POINCARÉ SERIES AND NEAR ORTHOGONALITY

Prerequisites. Modular forms for $\mathrm{SL}_2(\mathbb{Z})$: q-expansions, the cusp form Δ and the dimension formula [SeA, Ch. VII, §3]. The residue theorem and the mean value property of holomorphic functions. The power series $J_\nu(x) = \sum_{j\ge0}(-1)^j(x/2)^{\nu+2j}/(j!\,\Gamma(\nu+j+1))$. Gershgorin's theorem on the eigenvalues of a matrix. From Chapter 10, *The rank lemma and the zero estimate*, Section 10.1 and the bound (10.1.1), and Theorem 7.3.1 of Chapter 7, for the closing paragraphs only.

Chapter 10 asked for a family of vectors in a space of automorphic forms whose Gram matrix is close to a scalar. Equidistribution of CM points gives such families only with a rate that amounts to a subconvex bound. There is a cheaper source. If the weight is large, Poincaré series of small index are almost orthogonal, and Petersson's formula proves it with nothing more than a bound for a Bessel function. This chapter works it out on $\mathrm{SL}_2(\mathbb{Z})$, where the vectors are holomorphic Poincaré series and their inner products are Fourier coefficients.

Throughout, $k \ge 4$ is even, $\Gamma := \mathrm{SL}_2(\mathbb{Z})$, $\Gamma_\infty := \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\}$, $j(\gamma, z) := cz + d$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $e(z) := e^{2\pi i z}$, and $d\nu := y^{-2}\,dx\,dy$ on the upper half-plane $\mathbb{H}$.

11.1. Poincaré series

Let S_k be the space of cusp forms of weight k for Γ, with the Petersson inner product $\langle f, g\rangle := \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)}\,y^k\,d\nu(z)$. For $f \in S_k$ write $f(z) = \sum_{n\ge1} a_f(n)\,e(nz)$. For $m \ge 1$ the *Poincaré series*

$$P_m(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, z)^{-k}\,e(m\gamma z)$$

converges absolutely and uniformly on compact sets, and $P_m \in S_k$ [Iw, §§3.1–3.2].

Problem 11.1.1. For $f \in S_k$ and $m \ge 1$,

$$\langle f, P_m\rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\,a_f(m).$$

Problem 11.1.2. The series P_m, $m \ge 1$, span S_k.

So $P_m = 0$ for $k \in \{4, 6, 8, 10, 14\}$, where $S_k = 0$, although no term of the series vanishes. The first nonzero case is $k = 12$, where $S_{12} = \mathbb{C}\Delta$ with $\Delta = \sum_{n\ge1}\tau(n)\,e(nz)$.

Problem 11.1.3. For $k = 12$,

$$P_m = \frac{10!\,\tau(m)}{(4\pi m)^{11}\,\|\Delta\|^2}\,\Delta.$$

In particular P_1 and P_2 are proportional.

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So distinct Poincaré series need not be independent. Petersson's formula, in the next section, measures how far they are from orthogonal.

11.2. Petersson's formula

What is the Fourier expansion of P_m? For integers m, n and $c \geq 1$ let

$$S(m,n;c) := \sum_{\substack{d \bmod c \\ (d,c)=1}} e\Big(\frac{m\bar{d}+nd}{c}\Big), \qquad d\bar{d} \equiv 1 \pmod{c},$$

be the Kloosterman sum. It is real, and $S(m,n;c) = S(n,m;c)$.

Problem 11.2.1. The map $\gamma \mapsto (c,d)$ induces a bijection from $\Gamma_\infty\backslash\Gamma$ onto the set of pairs of coprime integers modulo ± 1. If $c \neq 0$, then $\gamma z = \frac{a}{c} - \frac{1}{c(cz+d)}$.

Problem 11.2.2. Let $\ell \geq 2$ be an integer, $\beta \in \mathbb{R}$ and $y > 0$. Then

$$\int_{\mathrm{i}y-\infty}^{\mathrm{i}y+\infty} z^{-\ell}\, e(-\beta z)\, dz = \begin{cases} (-2\pi\mathrm{i})^\ell\, \beta^{\ell-1}/(\ell-1)! & \beta > 0, \\ 0 & \beta \leq 0. \end{cases}$$

Problem 11.2.3. Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $y > 0$. Then $\int_{\mathrm{i}y-\infty}^{\mathrm{i}y+\infty} z^{-k}\, e(-\alpha/z - \beta z)\, dz$ vanishes for $\beta \leq 0$, and for $\beta > 0$ equals

$$2\pi\mathrm{i}^{-k}\Big(\frac{\beta}{\alpha}\Big)^{(k-1)/2} J_{k-1}\big(4\pi\sqrt{\alpha\beta}\big).$$

Problem 11.2.4. $P_m(z) = \sum_{n\geq 1} p_m(n)\, e(nz)$, where

$$p_m(n) = \delta(m,n) + 2\pi\mathrm{i}^{-k}\Big(\frac{n}{m}\Big)^{(k-1)/2} \sum_{c\geq 1} \frac{S(m,n;c)}{c}\, J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big).$$

Problems 11.1.1 and 11.2.4 together give Petersson's formula [Pe], [Iw, §3.2], [ILS, §2]. Let

$$T_k(m,n) := 2\pi\mathrm{i}^{-k} \sum_{c\geq 1} \frac{S(m,n;c)}{c}\, J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

a real symmetric function of $m, n \geq 1$.

Theorem 11.2.1 (Petersson)**.** *For every orthonormal basis (f_j) of S_k and all $m, n \geq 1$,*

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_j \overline{a_{f_j}(m)}\, a_{f_j}(n) = \delta(m,n) + T_k(m,n).$$

Problem 11.2.5. For $k \in \{4, 6, 8, 10, 14\}$ and all $m, n \geq 1$,

$$\sum_{c\geq 1} \frac{S(m,n;c)}{c}\, J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big) = -\frac{\mathrm{i}^k}{2\pi}\, \delta(m,n).$$

Numerically, for $m = n = 1$ the partial sums over $c \leq 800$ reproduce these values to within $3 \cdot 10^{-8}$ for $k = 4$ and 10^{-13} for $k \geq 6$. For $k = 12$ one finds $1 + T_{12}(1,1) = 2.84028\ldots$, which by Theorem 11.2.1 is $10!/\big((4\pi)^{11}\|\Delta\|^2\big)$, so that $\|\Delta\|^2 = 1.03536\ldots \cdot 10^{-6}$.

11.3. Near orthogonality

Normalize the Poincaré series by

$$\widetilde{P}_m := \frac{(4\pi m)^{(k-1)/2}}{\Gamma(k-1)^{1/2}}\, P_m,$$

and for $M \geq 1$ let G_M be the $M \times M$ Gram matrix $\big(\langle \widetilde{P}_m, \widetilde{P}_n \rangle\big)_{1 \leq m,n \leq M}$.

Problem 11.3.1. $\langle \widetilde{P}_m, \widetilde{P}_n \rangle = \delta(m,n) + T_k(m,n)$.

So G_M is the identity plus the matrix of Kloosterman terms, and the question is how small J_{k-1} is. We use Poisson's integral

$$J_\nu(x) = \frac{(x/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\nu-1/2} \cos(xt)\, dt \qquad (\nu > -\tfrac{1}{2},\ x > 0),$$

see [DLMF, (10.9.4)].

Problem 11.3.2. For $\nu > -\frac{1}{2}$ and $x > 0$, $|J_\nu(x)| \leq (x/2)^\nu / \Gamma(\nu+1)$.

Problem 11.3.3. $|T_k(m,n)| \leq 2\pi\, \zeta(k-1)\, (2\pi\sqrt{mn})^{k-1}/(k-1)!$.

Problem 11.3.4. If $1 \leq M \leq (k-1)/(4\pi e)$, then every eigenvalue of G_M lies within $k\, 2^{-k}$ of 1.

Problems 11.3.1–11.3.4 prove the following.

Theorem 11.3.1. *If $1 \leq M \leq (k-1)/(4\pi e)$, then $\|G_M - 1\| \leq k\, 2^{-k}$. In particular $P_1, \dots, P_M$ are linearly independent, and $\dim S_k \geq \lfloor (k-1)/(4\pi e) \rfloor$.*

Here $e = 2.718\dots$ is Euler's number, and $4\pi e = 34.15\dots$. The threshold is not the true one. For $x \leq r\nu$ with $r < 1$ fixed, the function $J_\nu(x)$ is exponentially small in ν, and Siegel's inequality makes this precise: for $\nu \geq 0$ and $0 < x \leq 1$,

$$|J_\nu(\nu x)| \leq \Big(\frac{x\, e^{\sqrt{1-x^2}}}{1+\sqrt{1-x^2}} \Big)^\nu$$

[DLMF, (10.14.5)]. The base is increasing in x and equals 1 at $x = 1$.

Problem 11.3.5. Let $0 < \theta < 1/(4\pi)$. There are $\varrho < 1$ and C, depending only on θ, such that $\|G_M - 1\| \leq Ck\varrho^k$ whenever $1 \leq M \leq \theta(k-1)$.

So for large k the $\widetilde{P}_m$ with $m \leq \theta(k-1)$, for a fixed $\theta < 1/(4\pi)$, are almost orthonormal. For m near $(k-1)/(4\pi)$ the argument $4\pi m$ of the term $c = 1$ of $T_k(m,m)$ reaches the transition region $x \approx \nu$ of J_ν, where $J_\nu(x)$ is of size about $\nu^{-1/3}$, and the diagonal entry is no longer exponentially close to 1. For $k = 100$, where $(k-1)/(4\pi) = 7.87\dots$ and $\dim S_{100} = 8$, a computation of the Kloosterman terms gives the following. The eigenvalues of G_7 lie in $[0.99999, 1.0094]$. Those of G_8 lie in $[0.982, 1.808]$, so $P_1, \dots, P_8$ are a basis of S_{100} but far from an orthogonal one. G_9 is singular, as it must be.

11.4. The dimension as formal degree times covolume

How large a proportion of S_k does Theorem 11.3.1 fill? Let $\Gamma\backslash\mathbb{H}$ carry the measure ν.

Problem 11.4.1. $\nu(\Gamma\backslash\mathbb{H}) = \pi/3$.

Problem 11.4.2. $\left|\dim S_k - \frac{k-1}{12}\right| \le \frac{13}{12}$ for every even $k \ge 4$.

So the series of Problem 11.3.5 span a proportion close to $12/(4\pi) = 3/\pi = 0.954\ldots$ of S_k, and those of Theorem 11.3.1 a proportion $3/(\pi e) = 0.351\ldots$. Where does $(k-1)/12$ come from? It is the product of $\pi/3$ and a constant attached to the weight alone.

Let $G := \mathrm{PSL}_2(\mathbb{R})$, acting on $\mathbb{H}$, and fix the Haar measure dg on G with $\int_G F(g\mathrm{i})\,dg = \int_{\mathbb{H}} F\,d\nu$; then $\mathrm{PSO}(2)$ has volume 1, and $\mathrm{vol}(\mathrm{PSL}_2(\mathbb{Z})\backslash G) = \pi/3$. Let $\mathcal{A}_k$ be the space of holomorphic functions f on $\mathbb{H}$ with $\|f\|^2 := \int_{\mathbb{H}} |f|^2 y^k\,d\nu < \infty$, on which G acts unitarily by $(\pi(g)f)(z) := f(g^{-1}z)\,j(g^{-1},z)^{-k}$. This is the holomorphic discrete series of weight k. For $w \in \mathbb{H}$ put

$$h_w(z) := \Big(\frac{z-\bar{w}}{2\mathrm{i}}\Big)^{-k}.$$

For a unit vector u of an irreducible unitary representation π with square-integrable matrix coefficients, the *formal degree* d_π is defined by $\int_G |\langle \pi(g)u, u\rangle|^2\,dg = 1/d_\pi$. It does not depend on u, by the Schur orthogonality relations for such representations [HC]. We compute it with u proportional to h_{i}.

Problem 11.4.3. $h_w \in \mathcal{A}_k$, and $\langle f, h_{\mathrm{i}}\rangle = \frac{4\pi}{k-1}\,f(\mathrm{i})$ for every $f \in \mathcal{A}_k$.

Problem 11.4.4. $\pi(g)h_w = \overline{j(g,w)}^{\,-k}\,h_{gw}$ for $g \in \mathrm{SL}_2(\mathbb{R})$ and $w \in \mathbb{H}$.

Problem 11.4.5. Let $u := h_{\mathrm{i}}/\|h_{\mathrm{i}}\|$, and $w := g\mathrm{i}$. Then $|\langle \pi(g)u, u\rangle|^2 = \big(4\,\mathrm{Im}\,w/|w+\mathrm{i}|^2\big)^k$, and the formal degree of π is $(k-1)/(4\pi)$.

With Problem 11.4.1 this gives

$$d_\pi \cdot \mathrm{vol}(\mathrm{PSL}_2(\mathbb{Z})\backslash G) = \frac{k-1}{4\pi}\cdot\frac{\pi}{3} = \frac{k-1}{12},$$

which is $\dim S_k$ up to the bounded error of Problem 11.4.2. This is not a coincidence. As in Problem 11.1.1, and using Problems 11.4.3 and 11.4.4, one checks that the reproducing kernel $B(z,w) := \sum_j f_j(z)\overline{f_j(w)}$ of S_k, for an orthonormal basis (f_j), is

$$B(z,w) = \frac{k-1}{4\pi}\sum_{\gamma\in\mathrm{PSL}_2(\mathbb{Z})} j(\gamma,z)^{-k}\,h_w(\gamma z).$$

Since $\dim S_k = \int_{\Gamma\backslash\mathbb{H}} B(z,z)\,y^k\,d\nu(z)$, the term $\gamma = 1$ contributes exactly the formal degree times the covolume, and by Problem 11.4.2 all other terms together contribute a bounded amount. On a definite unitary group the formal degree is the Weyl dimension of the representation at infinity, and the count of forms in Chapter 21 is the same product.

In the notation of Section 10.1, take $\mathcal{H} = S_k$, $D' = \dim S_k$, and the maps $B_m\colon \mathbb{C} \to S_k$, $1 \mapsto \widetilde{P}_m/\|\widetilde{P}_m\|$, for m in the range of Theorem 11.3.1, where $\|\widetilde{P}_m\|$ is exponentially close to 1. The bound (10.1.1) asks for $|\langle B_m(1), B_n(1)\rangle|^2 \le (1+\eta_0)/D'$ for $m \ne n$, about $12/k$. Theorem 11.3.1 gives far more, an exponentially small bound, but only for m up to a fixed proportion of D'. The images of these maps therefore do not fill S_k, and the Poincaré series of larger index, which do fill it, are not nearly orthogonal. So Theorem 11.3.1 alone does not give the hypothesis of Theorem 7.3.1 for S_k. The argument uses many families at once, each of small rank, whose mutual correlations are only as small as those of random subspaces.

The analogue in the argument. In the reconstruction [Adr, §14] of Alpöge's argument [Alp], see also [Zh], SL_2 is replaced by a unitary group $G = \mathrm{U}(V)$ of a hermitian space V over a CM extension E/F, and $\mathcal{A}_k$ by a holomorphic discrete series of large weight at each real place. The Fourier coefficient $a_f(m)$, an integral of f against a character of the upper unipotent subgroup of $\mathrm{SL}_2(\mathbb{R})$, becomes a Bessel period: an integral against a character of a subgroup $S = \mathrm{U}(W) \ltimes N$, where $W \subset V$ is a nondegenerate subspace and N is unipotent, as in the conjectures of Gan, Gross and Prasad [GGP]. To each word u of the construction corresponds a space of Bessel Poincaré series, obtained by summing over $S(F)\backslash G(F)$ the analogue of $j(\gamma, z)^{-k} e(m\gamma z)$, and unfolding their inner products is the analogue of Problem 11.1.1 [Adr, Prop. 14.5]. The near orthogonality of the Gram operators [Adr, Thm. 14.1], the bound (10.1.1) for families whose size grows with a parameter x and with $\eta_0 \leq \exp(-x^c)$ for some $c > 0$, is the analogue of Theorem 11.3.1, and as there no subconvex bound enters. The decay that replaces Problem 11.3.2 is that of the matrix coefficients of the discrete series, as in Problem 11.4.5, and the number D^* of [Adr, §14.1] is the formal degree times the volume of $G(F)\backslash G(\mathbb{A})$ times the value at 1 of the level idempotent. The deduction of the near orthogonality, and of $\dim \mathcal{H} \approx D^*$, from counting and pre-trace statements is carried out there, but these statements themselves are assumed there: in [Adr] they are [Adr, Hyps. 14.61 and 14.71–14.77]; see the ledger in Section 58.7 of Chapter 58. This completes the method. It needs maps whose values are divisible at the inert primes and of controlled size; the next part, from Chapter 12 on, finds the divisibility in theta functions with complex multiplication.

Hints

§11.1.

11.1.1: Unfold the sum over $\Gamma_\infty\backslash\Gamma$ to the strip $0 \leq x \leq 1$, $y > 0$, and integrate in x first.

11.1.2: Problem 11.1.1.

11.1.3: Expand P_m in the basis Δ and use Problem 11.1.1. Note that $\tau(1) = 1$, $\tau(2) = -24$.

§11.2.

11.2.1: The bottom row of a matrix in Γ is coprime, and two matrices with the same bottom row differ by left multiplication by an element of Γ_∞.

11.2.2: Close the contour in the half-plane where $e(-\beta z)$ decays.

11.2.3: Expand $e(-\alpha/z)$ in its power series and apply Problem 11.2.2 term by term. For $\beta \leq 0$, $|e(-\alpha/z)| \leq 1$ in $\mathbb{H}$.

11.2.4: The terms with $c > 0$ and fixed c are indexed by $d \in \mathbb{Z}$ coprime to c; group them by d mod c. Each group is a sum over translates $z \mapsto z + r$, $r \in \mathbb{Z}$, so its n-th Fourier coefficient is an integral over a whole horizontal line. Use Problems 11.2.1 and 11.2.3.

11.2.5: Theorem 11.2.1.

§11.3.

11.3.1: Problem 11.1.1 with $f = P_m$, and Problem 11.2.4.

11.3.2: Bound $|\cos(xt)|$ by 1 and evaluate the Beta integral.

11.3.3: $|S(m, n; c)| \leq c$.

11.3.4: $(k-1)! \geq ((k-1)/e)^{k-1}$. Then Gershgorin's theorem.

11.3.5: Treat the term $c = 1$ with Siegel's inequality and the terms $c \geq 2$ with Problem 11.3.2.

§11.4.

11.4.1: The standard fundamental domain $|x| \leq \frac{1}{2}$, $|z| \geq 1$; integrate in y first.

11.4.2: Write $k = 12j + r$ with $0 \leq r < 12$ and use the dimension formula.

11.4.3: Put $\zeta := (z - \mathrm{i})/(z + \mathrm{i})$ and $F(\zeta) := f(z)\big((z + \mathrm{i})/2\mathrm{i}\big)^k$. Express $|f|^2 y^k \, d\nu$ in terms of F and ζ, and use the mean value property on the circles $|\zeta| = r$. For $h_w \in \mathcal{A}_k$, use $4 \operatorname{Im} z \operatorname{Im} w/|z - \bar{w}|^2 = 1 - |\zeta_w|^2$ with $\zeta_w := (z - w)/(z - \bar{w})$.

11.4.4: $gz - \overline{gw} = (z - \bar{w})/\big(j(g, z)\overline{j(g, w)}\big)$.

11.4.5: Problems 11.4.3 and 11.4.4. For the integral, use geodesic polar coordinates about i, in which $4 \operatorname{Im} w/|w + \mathrm{i}|^2 = \cosh^{-2}(\rho/2)$.

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