THE KRONECKER LIMIT FORMULA AND CHOWLA–SELBERG

Prerequisites. Euler's Gamma function: the reflection and duplication formulas, the Beta integral, and $\psi := \Gamma'/\Gamma$ with $\psi(1) = -\gamma$. Poisson summation. Dirichlet L-functions: the functional equation [Dav, Ch. 9], the class number formula and binary quadratic forms [Dav, Ch. 6]. The Hurwitz zeta function [Ap, Ch. 12]. Compact Riemann surfaces and their holomorphic differentials, for Section 20.4 only. From Chapter 12, *Complex tori, theta functions and periods*: Section 12.3, in particular Problems 12.1.9, 12.3.4 and 12.3.7 and Theorem 12.3.1, with Problems 12.3.5 and 12.3.6.

Chapter 12 took Kronecker's limit formula from the literature and computed $L'(1,\chi)/L(1,\chi)$ for the character χ modulo 4 through the discriminant of the lemniscatic lattice. Here we prove the limit formula and reach the same value through $s = 0$, where no elliptic functions appear. Comparing the two routes is the method of Chowla and Selberg [CS]. It shows that, for the character χ of every imaginary quadratic field, $e^{\gamma - L'(1,\chi)/L(1,\chi)}$ is, up to an algebraic factor, the square of a period of an elliptic curve with complex multiplication divided by π^2.

20.1. Eisenstein series and Kronecker's limit formula

What is the constant term at $s = 1$ of $\sum'_{\mu \in \mathbb{Z}[\mathrm{i}]} |\mu|^{-2s}$? The answer comes from letting the lattice vary. For $\tau = x + \mathrm{i}y$ with $y > 0$ and $\operatorname{Re} s > 1$, let

$$E(\tau, s) := \sum_{(m,n) \neq (0,0)} \frac{y^s}{|m + n\tau|^{2s}}.$$

Let $q := e^{2\pi \mathrm{i} \tau}$, let $\eta(\tau) := q^{1/24} \prod_{n \geq 1} (1 - q^n)$ be Dedekind's function, and let $\sigma_t(n) := \sum_{e|n} e^t$. For $z > 0$, let $K_\nu(z) := \frac{1}{2} \int_0^\infty t^{\nu - 1} e^{-z(t + t^{-1})/2} \, dt$.

Problem 20.1.1. $E(g\tau, s) = E(\tau, s)$ for every $g \in \mathrm{SL}_2(\mathbb{Z})$.

Problem 20.1.2. For real x, $y > 0$ and $\operatorname{Re} s > \frac{1}{2}$,

$$\sum_{m \in \mathbb{Z}} \frac{1}{((m + x)^2 + y^2)^s} = \frac{\sqrt{\pi}\, \Gamma(s - \frac{1}{2})}{\Gamma(s)\, y^{2s-1}} + \frac{4\pi^s}{\Gamma(s)} \sum_{k \geq 1} \Big(\frac{k}{y}\Big)^{s - \frac{1}{2}} K_{s - \frac{1}{2}}(2\pi k y) \cos(2\pi k x).$$

Problem 20.1.3. $E(\tau, s)$ equals

$$2\zeta(2s) y^s + \frac{2\sqrt{\pi}\, \Gamma(s - \frac{1}{2})}{\Gamma(s)} \zeta(2s - 1) y^{1 - s} + \frac{8\pi^s \sqrt{y}}{\Gamma(s)} \sum_{n \geq 1} n^{s - \frac{1}{2}} \sigma_{1 - 2s}(n) K_{s - \frac{1}{2}}(2\pi n y) \cos(2\pi n x).$$

In particular $E(\tau, s)$ continues meromorphically to $\operatorname{Re} s > \frac{1}{2}$, with only a simple pole at $s = 1$.

Problem 20.1.4. $K_{1/2}(z) = \sqrt{\pi/2z}\, e^{-z}$.

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Problem 20.1.5. At $s=1$ the last term in Problem 20.1.3 is $-4\pi\log\bigl|\prod_{n\ge1}(1-q^n)\bigr|$.

Problem 20.1.6. As $s\to1$,

$$\frac{2\sqrt{\pi}\,\Gamma(s-\frac12)}{\Gamma(s)}\zeta(2s-1)y^{1-s}=\frac{\pi}{s-1}+2\pi\bigl(\gamma-\log2-\tfrac12\log y\bigr)+O(s-1).$$

Since $2\zeta(2)y=\pi^2y/3=-4\pi\log|q^{1/24}|$, Problems 20.1.3–20.1.6 prove the following [We], [La, Ch. 20, 21].

Theorem 20.1.1 (Kronecker's first limit formula)**.** $E(\tau,s)=\dfrac{\pi}{s-1}+2\pi\bigl(\gamma-\log2-\log(\sqrt{y}\,|\eta(\tau)|^2)\bigr)+O(s-1)$.

Problem 20.1.7. $|\eta(-1/\tau)|=|\tau|^{1/2}|\eta(\tau)|$.

Problem 20.1.8. Let $Z(s):=\sum'_{\mu\in\mathbb{Z}[\mathrm{i}]}|\mu|^{-2s}$, and let χ be the non-trivial character modulo 4. Then $Z(s)=\pi/(s-1)+2\pi(\gamma-\log2-2\log\eta(\mathrm{i}))+O(s-1)$, and

$$\frac{L'(1,\chi)}{L(1,\chi)}=\gamma-2\log2-4\log\eta(\mathrm{i}).$$

This is Problem 12.3.5, now with a proof. It expresses $L'/L(1,\chi)$ through a value of a modular form. To see Gamma values instead, we move to $s=0$, where $L(s,\chi)$ is a finite combination of Hurwitz zeta functions.

20.2. Lerch's formula and the functional equation

For $x>0$ and $\operatorname{Re}s>1$, let $\zeta(s,x):=\sum_{n\ge0}(n+x)^{-s}$. Euler–Maclaurin summation gives

$$\zeta(s,x)=\frac{x^{1-s}}{s-1}+\frac{x^{-s}}{2}-s\int_0^\infty\frac{\{t\}-\frac12}{(t+x)^{s+1}}\,dt,$$

which continues $\zeta(s,x)$ to $\operatorname{Re}s>-1$ with a single pole at $s=1$. Write $\zeta'(s,x)$ for $\partial_s\zeta(s,x)$.

Problem 20.2.1. $\zeta(0,x)=\frac12-x$, and $\zeta(s,\frac12)=(2^s-1)\zeta(s)$.

Problem 20.2.2. $\partial_x^2\,\zeta'(0,x)=\zeta(2,x)=(\log\Gamma)''(x)$, and $\zeta'(0,x+1)-\zeta'(0,x)=\log x$.

Problems 20.2.1 and 20.2.2, with $\zeta(0)=-\frac12$ and $\Gamma(\frac12)=\sqrt{\pi}$, prove the following.

Theorem 20.2.1 (Lerch)**.** $\zeta'(0,x)=\log\Gamma(x)-\frac12\log2\pi$ *for* $x>0$.

Problem 20.2.3. Let χ be an odd character modulo N. Then $L(0,\chi)=-\frac1N\sum_{a=1}^N a\,\chi(a)$, and

$$L'(0,\chi)=\sum_{a=1}^{N-1}\chi(a)\log\Gamma(a/N)-L(0,\chi)\log N.$$

Let χ be a primitive odd character modulo N, and let

$$\Lambda(s,\chi):=(N/\pi)^{(s+1)/2}\,\Gamma\bigl(\tfrac{s+1}{2}\bigr)L(s,\chi).$$

Then $\Lambda(s,\chi)=W(\chi)\Lambda(1-s,\bar\chi)$ with $|W(\chi)|=1$ [Dav, Ch. 9]; moreover $W(\chi)=\tau(\chi)/(\mathrm{i}\sqrt{N})$, which is 1 if χ is real, because then the Gauss sum is $\tau(\chi)=\mathrm{i}\sqrt{N}$ [Dav, Ch. 6].

Problem 20.2.4. $\dfrac{L'(1,\chi)}{L(1,\chi)} + \dfrac{L'(0,\bar\chi)}{L(0,\bar\chi)} = \gamma + \log\dfrac{2\pi}{N}$.

Problem 20.2.5. For the character modulo 4, Problems 20.2.3 and 20.2.4 give Theorem 12.3.1:

$$\frac{L'(1,\chi)}{L(1,\chi)} = \gamma + 2\log 2 + 3\log\pi - 4\log\Gamma(\tfrac14) = 0.24560\ldots$$

Problem 20.2.6. $\eta(\mathrm{i}) = \Gamma(\frac14)/(2\pi^{3/4})$.

Chapter 12 found this value from $\Delta(\varpi\mathbb{Z}[\mathrm{i}]) = 64$ (Problem 12.3.6). Here it comes from computing one constant, $L'(1,\chi)/L(1,\chi)$, in two ways: through $s=1$ by Theorem 20.1.1, and through $s=0$ by Theorem 20.2.1. The argument used only that $Z(s)$ is both an Eisenstein series and $4\zeta(s)L(s,\chi)$; the next section runs it for every imaginary quadratic field.

20.3. The general odd character

Let K be an imaginary quadratic field of discriminant $-d$, with class number h and w roots of unity, and let χ be its character, a primitive real odd character modulo d. The class number formula is $L(1,\chi) = 2\pi h/(w\sqrt{d})$. For each class of positive definite forms of discriminant $-d$, choose the reduced representative $Q=[a,b,c]$, so that $Q(m,n) = am^2+bmn+cn^2$, $b^2-4ac=-d$, $|b|\le a\le c$, and $b\ge 0$ if $|b|=a$ or $a=c$. Let $\tau_Q := (-b+\mathrm{i}\sqrt{d})/2a$ and $y_Q := \operatorname{Im}\tau_Q$. The forms Q together represent each $n\ge1$ exactly $w\sum_{e|n}\chi(e)$ times [Dav, Ch. 6], so

$$\sum_Q \sum_{(m,n)\neq(0,0)} Q(m,n)^{-s} = w\,\zeta(s)L(s,\chi) \qquad (\operatorname{Re} s>1).$$

The sum over Q has h terms. When $h>1$, can one class be taken alone?

Problem 20.3.1. For $d=20$ the reduced forms are $Q_1=[1,0,5]$ and $Q_2=[2,2,3]$. The form Q_1 represents 1 twice and 3 not at all, Q_2 represents 1 not at all and 3 four times, and $w\sum_{e|n}\chi(e)$ is 2 for $n=1$ and 4 for $n=3$. Hence neither $\sum'_{(m,n)} Q_i(m,n)^{-s}$ is a constant multiple of $\zeta(s)L(s,\chi)$.

So a formula through $L(s,\chi)$ can see only the sum over classes. At $s=1$ this sum is a sum of logarithms, and what one can hope to evaluate is the product over Q of the quantities $\sqrt{y_Q}\,|\eta(\tau_Q)|^2$, not each one.

Problem 20.3.2. $L(0,\chi) = 2h/w$. For $d=3,4,7,23$ this is $\frac13,\frac12,1,3$.

Problem 20.3.3. $\dfrac{L'(1,\chi)}{L(1,\chi)} = \gamma + \log 2\pi - \dfrac{w}{2h}\displaystyle\sum_{a=1}^{d-1}\chi(a)\log\Gamma\Big(\frac{a}{d}\Big)$. For $d=3,7,23$ it is $0.36828\ldots$, $0.01563\ldots$ and $-0.42212\ldots$.

Problem 20.3.4. $Q(m,n) = \dfrac{\sqrt{d}}{2y_Q}|m-n\tau_Q|^2$, and so

$$\sum_Q E(\tau_Q,s) = w\Big(\frac{\sqrt{d}}{2}\Big)^s\zeta(s)L(s,\chi).$$

Problem 20.3.5. $\dfrac{L'(1,\chi)}{L(1,\chi)} = \gamma - \log 2 - \dfrac12\log d - \dfrac{2}{h}\displaystyle\sum_Q \log\big(\sqrt{y_Q}\,|\eta(\tau_Q)|^2\big)$.

Problems 20.3.3 and 20.3.5 prove the formula of Chowla and Selberg [CS].

Theorem 20.3.1 (Chowla–Selberg)**.**

$$\prod_Q \sqrt{y_Q}\,|\eta(\tau_Q)|^2 = (4\pi\sqrt{d})^{-h/2}\prod_{a=1}^{d-1}\Gamma\Big(\frac{a}{d}\Big)^{w\chi(a)/4}.$$

Problem 20.3.6. For $d = 20$, $y_{Q_1} = \sqrt{5}$ and $y_{Q_2} = \sqrt{5}/2$. The geometric mean of $\sqrt{y_Q}\,|\eta(\tau_Q)|^2$ over the two classes is

$$(8\pi\sqrt{5})^{-1/2}\Big(\frac{\Gamma(\frac{1}{20})\Gamma(\frac{3}{20})\Gamma(\frac{7}{20})\Gamma(\frac{9}{20})}{\Gamma(\frac{11}{20})\Gamma(\frac{13}{20})\Gamma(\frac{17}{20})\Gamma(\frac{19}{20})}\Big)^{1/4} = 0.52301\ldots,$$

while the two values themselves are $0.46373\ldots$ for Q_1 and $0.58987\ldots$ for Q_2.

Problem 20.3.7. Let $\rho := e^{2\pi\mathrm{i}/3}$. Then $|\eta(\rho)|^2 = 3^{1/4}\,\Gamma(\frac{1}{3})^3/4\pi^2$.

20.4. Periods

Why should $\prod_a \Gamma(a/d)^{\chi(a)}$ be a period? Because values of Γ at rational numbers come, up to algebraic numbers and powers of π, from integrals of algebraic differentials on Fermat curves.

Let $N \geq 3$, and let F_N be the smooth projective curve with affine equation $u^N + v^N = 1$. For integers $a, b \geq 1$ with $a + b \leq N - 1$, the differential $\omega_{a,b} := u^{a-1}v^{b-N}\,du$ is holomorphic on F_N, as one checks at the points with $v = 0$ and at infinity. Let κ be the path $u \mapsto (u, (1-u^N)^{1/N})$, $0 \leq u \leq 1$, from $(0,1)$ to $(1,0)$. Let $\zeta_N := e^{2\pi\mathrm{i}/N}$, and let $A(u,v) := (\zeta_N u, v)$ and $B(u,v) := (u, \zeta_N v)$.

Problem 20.4.1. $\int_\kappa \omega_{a,b} = \frac{1}{N}\mathrm{B}(\frac{a}{N}, \frac{b}{N})$.

Problem 20.4.2. The chain $(1-A)(1-B)\kappa$ is a closed cycle on F_N, and the integral of $\omega_{a,b}$ over it is $\frac{1}{N}(1-\zeta_N^a)(1-\zeta_N^b)\,\mathrm{B}(\frac{a}{N}, \frac{b}{N})$.

So each $\mathrm{B}(\frac{a}{N}, \frac{b}{N})$ with $a, b \geq 1$ and $a + b \leq N - 1$ is, up to an algebraic factor, a period of a holomorphic differential on F_N defined over $\mathbb{Q}$.

Problem 20.4.3. For integers $a \geq 1$, $\Gamma(\frac{a}{N})^N = \Gamma(a)\prod_{j=1}^{N-1}\mathrm{B}(\frac{a}{N}, \frac{ja}{N})$.

Since $\mathrm{B}(x, y+1) = \mathrm{B}(x,y)\,y/(x+y)$, every $\mathrm{B}(\frac{a}{N}, \frac{b}{N})$ with $a, b \geq 1$ is a rational multiple of one with $1 \leq a, b \leq N$, which is N times an integral of the algebraic differential $\omega_{a,b}$ along κ (Problem 20.4.1). So by Problem 20.4.3, $\Gamma(\frac{a}{N})^N$ is, up to a rational factor, a product of such integrals.

Problem 20.4.4. The map $(u,v) \mapsto (u, v^2)$ from F_4 to the curve $y^2 = 1 - t^4$ pulls dt/y back to $\omega_{1,2}$, and $\mathrm{B}(\frac{1}{4}, \frac{1}{2}) = 2\varpi$.

For K as in Section 20.3, let

$$\Omega_K := \sqrt{\pi}\prod_{a=1}^{d-1}\Gamma\Big(\frac{a}{d}\Big)^{w\chi(a)/4h}.$$

For $K = \mathbb{Q}(\mathrm{i})$, $\Omega_K = 2\varpi$ (Problem 12.3.7, where Ω denotes $\Omega_K^2/2\pi = 2\varpi^2/\pi$), which by Problems 20.4.2 and 20.4.4 is a period of dt/y on $y^2 = 1 - t^4$ up to a factor in $\bar{\mathbb{Q}}^\times$.

Problem 20.4.5. For $K = \mathbb{Q}(\sqrt{-3})$,

$$\Omega_K = \sqrt{\pi}\,\big(\Gamma(\tfrac{1}{3})/\Gamma(\tfrac{2}{3})\big)^{3/2} = 3^{1/4}2^{-1/2}\,\mathrm{B}(\tfrac{1}{3},\tfrac{1}{3}).$$

Here F_3 is itself a curve of genus one, $\omega_{1,1}$ is its holomorphic differential, and A multiplies $\omega_{1,1}$ by ζ_3: the curve has complex multiplication by $\mathbb{Z}[\rho]$. This is the general pattern. Chowla and Selberg deduced the following from Theorem 20.3.1 [CS]; we give it in the formulation of Gross [Gr]. Every nonzero period of a nonzero holomorphic differential defined over $\bar{\mathbb{Q}}$ on an elliptic curve over $\bar{\mathbb{Q}}$ with complex multiplication by $\mathcal{O}_K$ lies in $\bar{\mathbb{Q}}^\times\,\Omega_K$. Gross proved this period relation geometrically, by relating such curves to Jacobians of Fermat curves, whose periods are Beta values by Problem 20.4.2 [Gr].

Problem 20.4.6. $e^{\gamma - L'(1,\chi)/L(1,\chi)} = \Omega_K^2/2\pi^2$. For $K = \mathbb{Q}(\mathrm{i})$ this is $2\varpi^2/\pi^2 = \Gamma(\frac{1}{4})^4/4\pi^3 = 1.39320\ldots$

Thus $e^{\gamma - L'(1,\chi)/L(1,\chi)} = -2\,(\Omega_K/2\pi\mathrm{i})^2$: up to a factor in $\bar{\mathbb{Q}}^\times$, it is the square of the ratio of a period of an elliptic curve with complex multiplication by $\mathcal{O}_K$ to $2\pi\mathrm{i}$, the period of dz/z on the multiplicative group. In Chapter 28, *Covolumes and periods*, the two constants enter in this form for $K = \mathbb{Q}(\mathrm{i})$, through the Faltings height of such a curve (Problem 28.1.10).

Hints

§20.1.

20.1.1: For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, write $m+n\,g\tau$ as $(m'+n'\tau)/(c\tau+d)$, and use $\mathrm{Im}(g\tau) = y/|c\tau+d|^2$.

20.1.2: Apply Poisson summation in m. To find the Fourier transform of $(t^2+y^2)^{-s}$, write it as an integral of Gaussians in t by means of $\Gamma(s)$.

20.1.3: Separate $n = 0$. For $n \neq 0$ apply Problem 20.1.2 with nx and $|n|y$, and collect the terms with $kn = N$. For the continuation, $K_\nu(z)$ decays exponentially as $z \to \infty$, uniformly for ν in a compact set.

20.1.4: Substitute $t = e^{2v}$ and then $w = \sinh v$, and reduce to a Gaussian integral.

20.1.5: $\sum_{n\geq 1} \sigma_{-1}(n)\,q^n = -\sum_{m\geq 1} \log(1-q^m)$.

20.1.6: $\zeta(2s-1) = \frac{1}{2(s-1)} + \gamma + O(s-1)$, and $\psi(\frac{1}{2}) = -\gamma - 2\log 2$ by the duplication formula.

20.1.7: Theorem 20.1.1 and Problem 20.1.1.

20.1.8: $Z(s) = E(\mathrm{i}, s)$ and $\eta(\mathrm{i}) > 0$. Then compare with the Laurent expansion of $4\zeta(s)L(s,\chi)$, using $L(1,\chi) = \pi/4$.

§20.2.

20.2.1: Set $s = 0$ in the continuation formula. For the second statement, split $\zeta(s)$ into even and odd n.

20.2.2: $\partial_x \zeta(s,x) = -s\,\zeta(s+1,x)$. Differentiate twice in x, then in s at $s = 0$. For the second statement, $\zeta(s,x+1) = \zeta(s,x) - x^{-s}$.

20.2.3: $L(s,\chi) = N^{-s}\sum_{a=1}^{N} \chi(a)\,\zeta(s,a/N)$, and $\sum_a \chi(a) = 0$.

20.2.4: Take the logarithmic derivative of the functional equation at $s = 1$, and use $\psi(\frac{1}{2})$ as in Problem 20.1.6. Here $L(0,\bar{\chi}) \neq 0$, since $\Lambda(0,\bar{\chi})$ is a nonzero multiple of $\Lambda(1,\chi)$.

20.2.5: $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \pi\sqrt{2}$.

§20.3.

20.3.2: Compare $\Lambda(0,\chi)$ with $\Lambda(1,\chi)$, and use the class number formula.
20.3.3: Problems 20.2.3, 20.2.4 and 20.3.2.
20.3.4: Expand $a|m - n\tau_Q|^2$. Then replace n by $-n$.
20.3.5: Expand both sides of Problem 20.3.4 at $s = 1$ by Theorem 20.1.1. The residues agree by the class number formula; compare the constant terms.
20.3.1: List the small values of each form. For the right side, $\chi(3) = (\frac{-20}{3})$.
20.3.6: Theorem 20.3.1 with $h = w = 2$; list the $a < 20$ with $\chi(a) = 1$. For the numerical values, sum the product for η to a few terms.
20.3.7: Theorem 20.3.1 with $d = 3$, and the reflection formula.

§20.4.

20.4.1: Substitute $t = u^N$ in $\mathrm{B}(\frac{a}{N}, \frac{b}{N}) = \int_0^1 t^{a/N-1}(1-t)^{b/N-1}\,dt$.
20.4.2: $A^*\omega_{a,b} = \zeta_N^a\,\omega_{a,b}$ and $B^*\omega_{a,b} = \zeta_N^b\,\omega_{a,b}$. For the boundary, A fixes the points with $u = 0$ and B those with $v = 0$.
20.4.3: $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$; the product telescopes.
20.4.4: Problem 20.4.1 and the definition of ϖ in Chapter 12.
20.4.5: $h = 1$, $w = 6$, and the reflection formula as in Problem 20.3.7.
20.4.6: Problem 20.3.3.

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