COVOLUMES OF LATTICES OF SECTIONS

Prerequisites. Euler–Maclaurin summation and Stirling's formula. Arithmetic intersection theory is used only through the statement of Theorem 26.2.1. From Chapter 6, *The slope method*: Section 6.3 (normed spaces, degrees and slopes, $\mu^\sharp$), and the definition of $\mu_{\max}$ in Section 6.4. From Chapter 21, *Masses and the count of forms*: Sections 21.1–21.3, in Section 26.3.

The slope method of Chapter 6 needs a lower bound for the slope of its source, a lattice of sections or of automorphic forms. In the argument the lattice of automorphic forms is the one whose dual is the target of the maps [Adr, §15.4]. How is its covolume computed? We compute it exactly for polynomials on a projective space and for forms of weight (a) on a definite unitary group of rank at most two, compare with the arithmetic Hilbert–Samuel theorem, and ask which uniformity the argument needs. The period that enters through the integral structure is the subject of Chapter 28. Degrees and slopes are those of Section 6.3; over $\mathbb{Q}$, $\deg L = -\log \operatorname{covol}(L)$ (Problem 6.3.1).

26.1. Projective spaces

Let $E_D \subset \mathbb{Z}[x,y]$ be the lattice of binary forms of degree D with the Fubini–Study L^2 norm of Section 6.3. The monomials are orthogonal, with $\|x^a y^{D-a}\|^2 = a!\,(D-a)!/(D+1)!$ (Problem 6.3.4), so the covolume has a closed form.

Problem 26.1.1. $\deg E_D = \frac{1}{2}(D+1)\log(D+1) + \frac{1}{2}\sum_{k=1}^{D}(2k-D-1)\log k$.

Problem 26.1.2. $\deg E_D = \frac{1}{4}D^2 + \frac{1}{4}D\log D + \frac{1}{2}\big(1 - \frac{1}{2}\log 2\pi\big)D + O(\log D)$. For $D = 10$ the exact value is $\deg E_{10} = 32.1969\ldots$

Problems 26.1.1 and 26.1.2 prove the following, which refines Problem 6.3.5.

Theorem 26.1.1. *The lattice $H^0(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(D)) = E_D$, with the L^2 norm of the Fubini–Study metric and measure, has*

$$\deg E_D = \tfrac{1}{4}D^2 + \tfrac{1}{4}D\log D + O(D).$$

The term $\frac{1}{4}D\log D$ is the sum of $\frac{1}{2}D\log D$ from the factor $D+1$ in $\|x^a y^{D-a}\|^{-2} = (D+1)\binom{D}{a}$, which comes from the normalization of the measure, and $-\frac{1}{4}D\log D$ from the binomial coefficients. How far do the slopes of sublattices spread? Recall that $\mu_{\max}(L)$ is the largest slope of a nonzero sublattice of L, and let $\mu_{\min}(L)$ be the smallest slope of a nonzero quotient of L.

Problem 26.1.3. Let $L = L_1 \oplus L_2$ be an orthogonal sum of Euclidean lattices. Then $\mu_{\max}(L) = \max(\mu_{\max}(L_1), \mu_{\max}(L_2))$.

Problem 26.1.4. $\mu_{\max}(E_D) = \frac{1}{2}\log\big((D+1)\binom{D}{\lfloor D/2\rfloor}\big) = \frac{1}{2}D\log 2 + O(\log D)$, and $\mu_{\min}(E_D) = \frac{1}{2}\log(D+1)$.

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A lower bound for $\mu(E_D)$ is all the slope method uses, so the sublattices of large slope cost nothing. Let $\|f\|_\infty := \sup_z |f(z,1)|(1+|z|^2)^{-D/2}$.

Problem 26.1.5. For $f \in E_D \otimes \mathbb{C}$, $\|f\| \le \|f\|_\infty \le (D+1)^{1/2}\|f\|$. The second inequality is an equality for $f = (\bar{w}x + y)^D$.

In n variables, let $E_{n,a} \subset \mathbb{Z}[x_1, \dots, x_n]$ be the lattice of forms of degree a, with $\|f\|^2 := \int_{S^{2n-1}} |f|^2 \, d\sigma$ for the invariant probability measure σ on the unit sphere of $\mathbb{C}^n$. For $n = 2$ this is the norm of E_a; in general it is the Fubini–Study L^2 norm on $\mathbb{P}^{n-1}$.

Problem 26.1.6. The monomials are orthogonal in $E_{n,a}$, and $\|x^\nu\|^2 = \nu!\,(n-1)!/(a+n-1)!$.

Problem 26.1.7. Let V be a normed space over K, and K'/K a finite extension. Give $V \otimes_K K'$ the unit lattices $L_v \otimes \mathcal{O}_{K'_w}$ and the norms extended to $V \otimes_K K'_w$. Then $\mu^\sharp(V \otimes_K K') = \mu^\sharp(V)$. In particular $\mu^\sharp(E_D \otimes \mathcal{O}_K) = 2\mu(E_D)$.

26.2. The arithmetic Hilbert–Samuel theorem

Let X be a projective flat integral scheme over $\mathbb{Z}$ of dimension $d+1$ with smooth generic fiber, $\bar{L}$ an ample line bundle on X with a smooth, conjugation-invariant Hermitian metric of positive curvature, and dV a smooth conjugation-invariant volume form on $X(\mathbb{C})$. Give $H^0(X, L^{\otimes n})$ the norm $\|s\|^2 := \int_{X(\mathbb{C})} |s|^2 \, dV$, and let $\hat{c}_1(\bar{L})^{d+1}$ be the arithmetic self-intersection of Gillet and Soulé [GS1].

Theorem 26.2.1 (Gillet–Soulé, Abbes–Bouche)**.** *As $n \to \infty$,*

$$\deg H^0(X, L^{\otimes n}) = \frac{\hat{c}_1(\bar{L})^{d+1}}{(d+1)!}\, n^{d+1} + o(n^{d+1}).$$

This is due to Gillet and Soulé for regular X [GS2] and to Abbes and Bouche in general [AB, §1, Théorème principal]; the error depends on X, $\bar{L}$ and dV, and is not explicit. For $\mathbb{P}^N_{\mathbb{Z}}$ with the Fubini–Study metric, Bost, Gillet and Soulé computed

$$\hat{c}_1(\bar{\mathcal{O}}(1))^{N+1} = \frac{1}{2} \sum_{k=1}^{N} \sum_{m=1}^{k} \frac{1}{m} \tag{1}$$

[BGS].

Problem 26.2.1. Theorem 26.1.1 and Theorem 26.2.1 give $\hat{c}_1(\bar{\mathcal{O}}(1))^2 = \frac{1}{2}$ on $\mathbb{P}^1_{\mathbb{Z}}$, in agreement with (1). Replacing the metric of $\mathcal{O}(1)$ by $e^{-c}|\cdot|$ adds $cD(D+1)$ to $\deg E_D$ and $2c$ to $\hat{c}_1(\bar{\mathcal{O}}(1))^2$.

Problem 26.2.2. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be Euclidean norms on a lattice L of rank r, with $\|v\|_1 \le C\|v\|_2$ for all v. Then $\deg_1 L \ge \deg_2 L - r \log C$. Hence, since $\operatorname{rank} H^0(X, L^{\otimes n}) = O(n^d)$, the leading term in Theorem 26.2.1 does not depend on dV.

26.3. Definite unitary groups

On a definite group the Shimura variety is a finite set, and the covolume of the lattice of integral forms can be computed by hand. The argument needs the covolume on unitary groups of signature (p,q) with $p, q > 0$, whose forms have discrete series at infinity and whose Shimura varieties have positive dimension [Adr, §15.14 and

Hyp. 15.58]; the definite case is a model. We use the notation of Chapter 21, *Masses and the count of forms*: $\mathcal{O} = \mathbb{Z}[\mathrm{i}]$, the class set Cl_n, the groups Γ_L, the mass M_n, the spaces $V_\lambda(L)$, and the forms $\mathcal{A}_\lambda(n)$ with the norm $\|\varphi\|^2 = \sum_L |\Gamma_L|^{-1}\|\varphi(L)\|^2$. Let $V_\lambda(L)_\mathbb{Z}$ be the image of $L^{\otimes|\lambda|}$ under the orthogonal projection onto $V_\lambda(L)$; for $\lambda = (a)$ it is the $\mathcal{O}$-span of the e^ν. The *integral forms* make up the $\mathcal{O}$-lattice $\mathcal{A}_\lambda(n)_\mathbb{Z}$ of φ with $\varphi(L) \in V_\lambda(L)_\mathbb{Z}$ for every L. Degrees are over $\mathbb{Q}(\mathrm{i})$, so $\mu^\sharp = \mu$.

Problem 26.3.1. $\deg \mathcal{A}_\lambda(n)_\mathbb{Z} = \sum_{L\in\mathrm{Cl}_n} \Big(\deg V_\lambda(L)_\mathbb{Z}^{\Gamma_L} + r_L \log|\Gamma_L|\Big)$, where r_L is the rank of the invariants $V_\lambda(L)_\mathbb{Z}^{\Gamma_L}$.

Problem 26.3.2. For $n = 1$ and $4 \mid a$, $\mathcal{A}_{(a)}(1)_\mathbb{Z}$ has rank one and slope $\log 4 = -\log M_1$.

Problem 26.3.3. With the norm of Problem 21.3.2, $\|e^\nu\|^2 = \binom{a+n-1}{n-1}\|x^\nu\|^2$, where $\|x^\nu\|$ is the norm of Problem 26.1.6. Hence $V_{(a)}(I_n)_\mathbb{Z} = E_{n,a} \otimes \mathcal{O}$, with its norm multiplied by $\binom{a+n-1}{n-1}^{1/2}$.

For $n = 2$ the class set is $\{I_2\}$, and $|\Gamma_{I_2}| = 32$ (Problems 21.1.4 and 21.1.2).

Problem 26.3.4. Let $4 \mid a$. The invariants $V_{(a)}(I_2)_\mathbb{Z}^\Gamma$ have the basis $\sum_{\mu\in\{\nu,\nu'\}} e^\mu$, where $\nu = (\nu_1, \nu_2)$ runs over the pairs with $4 \mid \nu_1$ and $\nu_1 \le \nu_2$, and $\nu' = (\nu_2, \nu_1)$. Consequently

$$\deg \mathcal{A}_{(a)}(2)_\mathbb{Z} = \sum_\nu \Big(\log 32 - \log|\{\nu,\nu'\}| + \log\tbinom{a}{\nu_1}\Big),$$

and $\deg \mathcal{A}_{(8)}(2)_\mathbb{Z} = \log 35840$.

Problem 26.3.5. $\mu\big(\mathcal{A}_{(a)}(2)_\mathbb{Z}\big) = \frac{1}{2}a + O(\log a)$ as $a \to \infty$ through multiples of 4.

So for $n = 2$, $\lambda = (a)$ and $4 \mid a$ the rank is about $4M_2 \dim V_{(a)}$ (Problem 21.3.5), each form contributes $\log 32 = -\log M_2$ through the stabilizer, and the slope has the same leading term $a/2$ as that of $E_a \otimes \mathcal{O}$. We do not treat larger n.

26.4. Uniformity

Theorem 26.2.1 concerns a fixed variety and a growing power of a fixed line bundle. In Alpöge's argument [Alp], [Zh] the varieties are Shimura varieties of unitary groups, or flag bundles over them, whose dimension grows; the level and the field vary; and the weight is given rather than tending to infinity. By Problem 26.1.7, $\mu^\sharp$ of a fixed lattice does not change when its field is extended. The other parameters are harder.

Problem 26.4.1. $1 \le \binom{a+n-1}{n-1} \Big/ \dfrac{a^{n-1}}{(n-1)!} \le e^{n^2/2a}$, and the ratio tends to infinity if n^2/a does.

So the leading term $a^{n-1}/(n-1)!$ of the Hilbert polynomial approximates the rank $\binom{a+n-1}{n-1}$ of $E_{n,a}$ only when a is large compared with n^2, the square of the dimension.

Problem 26.4.2. $\mu(E_{n,a}) \ge \frac{1}{2}\log\binom{a+n-1}{n-1}$.

This bound is uniform but misses the main term: for fixed n, (1) gives $\mu(E_{n,a}) \sim c_n a$ with $c_n = \frac{1}{2n}\sum_{k<n}\sum_{m\le k} 1/m$. Chow forms give lower bounds in which every constant is explicit. For an irreducible $X \subset \mathbb{P}^n_{\mathbb{Q}}$ of dimension $r-1$ with homogeneous ideal $I \subset \mathbb{Z}[x_0,\dots,x_n]$, David and Philippon bound $-\log \operatorname{covol}\big((\mathbb{Z}[x_0,\dots,x_n]/I)_a\big)$ from below by $h(I)\,\big((a-c)/r\big)^r$ minus an explicit error, for every a above an explicit threshold c, where $h(I)$ is a height of X [DPh, Prop. 4.10 and Cor. 4.11]. They do not aim at the leading term: the factor r^{-r} stands where Theorem 26.2.1 has $1/r!$, which is larger by a factor exponential in r.

Problem 26.4.3. Let $N \ge 1$, and let $E_D(N) \subseteq E_D$ be the sublattice of f whose coefficient of x^D is divisible by N, that is, $f(1,0) \equiv 0 \pmod N$. Then $\mu(E_D(N)) = \mu(E_D) - (\log N)/(D+1)$.

For automorphic forms, a level structure of index I on which every Γ_L acts freely also removes the terms $r_L \log|\Gamma_L|$ of Problem 26.3.1, and the rank becomes $IM_n \dim V_\lambda$.

Questions. In [Adr] the covolume bound the argument needs is [Adr, Hyp. 15.58], assumed there; the Lean formalization proves a bound of this kind, and Chapters 49 and 51 outline its proof; see the ledger in Section 58.7 of Chapter 58. [Zh, §4.3] describes Alpöge's argument as applying Theorem 26.2.1 while keeping track of the losses from changing the weight and the level and from the boundary of the compactification. The two sources apply it to different lattices: [Zh, §4.3] to the initial lattices of the construction, the base lattices of Chapter 58, and [Adr, §15] to the target, with the base lattices treated by a computation with Weyl modules. The Lean formalization applies it to neither, but to the lattices of forms of the lifts, on Shimura varieties of growing dimension, whose bounds then control the target (Chapters 49 and 51); see the remark after the ledger in Section 58.7 of Chapter 58. We do not know a published form of Theorem 26.2.1 that is uniform in all the parameters of this section. We also do not know whether Problems 26.3.4 and 26.3.5 extend to $\mathcal{A}_\lambda(n)_{\mathbb{Z}}$ for all n with an error $O(\log|\lambda|)$ per form, uniformly in n.

Hints

Section 26.1.

26.1.1: Problem 6.3.4; collect the terms $\log k$.

26.1.2: Expand $\sum_{k\le D} k\log k$ and $\log D!$ by Euler–Maclaurin, to $O(\log D)$. Only Stirling's constant $\frac{1}{2}\log 2\pi$ survives, multiplied by $-(D+1)/2$.

26.1.3: For a sublattice $V \subseteq L$, project to L_2 along L_1. The quotient norm on $V/(V\cap L_1)$ is at least the norm of the projection. Use Problem 6.3.3.

26.1.4: Apply Problem 26.1.3 to the monomial lines. For $\mu_{\min}$, pass to the dual lattice.

26.1.5: Expand f in the orthonormal basis of normalized monomials and apply Cauchy–Schwarz at the point z. The sum of the squares of the basis functions at z is $D+1$ times $(1+|z|^2)^D$.

26.1.6: Integrate $|z^\nu|^2 e^{-|z|^2}$ over $\mathbb{C}^n$ in two ways: as a product of one-variable integrals, and in polar coordinates.

26.1.7: A basis of V over K is a basis of $V \otimes_K K'$. Use $\sum_{w|v}[K'_w : K_v] = [K' : K]$.

Section 26.2.

26.2.1: Here $d = 1$. For the second statement use Problem 6.3.2.

26.2.2: Diagonalize one norm with respect to the other. Two smooth volume forms on a compact manifold are bounded by constant multiples of each other.

Section 26.3.

26.3.1: The forms make up an orthogonal sum over Cl_n. Scaling a norm by $|\Gamma_L|^{-1/2}$ changes the degree as in Problem 6.3.2, with $n_v = 2$.

26.3.2: $\Gamma_{I_1} = \mu_4$, and $\|e^a\| = 1$.

26.3.3: Problem 26.1.6.

26.3.4: The diagonal subgroup acts on e^ν by $\mathrm{i}^{\nu_1 j_1 + \nu_2 j_2}$. Then use Problem 26.3.1.

26.3.5: The rank is $\lfloor a/8 \rfloor + 1$. Compare the sum of $\log \binom{a}{\nu_1}$ over $\nu_1 \equiv 0 \pmod 4$, $\nu_1 \leq a/2$, with one eighth of the full sum, which is computed in Problem 26.1.2; $k \mapsto \log \binom{a}{k}$ increases on $[0, a/2]$.

Section 26.4.

26.4.1: Write the ratio as a product of $n - 1$ factors.

26.4.2: The monomial basis and Problem 26.1.6.

26.4.3: Problem 6.3.2.

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