\ell^{1-a} r_{\ell,\iota}(\chi)|_{G_F}$;* (ii) *for every finite place $w \nmid \ell$, $\iota\,\mathrm{WD}(r|_{G_{F_w}})^{F\text{-}ss} \cong \mathrm{rec}_{F_w}(\pi_w \otimes |\det|^{(1-a)/2})$;* (iii) *for every $w \mid \ell$, $r|_{G_{F_w}}$ is potentially semistable, crystalline if π_w is unramified, with Hodge–Tate weights determined by π_∞, and* (ii) *holds for the Weil–Deligne representation attached to it by Fontaine.*

The representations are constructed in the cohomology of Shimura varieties attached to unitary groups, to which π is first descended, by Kottwitz, Clozel, Harris–Taylor, Shin and Chenevier–Harris [Cl, HT, Shi, CH]. Part (ii) is due to Taylor–Yoshida and Caraiani [TY, Car1], and part (iii) to Barnet-Lamb–Gee–Geraghty–Taylor and Caraiani [BLGGT, Thm. 2.1.1], [Car2]. For Alpöge's argument [Alp], the reconstruction [Adr, §17.5] lists Theorem 34.2.1, with local–global compatibility at all places, among the published inputs of its Galois-theoretic steps.

The smallest case is $a = 1$ over an imaginary quadratic field K, where the unitary group is $U(1) := \ker(N\colon \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \to \mathbb{G}_m)$.

Problem 34.2.1. $U(1)(\mathbb{R})$ is the unit circle. $U(1)(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times$ if p splits in K, and $U(1)(\mathbb{Q}_p)$ is compact otherwise.

Problem 34.2.2. Let ϕ be a character of $\mathbb{A}_K^\times/K^\times$ that is trivial on $\mathbb{A}_\mathbb{Q}^\times$. Then $\phi^c = \phi^{-1}$, and $\phi = \mu(x/x^c)$ for exactly one character μ of $U(1)(\mathbb{A})/U(1)(\mathbb{Q})$. Conversely each such μ gives such a ϕ.

Problem 34.2.3. If $\phi^c = \phi^{-1}$, then $\phi|_{\mathbb{A}_\mathbb{Q}^\times}$ is either trivial or the quadratic character $\omega_{K/\mathbb{Q}}$.

So the characters with $\phi^c = \phi^{-1}$ fall into two classes, and by Problem 34.2.2 only the first comes from $U(1)$.

34.3. The Hecke character of $\mathbb{Q}(\mathrm{i})$

What is the representation of Theorem 34.2.1 when $a = 1$, and how does it compare with the curve $E\colon y^2 = x^3 - x$? Let $K = \mathbb{Q}(\mathrm{i})$, $\mathcal{O} = \mathbb{Z}[\mathrm{i}]$ and $\mathfrak{f} = (2 + 2\mathrm{i})$.

Problem 34.3.1. Every ideal $\mathfrak{a}$ of $\mathcal{O}$ prime to $1 + \mathrm{i}$ has exactly one generator $\alpha \equiv 1 \pmod{\mathfrak{f}}$. Let $\psi(\mathfrak{a}) := \alpha$. Then ψ is multiplicative, $\psi(\mathfrak{a}^c) = \overline{\psi(\mathfrak{a})}$, and $\psi(\mathfrak{a})\psi(\mathfrak{a}^c) = N\mathfrak{a}$.

For $p \equiv 1 \pmod 4$ and $\mathfrak{p} \mid p$, one has $p+1-\#E(\mathbb{F}_p) = \psi(\mathfrak{p})+\psi(\mathfrak{p}^c)$ [IR, Ch. 18, §4].

Problem 34.3.2. $\psi((1 + 2\mathrm{i})) = -1 - 2\mathrm{i}$ and $\psi((3 + 2\mathrm{i})) = 3 + 2\mathrm{i}$, and $\#E(\mathbb{F}_5) = \#E(\mathbb{F}_{13}) = 8$.

Problem 34.3.3. Let $n \geq 1$. If ideals $\mathfrak{a}, \mathfrak{b}$ prime to 2ℓ have the same class in the ray class group modulo $\mathfrak{f}\ell^n$, then $\psi(\mathfrak{a}) \equiv \psi(\mathfrak{b}) \pmod{\ell^n\mathcal{O}}$.

Combining Problem 34.3.3 with Artin reciprocity for the ray class fields modulo $\mathfrak{f}\ell^n$ gives the following.

Theorem 34.3.1. *There is a continuous character $\psi_\ell\colon G_K \to (\mathcal{O} \otimes \mathbb{Z}_\ell)^\times$, unramified outside 2ℓ, with $\psi_\ell(\mathrm{Frob}_\mathfrak{p}) = \psi(\mathfrak{p})$ for every prime $\mathfrak{p} \nmid 2\ell$, where $\mathrm{Frob}_\mathfrak{p}$ is an arithmetic Frobenius.*

By Problem 34.3.1, $\psi\psi^c$ is the norm, so ψ is polarized, and ψ_ℓ, followed by an embedding $\mathcal{O} \otimes \mathbb{Z}_\ell \to \overline{\mathbb{Q}}_\ell$, is the representation of Theorem 34.2.1 for $a = 1$, up to the normalizations of the reciprocity maps. At the places not above ℓ the theorem is then class field theory; the next problem looks at the places above ℓ.

Problem 34.3.4. Let $\ell \equiv 1 \pmod 4$, $\ell = \lambda\bar{\lambda}$, and let $\psi_\lambda\colon G_K \to \mathbb{Z}_\ell^\times$ be ψ_ℓ followed by $\mathcal{O} \otimes \mathbb{Z}_\ell \to \mathcal{O}_\lambda = \mathbb{Z}_\ell$. Then $\psi_\lambda = \epsilon_\ell$ on an open subgroup of the inertia group at λ, and ψ_λ has finite image on the inertia group at $\bar{\lambda}$.

So ψ_λ has Hodge–Tate weights 1 and 0 at the two places above ℓ (in the convention where ϵ_ℓ has weight 1), and the polarization appears as $\mathrm{HT}_{\tau c} = 1 - \mathrm{HT}_\tau$.

The automorphism $[\mathrm{i}](x, y) = (-x, \mathrm{i}y)$ of E is defined over K, so $V_\ell(E) := T_\ell(E) \otimes \mathbb{Q}$ is free of rank one over $K \otimes \mathbb{Q}_\ell$, and G_K acts on it through a character $G_K \to (K \otimes \mathbb{Q}_\ell)^\times$.

Problem 34.3.5. Let $\mathfrak{p} \nmid 2\ell$. If $N\mathfrak{p} = p \equiv 1 \pmod 4$, then $\mathrm{Frob}_\mathfrak{p}$ acts on $V_\ell(E)$ as $\psi(\mathfrak{p})$. If $\mathfrak{p} = (p)$ with $p \equiv 3 \pmod 4$, it acts as $-p = \psi((p))$.

By the Chebotarev density theorem, Problem 34.3.5 proves the following; it is due to Deuring, see [Si2, Ch. II, §10].

Theorem 34.3.2. *The character of G_K on $V_\ell(E)$ is ψ_ℓ.*

Problem 34.3.6. Let $\psi_1, \psi_2\colon G_K \to \overline{\mathbb{Q}}_\ell^\times$ be ψ_ℓ followed by the two embeddings $\mathcal{O} \otimes \mathbb{Z}_\ell \to \overline{\mathbb{Q}}_\ell$. Then $\psi_2 = \psi_1^c$, $V_\ell(E) \otimes \overline{\mathbb{Q}}_\ell \cong \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \psi_1$, and $\mathrm{tr}(\mathrm{Frob}_p \mid V_\ell(E)) = 0$ for $p \equiv 3 \pmod 4$, $p \neq \ell$.

Restricted to G_K, $V_\ell(E)$ is $\psi_1 \oplus \psi_1^c$, which is reducible. So the weight-two form of E, base changed to K, is not cuspidal, and it enters Theorem 34.2.1 only through ψ, with $a = 1$. For a cuspidal example with $a = 2$ one needs a curve without complex multiplication by K. We take $X_0(11)$ and stay over $\mathbb{Q}$; its base change to K is cuspidal, and the local questions are the same.

34.4. Eichler–Shimura for $X_0(11)$

How does one get Frobenius eigenvalues out of the cohomology of a Shimura variety, and what does each method give? Let

$$f = q\prod_{n\geq 1}(1-q^n)^2(1-q^{11n})^2 = \sum_{n\geq 1} a_n q^n,$$

the normalized eigenform spanning $S_2(\Gamma_0(11))$ [DS, §3.9]. The curve $X_0(11)$ has genus one and the model $E\colon y^2+y = x^3-x^2-10x-20$ over $\mathbb{Q}$, with good reduction away from 11. For $p \neq 11$, the Hecke operator T_p acts on $E = J_0(11)$ as $[a_p]$, and by the Eichler–Shimura relation [DS, Thm. 8.7.2], [Sh, Thm. 7.9], the reduction of T_p modulo p is $F+V$, where F is the Frobenius endomorphism and $FV = VF = p$.

Problem 34.4.1. $a_2 = -2$, $a_3 = -1$, $a_5 = 1$, $a_7 = -2$, $a_{11} = 1$, $a_{13} = 4$.

Problem 34.4.2. For $p \notin \{11, \ell\}$, the arithmetic Frobenius at p on $V_\ell(E)$ satisfies $\mathrm{Frob}_p^2 - a_p\mathrm{Frob}_p + p = 0$.

For $p \notin \{11, \ell\}$, the arithmetic Frobenius at p has determinant p on $V_\ell(E)$, by the Weil pairing [Si1, §III.8], and $\#E(\mathbb{F}_{p^k}) = p^k + 1 - \mathrm{tr}(\mathrm{Frob}_p^k \mid V_\ell(E))$ for $k \geq 1$ [Si1, §V.2].

Problem 34.4.3. Let $\Phi \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ satisfy $\Phi^2 - a\Phi + p = 0$ with $a \in \mathbb{Z}$, and let $\det\Phi = p$. Then the characteristic polynomial of Φ is $X^2 - aX + p$. Hence, taking $\ell \neq p$, $\#E(\mathbb{F}_p) = p + 1 - a_p$ for $p \neq 11$.

The relation of Problem 34.4.2 kills Frobenius; it does not give the multiplicities of its roots. In dimension two the determinant supplies them. In dimension three, $\mathrm{diag}(\alpha,\alpha,\beta)$ and $\mathrm{diag}(\alpha,\beta,\beta)$ are killed by the same quadratic polynomial. The published constructions behind Theorem 34.2.1 obtain the whole characteristic polynomial by counting points, following Langlands and Kottwitz [Ko]. The simplest instance is the trace formula for $\#E(\mathbb{F}_{p^k})$ stated before Problem 34.4.3.

Problem 34.4.4. $\#E(\mathbb{F}_p) = p + 1 - a_p$ for $p = 2, 3, 5, 7, 13$ by direct count. The numbers $\#E(\mathbb{F}_p)$ and $\#E(\mathbb{F}_{p^2})$ determine the characteristic polynomial of Frob_p; for $p = 2$ both equal 5.

At 11 the curve has split multiplicative reduction. By Tate's theory [Si2, Ch. V], $E \cong \mathbb{G}_m/q^{\mathbb{Z}}$ over $\mathbb{Q}_{11}$ for some $q \in 11\mathbb{Z}_{11}$, and $T_\ell(E)$ is an extension of $\mathbb{Z}_\ell$ by $\mathbb{Z}_\ell(1)$ on which the inertia group at 11 acts through the Kummer classes of q^{1/ℓ^n}.

Problem 34.4.5. For $\ell \neq 11$, $\mathrm{WD}(V_\ell(E)|_{G_{\mathbb{Q}_{11}}}) \cong \mathrm{Sp}(2) \otimes \chi$ for an unramified character χ; in particular $N \neq 0$.

Correspondingly, π_{11} is an unramified twist of St, and the compatibility of Theorem 34.2.1(ii), in its version over $\mathbb{Q}$ due to Carayol [Ca], matches the monodromy at 11. The later parts, from Chapter 35 on, state each ingredient in the generality the proof uses and assemble them.

Hints

§34.1.

34.1.1: Take $w = \mathrm{Frob}$: N is conjugate to $q^{-1}N$.

34.1.2: $\ker N$ is a line stable under W_K; compare the characters on $\ker N$ and on $V/\ker N$. The extension is trivial on inertia, and splits because $|\mathrm{Frob}| \neq 1$.

34.1.3: $\exp(t_\ell(\sigma)N) = 1$ on an open subgroup of I_K exactly when $N = 0$.

§34.2.

34.2.1: $K \otimes \mathbb{Q}_p$ is $\mathbb{Q}_p \times \mathbb{Q}_p$ or a field.

34.2.2: Hilbert's Theorem 90 for K and for each $K \otimes \mathbb{Q}_v$, and for the units at the unramified places.

34.2.3: ϕ is trivial on $\mathbb{Q}^\times$ and on the norms xx^c; use $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times N(\mathbb{A}_K^\times) \cong \mathbb{Z}/2$.

§34.3.

34.3.1: The units $\pm 1, \pm \mathrm{i}$ map bijectively onto $(\mathcal{O}/\mathfrak{f})^\times$, which has order 4; and $\mathfrak{f}^c = \mathfrak{f}$.

34.3.3: Write $\mathfrak{a} = \mathfrak{b}(\beta)$ with $\beta \equiv 1 \pmod{{}^\times \mathfrak{f}\ell^n}$; then β is the generator of (β) that ψ picks.

34.3.4: Approximate an idèle $u \in 1 + \ell^m\mathbb{Z}_\ell$ placed at λ by a global element of $\mathcal{O}$. With uniformizers sent to arithmetic Frobenius, as in Theorem 34.3.1, the local reciprocity map sends u to an element acting on ℓ-power roots of unity by u^{-1}.

34.3.5: The p-power Frobenius of the reduction commutes with [i], so it is $[\alpha]$ with $\alpha \in \mathcal{O}$ of norm p. It is inseparable, so $\alpha \in \mathfrak{p}$; fix the unit with $a_p = \psi(\mathfrak{p}) + \psi(\mathfrak{p}^c)$. For $p \equiv 3 \pmod 4$, use $a_p = 0$.

34.3.6: Complex conjugation acts on $V_\ell(E)$ and anticommutes with [i].

§34.4.

34.4.1: Expand the product to order q^{13}.

34.4.2: Multiply $T_p = F + V$ by F, and use that Frob_p acts on T_ℓ of the reduction as F.

34.4.3: Cayley–Hamilton.

34.4.4: In $\mathbb{F}_4$, the equation becomes $y^2 + y = x^3 + x^2$. Use $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$.

34.4.5: Problems 34.1.2 and 34.1.3; $v_{11}(q) \neq 0$.

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