COVOLUMES AND PERIODS

Prerequisites. Weierstrass functions and theta functions of one variable, and $\Delta(\mathbb{Z}+\mathbb{Z}\tau) = (2\pi)^{12} q \prod_{n\geq 1}(1-q^n)^{24}$, $q = e^{2\pi i \tau}$ [SeA, §VII.4.4]. Modular forms for $\mathrm{SL}_2(\mathbb{Z})$, q-expansions and the valence formula [SeA, Ch. VII]. From Chapter 12: Sections 12.1, 12.2 and 12.3. From Chapter 6, *The slope method*: Section 6.3, and the definition of $\mu_{\max}$ in Section 6.4. From Chapter 26: Theorem 26.2.1, Problems 26.1.5, 26.1.7 and 26.2.2, and Section 26.3.

In Chapter 26 the leading term of a covolume was an arithmetic self-intersection. For theta functions on an elliptic curve it vanishes, and a period takes its place. Where does the period come from, and how does it enter lattices of modular forms and of forms on a definite group?

28.1. Theta functions on an elliptic curve

Let $\Lambda \subset \mathbb{C}$ be a lattice, with σ, $\wp$ and the quasi-period map η as in Section 12.1. Let $A(\Lambda)$ be the area of $\mathbb{C}/\Lambda$, and $\Delta(\Lambda) = g_2^3 - 27 g_3^2$.

Problem 28.1.1. There is a unique $s_2 \in \mathbb{C}$ with $\eta(\lambda) = s_2\lambda + \pi\bar{\lambda}/A(\Lambda)$ for all $\lambda \in \Lambda$.

Problem 28.1.2. The function

$$F(z) := |\sigma(z)| \exp\Big(-\tfrac{1}{2}\operatorname{Re}(s_2 z^2) - \frac{\pi|z|^2}{2A(\Lambda)}\Big)$$

is Λ-periodic, and $F(z)/|z| \to 1$ as $z \to 0$.

Problem 28.1.3. $F(-z) = F(z)$ and $F(2z) = |\wp'(z)|\, F(z)^4$.

Let $E = \mathbb{C}/\Lambda$, and let ω_E be its cotangent space at the origin. Since the fiber of $\mathcal{O}([0])$ at the origin is the tangent space, $\mathcal{O}([0]) \otimes \omega_E$ is trivial there. By Problem 28.1.2 it has a metric, trivial at the origin and with translation-invariant curvature, for which the canonical section (pulled back to σ on $\mathbb{C}$) tensored with dz has norm F; this is the cubist metric of Moret-Bailly [MB1].

For a nonzero differential $\omega = c\,dz \in \omega_E$, let $\Lambda_\omega = c\Lambda$ be its period lattice, $x_\omega := c^{-2}\wp$ and $y_\omega := c^{-3}\wp'/2$, so that $dx_\omega/2y_\omega = \omega$ and the Weierstrass equation of (x_ω, y_ω) has discriminant $\Delta(\Lambda_\omega)$. Let $f_0 := 1$, $f_2 := x_\omega$, $f_3 := y_\omega$, $f_4 := x_\omega^2$, and so on, with f_j of pole order j; for $k \geq 1$, $f_0, f_2, \ldots, f_k$ span the elliptic functions with poles of order at most k at Λ only. Put

$$\|f\,\omega^k\|^2 := \frac{|c|^{2k}}{A(\Lambda)} \int_{\mathbb{C}/\Lambda} |f|^2 F^{2k}\, dA,$$

the L^2 norm of a section of $(\mathcal{O}([0]) \otimes \omega_E)^{\otimes k}$, and let $\mathrm{Gr}_k(\omega)$ be the Gram determinant of $f_0\omega^k, f_2\omega^k, \ldots, f_k\omega^k$.

For the next problem let $\Lambda = \mathbb{Z}+\mathbb{Z}\tau$, let ϑ be as in Section 12.2, and for $0 \leq a < k$ let $\vartheta_{k,a}(z) := \sum_{n\in\mathbb{Z}} e^{\pi i k\tau(n+a/k)^2 + 2\pi i k z(n+a/k)}$.

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Problem 28.1.4. The functions $\vartheta_{k,a}$, $0 \le a < k$, are orthogonal for the weight $e^{-2\pi k(\operatorname{Im} z)^2/\operatorname{Im}\tau}$ on $\mathbb{C}/\Lambda$ with the measure $dA/\operatorname{Im}\tau$, and each has norm squared $(2k\operatorname{Im}\tau)^{-1/2}$.

Jacobi's derivative formula reads $|\vartheta'(\frac{1+\tau}{2})| = e^{\pi\operatorname{Im}\tau/4}(2\pi)^{-1/2}|\Delta(\mathbb{Z}+\mathbb{Z}\tau)|^{1/8}$ [MuT, Ch. I, §§13–14].

Problem 28.1.5. $\mathrm{Gr}_1(\omega) = \|\omega\|^2 = \sqrt{2}\,\pi\, A(\Lambda_\omega)^{-1/2}|\Delta(\Lambda_\omega)|^{-1/4}$.

Theorem 28.1.1. *For every nonzero differential ω on E and every $k \ge 1$,*

$$-\tfrac{1}{2}\log \mathrm{Gr}_k(\omega) = \frac{k}{4}\log A(\Lambda_\omega) + \frac{k^2+2}{24}\log|\Delta(\Lambda_\omega)| + \frac{k}{4}\log\frac{k}{2\pi^2}.$$

For $k=1$ this is Problem 28.1.5; for general k it follows from Moret-Bailly's key formula [MB1, Ch. VIII], [MB2], in the metrized form of Bost [Bo, Thm. 4.10(v)]. We have checked it numerically for $k \le 6$.

Problem 28.1.6. $A(c\Lambda)|\Delta(c\Lambda)|^{1/6} = A(\Lambda)|\Delta(\Lambda)|^{1/6}$. Moreover $\mathrm{Gr}_k(c\,\omega) = |c|^{2N_k}\mathrm{Gr}_k(\omega)$ with $N_k = (k^2-k+2)/2$, and the right side of Theorem 28.1.1 also changes by $-N_k\log|c|$ when ω is replaced by $c\,\omega$.

Now let E be an elliptic curve over a number field K with a Weierstrass equation over $\mathcal{O}_K$ of unit discriminant Δ_E, and $\omega = dx/(2y+a_1x+a_3)$; a curve with complex multiplication has such an equation over a finite extension [ST], [Si, §VIII.8]. Let Λ_σ be the period lattice of ω^σ for $\sigma\colon K \to \mathbb{C}$. The *Faltings height*, in Deligne's normalization [De], is

$$h_{\mathrm{F}}(E) := -\frac{1}{2[K:\mathbb{Q}]}\sum_\sigma \log A(\Lambda_\sigma) = -\frac{1}{2[K:\mathbb{Q}]}\sum_\sigma \log\bigl(A(\Lambda_\sigma)|\Delta(\Lambda_\sigma)|^{1/6}\bigr),$$

the second form by the product formula; by Problem 28.1.6 it does not change under extension of K. Let M_k be the $\mathcal{O}_K$-lattice of sections of $(\mathcal{O}([0])\otimes\omega_E)^{\otimes k}$ on the Weierstrass model; a basis is formed by the monomials in the coordinates x, y of the model of pole orders $0, 2, \dots, k$, multiplied by ω^k. The $f_j\omega^k$ are obtained from this basis by a triangular change of basis over K with diagonal entries 1, which may be nonintegral at 2 and 3; so the two bases have the same Gram determinant at every embedding, and Theorem 28.1.1 gives:

Theorem 28.1.2. $\dfrac{\deg M_k}{[K:\mathbb{Q}]} = -\dfrac{k}{2}\,h_{\mathrm{F}}(E) + \dfrac{k}{4}\log\dfrac{k}{2\pi^2}$, *and hence* $\mu^\sharp(M_k) = -h_{\mathrm{F}}(E) + \tfrac{1}{2}\log\dfrac{k}{2\pi^2}$.

The degree has no term in k^2. Why? Let $\bar L$ be $\mathcal{O}([0])\otimes\omega_E$ on the Weierstrass model, with the cubist metric at every embedding, and let $I := \hat c_1(\bar L)^2/[K:\mathbb{Q}]$. Multiplication by 2 is finite and flat of degree 4, so $\hat c_1([2]^*\bar M)^2 = 4\,\hat c_1(\bar M)^2$ for every metrized line bundle $\bar M$; and there is an isomorphism $[2]^*\bar L \cong \bar L^{\otimes 4}$ which is an isometry [MB1], at the archimedean places the identity of Problem 28.1.3.

Problem 28.1.7. $I = 0$.

So by Theorem 26.2.1, applied to the Weierstrass model as a scheme over $\mathbb{Z}$ (regarding M_k as a lattice over $\mathbb{Z}$ changes its degree by $O(k)$), the coefficient of k^2 in $\deg M_k$ must vanish, as Theorem 28.1.2 shows it does. What does the normalization of the metric contribute? Let $\bar L_t$ be $\bar L$ with all norms multiplied by e^{-t}, and $M_k(t)$

the lattice M_k with the norms of $\bar{L}_t^{\otimes k}$. For the trivial bundle $\bar{\mathcal{O}}(a)$ with $\|1\| = e^{-a}$ at every embedding, $\hat{c}_1(\bar{M} \otimes \bar{\mathcal{O}}(a))^2 = \hat{c}_1(\bar{M})^2 + 2a[K:\mathbb{Q}] \deg M_K$, where $\deg M_K$ is the degree on the generic fiber [GS].

Problem 28.1.8. $\deg M_k(t)/[K:\mathbb{Q}] = \deg M_k/[K:\mathbb{Q}] + tk^2$. The isomorphism $[2]^*\bar{L}_t \cong \bar{L}_t^{\otimes 4}$ multiplies norms by e^{-3t}, and computing $\hat{c}_1([2]^*\bar{L}_t)^2$ in the two ways gives $\hat{c}_1(\bar{L}_t)^2/[K:\mathbb{Q}] = 2t$, in agreement with Theorem 26.2.1.

A constant rescaling of the metric moves the leading term of the degree; the cubist metric is the one normalization for which multiplication by 2 is an isometry, and only for it does the k^2 term vanish. The period takes the place of the leading term of Theorem 26.2.1.

Problem 28.1.9. Let E have complex multiplication by $\mathbb{Z}[\mathrm{i}]$, and let Ω be as in Problem 12.3.7. Then $h_{\mathrm{F}}(E) = -\frac{1}{2}\log(\pi\Omega) = -2\log\Gamma(\frac{1}{4}) + \frac{1}{2}\log 4\pi = -1.31053\ldots$

Problem 28.1.10. For E as in Problem 28.1.9, $h_{\mathrm{F}}(E) = \frac{1}{2}L'(1,\chi)/L(1,\chi) - \frac{1}{2}\gamma - \log\pi$, where χ is the character modulo 4. Hence $\mu^\sharp(M_k) = \frac{1}{2}\big(\log\frac{k}{2} + \gamma - L'(1,\chi)/L(1,\chi)\big)$.

So Euler's constant and $L'(1,\chi)/L(1,\chi)$, which enter the slope method through the heights at infinity (Section 7.4 of *The slope inequality in the mean*), also enter the slope of the lattice of forms whose dual is the target.

28.2. Cusp forms of level one

Let $M_k(\mathbb{Z})$ be the lattice of modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ with q-expansion in $\mathbb{Z}[[q]]$, $S_k(\mathbb{Z})$ its cusp forms, and $\Delta(\tau) := q\prod_{n\ge1}(1-q^n)^{24} = (2\pi)^{-12}\Delta(\mathbb{Z}+\mathbb{Z}\tau)$. The Petersson norm is $\|f\|^2 := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} |f|^2 y^{k-2}\,dx\,dy$.

Problem 28.2.1. $S_k(\mathbb{Z}) = \Delta \cdot M_{k-12}(\mathbb{Z})$, and $S_k(\mathbb{Z})$ is a lattice of rank $\dim S_k$.

Problem 28.2.2. $S_k(\mathbb{Z})$ has rank one exactly for $k \in \{12,16,18,20,22,26\}$, and then $S_k(\mathbb{Z}) = \mathbb{Z}\,\Delta E_{k-12}$, where $E_0 = 1$, E_4, E_6, $E_8 = E_4^2$, $E_{10} = E_4E_6$ and $E_{14} = E_4^2E_6$ are the normalized Eisenstein series.

For example $\|\Delta\|^2 = 1.035362\ldots\cdot 10^{-6}$ (Section 11.2 of Chapter 11), so $\deg S_{12}(\mathbb{Z}) = 6.89037\ldots$ How fast do the slopes grow with k?

Problem 28.2.3. Every nonzero $f \in S_k(\mathbb{Z})$ has $\|f\|^2 \ge 3e^{-\pi k/3}/(\pi k)$. Hence $\mu_{\max}(S_k(\mathbb{Z})) \le \pi k/6 + O(\log k)$.

Linear growth is also what Theorem 26.2.1 gives for the mean slope of the sections of powers of a line bundle with a smooth metric. For the Hodge bundle ω the Petersson metric is singular at the cusp; its self-intersection was computed by Kühn [Kü]. Problem 28.2.3 does not use this self-intersection: it compares the Petersson norm with a norm on the half-strip $y \ge 1$, as Problems 26.1.5 and 26.2.2 compare norms.

A form f gives the section $f(\tau)(2\pi\mathrm{i}\,dz)^k$ of ω^k on $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, and $2\pi\mathrm{i}\,dz$ is the integral differential dt/t of the Tate curve [Ka]; the Petersson metric is $\|dz\|^2 = y$.

Problem 28.2.4. The function $y^6|\Delta(\tau)|$ is invariant under $\mathrm{SL}_2(\mathbb{Z})$. If $\mathbb{Z}+\mathbb{Z}\tau$ is the ring of integers of an imaginary quadratic field of class number one and E_τ is as above, then

$$h_{\mathrm{F}}(E_\tau) = -\tfrac{1}{12}\log\big(y^6|\Delta(\tau)|\big) - \log 2\pi.$$

In particular $\Delta(\mathrm{i}) = (\Omega/4\pi)^6$.

So the Faltings height is $-\frac{1}{12}$ times the logarithm of the Petersson norm $y^6|\Delta(\tau)|$ of Δ at the point with complex multiplication, and the term $-\log 2\pi$ is the change from dz to $2\pi\mathrm{i}\,dz$. Such normalizations move terms linear in the weight, as the main terms are, and must be fixed once.

28.3. The period on a definite group

The lattice $V_\lambda(L)_\mathbb{Z}$ of Section 26.3 is defined by L, the homology of the torus $L_\mathbb{C}/L$; the algebraic structure is defined by differentials. For $L = I_n$ the torus is E^n with $E = \mathbb{C}/\mathcal{O}$; since $\mathcal{O}$ is a principal ideal domain, every L is free over $\mathcal{O}$ and every $L_\mathbb{C}/L$ is isomorphic to E^n. Let $\omega = c\,dz$ be the invariant differential of a Weierstrass equation for E, let $V_\lambda(L)^{\mathrm{alg}}$ be the image of the tensor powers of $c^{-1}L$ (for $L = I_n$, the $\mathcal{O}$-span of the $\partial_k := c^{-1}e_k$), and let $\mathcal{A}_\lambda(n)^{\mathrm{alg}}$ be the corresponding lattice of forms.

Problem 28.3.1. $\mu\big(\mathcal{A}_\lambda(n)^{\mathrm{alg}}\big) = \mu\big(\mathcal{A}_\lambda(n)_\mathbb{Z}\big) + |\lambda|\log A(c\,\mathcal{O})$. For $y^2 = x^3 - x$ and $\omega = dx/2y$, $A(c\,\mathcal{O}) = \varpi^2 = \pi\Omega/2$.

Over a field where E has an equation of unit discriminant, Problems 28.1.9 and 26.1.7 give instead an increase of $\mu^\sharp$ by $-2|\lambda|h_\mathrm{F}(E) = |\lambda|\log(\pi\Omega) = |\lambda|(\gamma + 2\log\pi - L'(1,\chi)/L(1,\chi))$; the difference $|\lambda|\log 2$ comes from the prime above 2. So the period enters the slope of the lattice of forms linearly in the weight. In Alpöge's argument [Alp], [Zh] the lattices are of forms on unitary groups of signature (p,q), and the period enters with coefficient $p-q$ times the weight [Adr, Hyp. 15.58]. These groups are indefinite, and their Shimura varieties have positive dimension; Section 28.2, on $X(1)$, is the closer model, and the definite case of this section is a model only. What the sizes do not account for are the denominators of maps that do not respect integral structures; the next part, from Chapter 29 on, measures them.

Hints

Section 28.1.

28.1.1: η is $\mathbb{Z}$-linear, so it extends to an $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{C}$, and such a map is $z \mapsto \alpha z + \beta\bar z$. Legendre's relation (Problem 12.1.3) determines β.

28.1.2: Use Problem 12.1.4 and Problem 28.1.1.

28.1.3: σ is odd. Let $v \to u$ in Problem 12.1.5 to get $\sigma(2u) = -\wp'(u)\sigma(u)^4$.

28.1.4: Integrate first over $\mathrm{Re}\,z \in [0,1]$, then unfold the sum over n against the integral over $\mathrm{Im}\,z \in [0, \mathrm{Im}\,\tau]$.

28.1.5: Both sides depend only on (E,ω), and change by the same factor when ω is multiplied by a constant. So one may take $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ and $\omega = dz$. Write σ through $\vartheta(z + \frac{1+\tau}{2})$ (Problem 12.2.3); the constant is fixed by $\sigma'(0) = 1$. Note $\eta(1) = s_2 + \pi/\mathrm{Im}\,\tau$ and $\mathrm{Re}(z^2) - |z|^2 = -2(\mathrm{Im}\,z)^2$.

28.1.6: $\Delta(c\Lambda) = c^{-12}\Delta(\Lambda)$, and f_j for $c\,\omega$ is c^{-j} times f_j for ω.

28.1.8: Each of the k sections $f_j\omega^k$ has its norm multiplied by e^{-tk} at every embedding; take the Gram determinant. For the self-intersection, write $[2]^*\bar L_t$ as $\bar L_t^{\otimes 4} \otimes \bar{\mathcal{O}}(a)$ and find a.

28.1.9: Every conjugate of E is isomorphic to $\mathbb{C}/\mathbb{Z}[\mathrm{i}]$. Take $\Lambda = \mathbb{Z}[\mathrm{i}]$ in the invariant form of h_F, and use Problems 12.3.6 and 12.3.7 (where η is Dedekind's function, not the quasi-period map).

28.1.10: Theorem 12.3.1.

Section 28.2.

28.2.1: $1/\Delta = q^{-1}\prod_{n\geq 1}(1-q^n)^{-24}$ has integral coefficients. For the rank, S_k has a basis of products of Δ, E_4 and E_6 [SeA, §VII.3], and $f \mapsto (a_1, \ldots, a_d)$, $d = \dim S_k$, is injective by the valence formula.

28.2.2: $\dim S_k$ is the dimension of the space of modular forms of weight $k-12$, which is spanned by a form with q-expansion $1 + O(q)$ when it is one-dimensional.

28.2.3: The half-strip $|x| \leq \frac{1}{2}$, $y \geq 1$ lies in the fundamental domain. If the first nonzero coefficient of f is a_m, integrate $|f|^2$ over x by Parseval and use $y^{k-2} \geq 1$. The valence formula gives $m \leq k/12$. For the slope, Minkowski's theorem gives a nonzero vector of length at most $\sqrt{s}\,\mathrm{covol}(V)^{1/s}$ in a lattice V of rank s.

28.2.4: Use the invariant form of h_{F} in Section 28.1, with $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$; every conjugate of E_τ is isomorphic to E_τ, since the class number is one. For Δ(i), Problem 12.3.7.

Section 28.3.

28.3.1: The period lattice of ω is $c\,\mathcal{O}$. Scale each basis vector by $c^{-|\lambda|}$. For the curve, use Problem 12.1.9: the period lattice of $dx/2y$ is $\varpi\mathbb{Z}[\mathrm{i}]$.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

[Alp] L. Alpöge, untitled note announcing a proof of the Riemann hypothesis, 20 November 2026, `https://alpo.ge/remark.pdf`.

[Bo] J.-B. Bost, *Intrinsic heights of stable varieties and abelian varieties*, Duke Math. J. **82** (1996), 21–70.

[De] P. Deligne, *Preuve des conjectures de Tate et de Shafarevitch (d'après G. Faltings)*, Séminaire Bourbaki, Exp. 616, Astérisque **121–122** (1985), 25–41.

[GS] H. Gillet and C. Soulé, *Arithmetic intersection theory*, Publ. Math. IHÉS **72** (1990), 93–174.

[Ka] N. M. Katz, *p-adic properties of modular schemes and modular forms*, in *Modular Functions of One Variable III*, Lecture Notes in Math. 350, Springer, 1973, 69–190.

[Kü] U. Kühn, *Generalized arithmetic intersection numbers*, J. reine angew. Math. **534** (2001), 209–236.

[MB1] L. Moret-Bailly, *Pinceaux de variétés abéliennes*, Astérisque **129** (1985).

[MB2] L. Moret-Bailly, *Sur l'équation fonctionnelle de la fonction thêta de Riemann*, Compositio Math. **75** (1990), 203–217.

[MuT] D. Mumford, *Tata Lectures on Theta I*, Progr. Math. 28, Birkhäuser, 1983.

[SeA] J.-P. Serre, *A Course in Arithmetic*, Grad. Texts in Math. 7, Springer, 1973.

[Si] J. H. Silverman, *The Arithmetic of Elliptic Curves*, 2nd ed., Grad. Texts in Math. 106, Springer, 2009.

[ST] J.-P. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. of Math. **88** (1968), 492–517.

[Zh] R. Zhang, *A preliminary note on Alpöge's proposed proof of the generalized Riemann hypothesis*, preprint, 2026, hexagon:2611.00809.