THE RANK LEMMA AND THE ZERO ESTIMATE

Prerequisites. Section 7.3 of Chapter 7, *The slope inequality in the mean*: Problem 7.3.1, Theorem 7.3.1, and their notation. Finite-dimensional spectral theory and Schur's lemma. Binary quadratic forms and their reduction [Cox, §2]. For Section 10.4 only, the spectral expansion of an automorphic kernel on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, where $\mathbb{H}$ is the upper half-plane [Iw2, Ch. 7].

Theorem 7.3.1 bounds the codimension of a sum of many images by a second moment. Where can such a bound come from? In this chapter it comes from Gram matrices, from evaluations at points, and from a relative trace formula, each in its simplest case.

10.1. Gram matrices

As in Section 7.3, let $\mathcal{H}$ be a complex inner product space, now of dimension $D' > 0$. Let $B_i\colon X_i \to \mathcal{H}$, $i \in I$, be linear maps from inner product spaces of dimensions r_i, with $R := \sum_i r_i > 0$, let $\gamma_i > 0$, and let $K_i := \gamma_i B_i^* B_i$ and $Q := \sum_i \gamma_i B_i B_i^*$. Let $X := \bigoplus_i X_i$ and $B\colon X \to \mathcal{H}$, $B(x_i)_i := \sum_i \sqrt{\gamma_i} B_i x_i$, so that $Q = BB^*$. The *Gram matrix* of the family is $G := B^*B$. Its blocks are $G_{ij} = \sqrt{\gamma_i\gamma_j}\, B_i^* B_j\colon X_j \to X_i$, and $G_{ii} = K_i$. When every X_i is a line spanned by a unit vector ξ_i, G is the Gram matrix of the vectors $\sqrt{\gamma_i} B_i \xi_i$.

Problem 10.1.1. $Q(\mathcal{H}) = \sum_i B_i(X_i)$.

Problem 10.1.2. Q and G have the same nonzero eigenvalues, with multiplicities. Hence $\operatorname{tr} Q = \sum_i \operatorname{tr} K_i$ and $\operatorname{tr} Q^2 = \sum_{i,j} \|G_{ij}\|_{\mathrm{HS}}^2$.

With Problem 7.3.1 this proves the following.

Theorem 10.1.1. *If some B_i is nonzero, then*

$$\dim \sum_i B_i(X_i) \geq \frac{\left(\sum_i \operatorname{tr} K_i\right)^2}{\sum_{i,j} \|G_{ij}\|_{\mathrm{HS}}^2}.$$

Unlike the left side, the right side changes little when the B_i are perturbed.

Problem 10.1.3. Let $v, w \in \mathbb{C}^2$ be unit vectors, $X_1 = X_2 = \mathbb{C}$, $B_1(1) = v$, $B_2(1) = w$ and $\gamma_1 = \gamma_2 = 1$. The right side of Theorem 10.1.1 is $2/(1 + |\langle v, w\rangle|^2)$.

So two nearly parallel lines count as about one direction. Suppose now that each weighted map is an isometry, $K_i = 1$, and that for some $\eta_0 \geq 0$

$$\|G_{ij}\|_{\mathrm{HS}}^2 \leq (1 + \eta_0)\frac{r_i r_j}{D'} \qquad (i \neq j). \tag{10.1.1}$$

This is hypothesis (G1) of [Adr, Prop. 13.11], with D' replaced there by a number close to $\dim \mathcal{H}$.

Problem 10.1.4. Under (10.1.1) and $K_i = 1$ for all i, $\operatorname{tr} Q^2 \leq (1 + \eta_0 + D'/R)(\operatorname{tr} Q)^2/D'$.

Date: November 25, 2026.

Combining Problem 10.1.4 with Theorem 7.3.1, for $e = 0$, gives a zero estimate from pairwise bounds.

Theorem 10.1.2. *Assume* (10.1.1) *and* $K_i = 1$ *for all* i. *Let* $Y_i \subseteq X_i$ *be subspaces with* $\sum_i \operatorname{codim} Y_i \le \alpha R$, *where* $0 \le \alpha \le 1$. *Then*

$$\operatorname{codim}_{\mathcal{H}} \sum_i B_i(Y_i) \le \Big(\eta_0 + \frac{D'}{R} + 4\alpha\Big) D'.$$

The constant $r_i r_j / D'$ in (10.1.1) is an average over relative positions.

Problem 10.1.5. Let $B\colon \mathbb{C}^r \to \mathcal{H}$ and $B'\colon \mathbb{C}^{r'} \to \mathcal{H}$ be isometries, and let u be distributed according to Haar measure on the unitary group of $\mathcal{H}$. Then $\mathbb{E}\|B^* u B'\|_{\mathrm{HS}}^2 = rr'/D'$.

So (10.1.1) asks that no two images overlap, in the Hilbert–Schmidt norm, by more than $1 + \eta_0$ times this average. The number of maps by itself proves nothing.

Problem 10.1.6. Let $\mathcal{H} = \mathbb{C}^{D'}$ with $D' \ge 2$ and standard basis $e_1, \dots, e_{D'}$. Let $X_0 = \mathbb{C}^{D'}$ with $B_0 = 1$, and for $1 \le i \le N$ let $X_i = \mathbb{C}$ with $B_i(1) = e_1$; let all $\gamma_i = 1$. Then B_0 is onto, but omitting X_0, a proportion $D'/(N + D')$ of the source directions, leaves a sum of images of codimension $D' - 1$. The ratio $D' \operatorname{tr} Q^2 / (\operatorname{tr} Q)^2$ equals $D'\big((N+1)^2 + D' - 1\big)/(N + D')^2$.

For large N this ratio is close to D', and Theorem 7.3.1 gives nothing, as it must. Here (10.1.1) fails by a factor D' for every pair $i, j \ge 1$.

10.2. Evaluation at points

The simplest maps are evaluations at points. Let ν be a probability measure on a space $\mathcal{X}$, and let $\mathcal{H} \subset L^2(\mathcal{X}, \nu)$ be a space of bounded continuous functions of dimension D', with an orthonormal basis $\varphi_1, \dots, \varphi_{D'}$. Its reproducing kernel is $K(x, y) := \sum_n \varphi_n(x)\overline{\varphi_n(y)}$. For points $x_1, \dots, x_N \in \mathcal{X}$ and weights $\gamma_i > 0$, let $X_i = \mathbb{C}$ and $B_i(1) := K(\cdot, x_i)$. Problem 7.3.2 is an example.

Problem 10.2.1. $B_i^* f = f(x_i)$, so that $\langle Qf, f\rangle = \sum_i \gamma_i |f(x_i)|^2$, and $G_{ij} = \sqrt{\gamma_i \gamma_j}\, K(x_i, x_j)$. Moreover $\int_{\mathcal{X}} K(x, x)\, d\nu(x) = D'$.

Problem 10.2.2. $\langle Q\varphi_n, \varphi_m\rangle = \sum_i \gamma_i (\varphi_n \overline{\varphi_m})(x_i)$. Hence $Q = 1$ exactly when the measure $\sum_i \gamma_i \delta_{x_i}$ integrates every $\varphi_n\overline{\varphi_m}$ as ν does.

Problem 10.2.3. The functions $f \in \mathcal{H}$ that vanish at every x_i form a space of dimension at most $\|Q - 1\|_{\mathrm{HS}}^2$.

In particular, if the weighted points integrate every $\varphi_n\overline{\varphi_m}$ exactly, no nonzero $f \in \mathcal{H}$ vanishes at all of them. The simplest families with Q exactly scalar are orbits of a group.

Problem 10.2.4. Let a finite group Γ act unitarily and irreducibly on $\mathcal{H}$, and let $Y \subseteq \mathcal{H}$ be a subspace of dimension r with inclusion ι. For $g \in \Gamma$ let $B_g := g\iota\colon Y \to \mathcal{H}$ with weight $1/|\Gamma|$. Then $Q = (r/D') \cdot 1$.

For instance, let ν be the normalized rotation-invariant measure on the sphere $S^2 \subset \mathbb{R}^3$, and let $\mathcal{H}$ be the space of restrictions of linear forms. Since $\int x_k^2\, d\nu = 1/3$, the functions $\sqrt{3}\, x_k$ are an orthonormal basis, and $K(x, y) = 3\langle x, y\rangle$. The rotation group of a regular tetrahedron acts irreducibly on $\mathcal{H}$, and the orbit of the line

spanned by $K(\cdot, x_0)$, for a vertex x_0, gives, up to the factor $K(x_0, x_0) = 3$, the evaluations at the four vertices.

Problem 10.2.5. Let $x_1, \dots, x_4$ be the vertices of a regular tetrahedron inscribed in S^2, and $\gamma_i = 1/4$. Then $G = 1 - J/4$, where J is the 4×4 matrix of ones, and $Q = 1$. For the points $\pm e_1, \pm e_2$ instead, with $\gamma_i = 1/4$, $\operatorname{rank} Q = (\operatorname{tr} Q)^2 / \operatorname{tr} Q^2 = 2$.

10.3. Heegner points

The simplest point sets with arithmetic meaning are CM points. On the modular curve their equidistribution is a theorem of Duke. Let $\mathcal{X} := \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ with the probability measure $d\nu = (3/\pi)\, y^{-2}\, dx\, dy$. A primitive positive definite form $[a, b, c] = ax^2 + bxy + cy^2$ of discriminant $b^2 - 4ac = -d$ has the root $z_{[a,b,c]} := (-b + \mathrm{i}\sqrt{d})/(2a)$ in $\mathbb{H}$, and the map from forms to roots is $\mathrm{SL}_2(\mathbb{Z})$-equivariant. For $d > 4$ let $\mathcal{H}_d \subset \mathcal{X}$ be the set of these roots. Every form is equivalent to exactly one reduced form [Cox, Thm. 2.8], so $\mathcal{H}_d$ has $h(-d)$ points, one for each reduced form, where $h(-d)$ is the class number, the number of classes of primitive positive definite forms of discriminant $-d$.

Problem 10.3.1. $h(-23) = 3$, and $\mathcal{H}_{23}$ consists of $(-1 + \mathrm{i}\sqrt{23})/2$ and $(\pm 1 + \mathrm{i}\sqrt{23})/4$.

Theorem 10.3.1 (Duke)**.** *Let $d \to \infty$ through integers for which $-d$ is a fundamental discriminant. Then for every continuous compactly supported function F on $\mathcal{X}$,*

$$\frac{1}{h(-d)} \sum_{z \in \mathcal{H}_d} F(z) \to \int_{\mathcal{X}} F\, d\nu.$$

See [Du, Thm. 1], where it is proved, with a rate, for convex sets; continuous F follow by approximation. The proof rests on Iwaniec's bound for the Fourier coefficients of forms of half-integral weight [Iw1].

Problem 10.3.2. Theorem 10.3.1 holds for every bounded continuous F.

Problem 10.3.3. Let $\mathcal{H}$ be as in Section 10.2 on $\mathcal{X}$, for instance the span of 1 and finitely many Maass cusp forms. Let Q_d be the operator of the evaluations at the points of $\mathcal{H}_d$ with weights $1/h(-d)$. Then $Q_d \to 1$ as $d \to \infty$ as in Theorem 10.3.1. Hence, for all large such d, no nonzero $f \in \mathcal{H}$ vanishes on $\mathcal{H}_d$.

10.4. What a relative trace formula supplies

How can one verify (10.1.1)? A block G_{ij} is a sum over an orthonormal basis of $\mathcal{H}$ of products of values, by Problem 10.2.1. A trace formula turns such a sum into a sum over the group. Begin with the circle. Let $\mathcal{X} = \mathbb{R}/\mathbb{Z}$ with Lebesgue measure, $e(t) := e^{2\pi i t}$, and let $\mathcal{H}$ be spanned by the $e(kx)$ with $|k| < n$. Then $D' = 2n - 1$, and $K(x, y) = D_n(x - y)$ with $D_n(t) := \sum_{|k|<n} e(kt)$. For points $x_1, \dots, x_N$ with $\gamma_i = 1/N$, let $W(k) := N^{-1} \sum_i e(kx_i)$; then $\operatorname{tr} Q = D'$.

Problem 10.4.1. $\operatorname{tr} Q^2 = \dfrac{1}{N^2} \displaystyle\sum_{i,j} |D_n(x_i - x_j)|^2 = \sum_{|k|<D'} (D' - |k|)|W(k)|^2.$

The middle term is a sum over pairs of points, and is large when many pairs lie at distance less than $1/D'$. The right side is a sum over frequencies. So Q is

almost scalar when the Weyl sums are small up to frequency D', or equivalently when there are not too many close pairs.

Problem 10.4.2. If the x_i are independent and uniformly distributed, then $\mathbb{E}\,\mathrm{tr}\,Q^2 = D' + D'(D'-1)/N$.

Here $K_i = D'/N$. After rescaling to $\gamma_i = 1/D'$, so that $K_i = 1$, the ratio $D'\,\mathrm{tr}\,Q^2/(\mathrm{tr}\,Q)^2$, which does not depend on the scaling, is $1 + (D'-1)/N$ in the mean, within the value $1 + D'/N$ that Problem 10.1.4 allows for $\eta_0 = 0$ and $R = N$.

In the argument the maps are periods, not evaluations. On $\mathcal{X} = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, with $\mathcal{H}$ as in Problem 10.3.3, let $P_d(f) := \sum_{z\in\mathcal{H}_d} f(z)$. For $d > 4$ let $X_d = \mathbb{C}$, let $\gamma_d > 0$, and let $B_d\colon \mathbb{C} \to \mathcal{H}$ be the map with $B_d^* = P_d$. The family (B_d), over a set of d, satisfies a zero estimate if few $f \in \mathcal{H}$ have all their periods zero.

Problem 10.4.3. $G_{dd'} = \sqrt{\gamma_d\gamma_{d'}} \sum_{z\in\mathcal{H}_d} \sum_{w\in\mathcal{H}_{d'}} K(z,w) = \sqrt{\gamma_d\gamma_{d'}} \sum_n P_d(\varphi_n)\overline{P_{d'}(\varphi_n)}$.

Now replace K by an automorphic kernel. Let ρ be the hyperbolic distance, $\cosh\rho(z,w) = 1 + |z-w|^2/(2\,\mathrm{Im}\,z\,\mathrm{Im}\,w)$, let k be a smooth compactly supported function on $[1,\infty)$, and let $K_k(z,w) := \sum_{g\in\mathrm{PSL}_2(\mathbb{Z})} k(\cosh\rho(z,gw))$.

Problem 10.4.4. For forms $[a,b,c]$ and $[a',b',c']$ of discriminants $-d$ and $-d'$, $\cosh\rho(z_{[a,b,c]}, z_{[a',b',c']}) = (2ac' + 2a'c - bb')/\sqrt{dd'}$.

Problem 10.4.5. For $d, d' > 4$,

$$\sum_{z\in\mathcal{H}_d} \sum_{w\in\mathcal{H}_{d'}} K_k(z,w) = \sum_{[a,b,c]} \sum_{[a',b',c']} k\Big(\frac{2ac' + 2a'c - bb'}{\sqrt{dd'}}\Big),$$

where $[a,b,c]$ runs over the reduced forms of discriminant $-d$, and $[a',b',c']$ over all primitive positive definite forms of discriminant $-d'$.

The right side counts integral points. The left side has a spectral expansion. By Selberg's pre-trace formula [Iw2, Thm. 7.4], $K_k(z,w) = \sum_j h(t_j)u_j(z)\overline{u_j(w)}$ plus an integral over the Eisenstein series, where u_j runs over a basis of Maass forms orthonormal for the measure $y^{-2}\,dx\,dy = (\pi/3)\,d\nu$, and h is the Selberg–Harish-Chandra transform of k for that measure. So, apart from the continuous spectrum, the left side of Problem 10.4.5 is $\sum_j h(t_j)P_d(u_j)\overline{P_{d'}(u_j)}$. If $h(t_j) > 0$ for the u_j spanning $\mathcal{H}$, and $\mathcal{H}$ is given the inner product for which the $\sqrt{h(t_j)}\,u_j$ are orthonormal, then by Problem 10.4.3, whose proof uses only that K is the reproducing kernel of $\mathcal{H}$, this is the Gram block $G_{dd'}$, up to the factor $\sqrt{\gamma_d\gamma_{d'}}$ and the contribution of the other u_j. A bound of the form (10.1.1) thus becomes a count of pairs of forms with $2ac' + 2a'c - bb'$ in a given range.

This is the simplest case of a relative trace formula. For a reductive group G over $\mathbb{Q}$ with subgroups H_1 and H_2, one integrates the automorphic kernel $\sum_{\gamma\in G(\mathbb{Q})} f(x^{-1}\gamma y)$ over $x \in H_1(\mathbb{Q})\backslash H_1(\mathbb{A})$ and $y \in H_2(\mathbb{Q})\backslash H_2(\mathbb{A})$. Spectrally one obtains a sum over automorphic forms φ of products of an H_1-period of φ and the conjugate of an H_2-period. Geometrically one obtains a sum over the double cosets $H_1(\mathbb{Q})\backslash G(\mathbb{Q})/H_2(\mathbb{Q})$ of orbital integrals. For $G = \mathrm{PGL}_2$ and H_1, H_2 tori, which contains Problem 10.4.5, Jacquet used this identity to prove Waldspurger's formula [Ja]. In Alpöge's argument [Alp], see also [Zh], the maps are Bessel periods on unitary groups [GGP], and the pairwise bounds do not come from equidistribution of CM points but from near orthogonality of Poincaré series of large weight; Chapter 11 treats it, and its last section describes the form it takes in the argument.

Hints

§10.1.

10.1.1: $\ker B^* = (\operatorname{im} B)^\perp$.

10.1.2: If $Gx = \lambda x$ with $\lambda \neq 0$, then $Q(Bx) = \lambda Bx$. Then expand $\operatorname{tr} G^2$ by blocks.

10.1.3: G is the 2×2 matrix with diagonal entries 1 and off-diagonal entries $\langle w, v\rangle$ and $\langle v, w\rangle$.

10.1.4: $\|G_{ii}\|_{\mathrm{HS}}^2 = r_i$, and $\sum_{i\neq j} r_i r_j \leq R^2$.

10.1.5: Schur's lemma.

10.1.6: Q is diagonal.

§10.2.

10.2.1: $\langle f, K(\cdot, x)\rangle = f(x)$ for $f \in \mathcal{H}$.

10.2.3: Compare Q with 1 on that space, using Problem 10.2.1.

10.2.4: Q commutes with Γ, and $\operatorname{tr} Q = r$.

10.2.5: Use Problem 10.1.2. The vertices satisfy $\langle x_i, x_j\rangle = -1/3$ for $i \neq j$. For the second family, consider the function x_3.

§10.3.

10.3.1: A reduced form has $|b| \leq a \leq c$, so $a \leq \sqrt{d/3}$.

10.3.2: Approximate 1 from below by compactly supported functions.

10.3.3: Apply Problem 10.3.2 to the functions of Problem 10.2.2, and then use Problem 10.2.3.

§10.4.

10.4.1: $|D_n(t)|^2 = \sum_{|k|<D'} (D' - |k|)e(kt)$.

10.4.2: $\mathbb{E}|W(k)|^2 = 1/N$ for $k \neq 0$.

10.4.3: By Problem 10.2.1, $B_d(1) = \sum_{w\in\mathcal{H}_d} K(\cdot, w)$.

10.4.4: $\operatorname{Im} z_{[a,b,c]} = \sqrt{d}/(2a)$. Expand $|z - z'|^2$ and use $b^2 - 4ac = -d$.

10.4.5: For $d' > 4$ a form has trivial stabilizer in $\mathrm{PSL}_2(\mathbb{Z})$. Use Problem 10.4.4.

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