FOURIER ANALYSIS ON THE GALOIS GROUP: THE TABLE AND AVERAGING OVER TRANSLATES

Prerequisites. Frobenius elements and inertia groups in abelian extensions of $\mathbb{Q}$ [Ne, Ch. I, §9]; Dirichlet characters as characters of Galois groups, with their conductors [Wa, Ch. 3]; orthogonality of characters of finite abelian groups [SeA, Ch. VI, §1]. From Chapters 1 and 2, *From zeros to a one-sided ramp sum* and *Inert primes and Estimate B*: the functions G, G_χ and constants $c_\chi = \frac{L'}{L}(1,\chi)$ of equations (1) and (3) there, Section 1.4, and Sections 2.1–2.2 with Estimate B.

In Chapter 2 an odd prime power was put in a cell of a table with two rows and n columns, and averaging over translates turned some of the cells into four ramp sums (Problem 2.1.6). Here the cells are the archimedean places of an abelian CM field, a weighted prime sum is an exact combination of the functions G_χ over the characters of its Galois group, and averaging over translates is the projection onto the characters trivial on the translations. We follow [Dup, Ch. 5–6], an account of Alpöge's argument [Alp].

Throughout, $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod 8$ are primes with $q \mid d+1$; $n \geq 2$ is a power of 2, $N := 4n$, $H := (\mathbb{Z}/N\mathbb{Z})^\times/\{\pm 1\}$, and $[r] \in H$ is the class of an odd r. Let $\chi_q(r) := \left(\frac{r}{q}\right)$ and $\chi_d(r) := \left(\frac{r}{d}\right)$ be the Legendre symbols, $\eta := \chi_q\chi_d$ and $v(r) := \frac{1}{2}\big(1+\chi_q(r)\big)$. The example is $q = 11$, $d = 1319$, $N = 16$, with $H = \{[1],[3],[5],[7]\}$.

4.1. The host field and its places

Why are the class of r modulo N up to sign and $\chi_q(r)$ the right labels for a prime power? Let

$$E := \mathbb{Q}\big(\zeta_N + \zeta_N^{-1}, \sqrt{-q}, \sqrt{-d}\big) \subset \mathbb{Q}(\zeta_{Nqd}), \qquad K := \mathbb{Q}(\sqrt{-d}),$$

$\Gamma := H \times \{\pm 1\} \times \{\pm 1\}$, and for r prime to $2qd$ let $\mathrm{Frob}_r \in \mathrm{Gal}(E/\mathbb{Q})$ be the restriction of $\zeta \mapsto \zeta^r$.

Problem 4.1.1. There is an isomorphism $\mathrm{Gal}(E/\mathbb{Q}) \to \Gamma$ with $\mathrm{Frob}_r \mapsto \big([r], \chi_q(r), \chi_d(r)\big)$ for every r prime to $2qd$. In particular $[E:\mathbb{Q}] = 4n$.

We identify $\mathrm{Gal}(E/\mathbb{Q})$ with Γ and H with $H \times 1 \times 1$. Then $\Phi := \mathrm{Gal}(E/K) = H \times \{\pm 1\} \times 1$, and E is a CM field whose complex conjugation is $c := ([1], -1, -1)$ under every embedding. Fix an embedding $\sigma_0 \colon E \to \mathbb{C}$.

Problem 4.1.2. The conjugate of $\sigma_0 \circ g$ is $\sigma_0 \circ gc$. Hence each of the $2n$ complex places of E contains exactly one embedding $\sigma_0 \circ \rho$ with $\rho \in \Phi$.

So the CM type Φ indexes the places. We call $(s, e_1, 1) \in \Phi$ the *cell* (v, s), in row $v := \frac{1}{2}(1-e_1)$ and column s; the group H translates the columns. In [Dup, Def. 6.1.3] a prime power r counts exactly when $\mathrm{Frob}_r \notin \Phi$, and it counts at the place of $\sigma_0 \circ \mathrm{Frob}_r$, represented in Φ by $\mathrm{Frob}_r\, c$.

Date: November 25, 2026.

Problem 4.1.3. Let $p \nmid 2qd$ be prime. Then $\mathrm{Frob}_{p^h} \notin \Phi$ if and only if p is inert in K and h is odd, and then Frob_{p^h} c is the cell $\big(v(p^h), [p^h]\big)$.

Problem 4.1.4. For $q = 11$, $d = 1319$, $N = 16$, the odd primes $p \le 61$, $p \ne 11$, with $\mathrm{Frob}_p \notin \Phi$ lie in the cells below; the others are squares modulo 1319.

row\column	[1]	[3]	[5]	[7]
0	17	13, 29, 61	–	41
1	47	–	37, 59	23

At $p \mid 2qd$ let $I_p \subset \Gamma$ be the inertia group and F_p the coset of I_p inducing the Frobenius on the fixed field of I_p [Ne, Ch. I, §9]; for $p \nmid 2qd$ let $I_p := 1$ and $F_p := \{\mathrm{Frob}_p\}$. For $r = p^h$ let $F(r) := F_p^h$, the h-th power of F_p in Γ/I_p (a coset of I_p, not the set of h-th powers of its elements), and $\Lambda_d(r) := \Lambda(r)\big(1 - \chi_d(r)\big)$.

Problem 4.1.5. (i) $F(2^h), F(q^h) \subset \Phi$. (ii) $F(d^h)$ has exactly one element g outside Φ, and gc is the cell $\big(v(d^h), [d^h]\big)$; for the example, d and d^2 lie in $(0, [7])$ and $(1, [1])$.

So $F(r) \smallsetminus \Phi$ is empty or a single element g_r; in the second case let $\mathrm{cell}_t(r) := t g_r c$ for $t \in H$, the cell of r with shift t.

Problem 4.1.6. For every prime power $r = p^h$, $2|I_p|^{-1} \#\big(F(r) \smallsetminus \Phi\big) \log p = \Lambda_d(r)$.

So $\Lambda_d(r) = 0$ when r has no cell, and a term with the factor $\Lambda_d(r)$ that involves $\mathrm{cell}_t(r)$ is then read as 0.

4.2. Characters and the balance identity

The characters of Γ are $\theta_{\xi,j,l}(s, e_1, e_2) := \xi(s) e_1^j e_2^l$, with ξ a character of H and $j, l \in \{0, 1\}$. Let χ_θ be the primitive character inducing $r \mapsto \theta(\mathrm{Frob}_r)$, let $\theta_K := \theta_{1,0,1}$, and let $G_1 := G$ and $G_\theta := G_{\chi_\theta}$, $c_\theta := c_{\chi_\theta}$ for $\theta \ne 1$. For a character ξ of H we also write G_ξ, $G_{\xi\chi_q}$, $G_{\xi\chi_d}$, $G_{\xi\eta}$ for G_θ with $\theta = \theta_{\xi,0,0}$, $\theta_{\xi,1,0}$, $\theta_{\xi,0,1}$, $\theta_{\xi,1,1}$.

Problem 4.2.1. $\chi_{\theta_{\xi,j,l}}$ induces $\xi \chi_q^j \chi_d^l$, with conductor $f(\xi) q^j d^l$, $f(\xi) \mid N$. For $N = 16$ and $\xi([3]) = i$, the conductors of ξ^k, $k = 0, 1, 2, 3$, are $1, 16, 8, 16$, and ξ^2 induces $\chi_8(r) = (-1)^{(r^2-1)/8}$.

Problem 4.2.2. For every prime power r, $t \in H$ and $\rho \in \Phi$,

$$\Lambda_d(r)\, \mathbf{1}\big[\mathrm{cell}_t(r) = \rho\big] = \frac{\Lambda(r)}{2n} \sum_\theta \overline{\theta(t^{-1}\rho c)}\, \chi_\theta(r).$$

Problem 4.2.3. $\sum_{\rho \in \Phi} \overline{\theta(t^{-1}\rho c)}$ is $2n$ if $\theta = 1$, $-2n$ if $\theta = \theta_K$, and 0 otherwise; and $\chi_{\theta_K} = \chi_d$.

The arithmetic of the argument gives a smoothed Legendre count over the prime powers with a cell, each weighted by its cell [Dup, Rem. 6.1.5]. For $a > 0$, $t \in H$ and $w \colon \Phi \to (0, \infty)$ let

$$P_{t,w}(a) := \sum_r \Lambda_d(r) \Big(\frac{a}{r} - w\big(\mathrm{cell}_t(r)\big) \Big)_+, \qquad M_w(a) := a\big(\log a - 1 - \gamma + c_{\chi_d} - \langle \log w \rangle\big),$$

where $\langle \log w \rangle$ is the mean of $\log w$ over Φ. The term $a\langle \log w \rangle$ is what an even spread of the primes over the cells would give, since the cell ρ tests primes up to $a/w(\rho)$.

Problem 4.2.4. The finite parts $\sum_{r\le y}\Lambda(r)\chi_\theta(r)(y/r-1)$ of the $G_\theta(y)$ satisfy

$$\frac{1}{2n}\sum_\rho w(\rho)\sum_\theta \overline{\theta(t^{-1}\rho c)}\sum_{r\le a/w(\rho)}\Lambda(r)\chi_\theta(r)\Big(\frac{a}{w(\rho)r}-1\Big)=P_{t,w}(a).$$

Problem 4.2.5. The linear parts, $-y\log y+(1+\gamma)y$ for $\theta=1$ and $c_\theta y$ otherwise, contribute $-M_w(a)$ in the same way.

Adding Problems 4.2.4 and 4.2.5 gives the case of [Dup, Prop. 6.1.4] needed here.

Theorem 4.2.1 (balance identity)**.** *For every $a>0$, $t\in H$ and $w\colon\Phi\to(0,\infty)$,*

$$P_{t,w}(a)-M_w(a)=\sum_{\rho\in\Phi}w(\rho)R_{t,\rho}\big(a/w(\rho)\big),\qquad R_{t,\rho}:=\frac{1}{2n}\sum_\theta\overline{\theta(t^{-1}\rho c)}\,G_\theta. \quad (1)$$

Both sides are finite sums; no prime number theorem enters. So an upper bound $P\le M+\mathcal{E}$, of the kind divisibility gives, bounds a combination of the G_θ. For w constant the right side is $b(G-G_{\chi_d})(a/b)$, as in Problem 2.1.3. In the example, with $t=[1]$, $a=10^4$, and w equal to 10 at $(1,[1])$ and 0.05 elsewhere, $P_{t,w}(a)=84701.0317$ and $M_w(a)=84463.3942$.

4.3. Averaging over translates

Why not use the cells one at a time?

Problem 4.3.1. $\sum_{\rho\in\Phi}\theta(t^{-1}\rho c)R_{t,\rho}=G_\theta-G_{\theta\theta_K}$. Hence $|G_\xi-G_{\xi\chi_d}|\le 2n\max_\rho|R_{t,\rho}|$ for every character $\xi\ne1$ of H.

Bounds for single cells at the scale x would thus bound $G_\xi(x)-G_{\xi\chi_d}(x)$ for the fixed characters $\xi\ne1$ of conductor dividing N. These involve the fixed functions $L(s,\xi)$, and a two-sided bound $|G_\xi(x)|\le x^{1/2+\varepsilon}$ at all large x, for every $\varepsilon>0$, is as strong as the Riemann hypothesis for $L(s,\xi)$ (Problem 2.2.1). The characters $\xi\ne1$ must be removed. For $u\in H$ let $w_u(\rho):=w(u^{-1}\rho)$.

Problem 4.3.2. Independently of t,

$$\frac{1}{n}\sum_{u\in H}\big(P_{t,w_u}(a)-M_{w_u}(a)\big)=\frac{1}{2n}\sum_\rho w(\rho)\sum_{j,l}\overline{\theta_{1,j,l}(\rho c)}\,G_{\theta_{1,j,l}}\big(a/w(\rho)\big).$$

Problem 4.3.3. For ρ in row 1 the inner sum is $R_{q,d}:=G+G_{\chi_q}-G_{\chi_d}-G_\eta$, the right side of Problem 2.1.7; in row 0 it is $R'_{q,d}:=G-G_{\chi_q}-G_{\chi_d}+G_\eta$.

Theorem 4.3.1. *For every $a>0$, $t\in H$ and $w\colon\Phi\to(0,\infty)$,*

$$\frac{1}{n}\sum_{u\in H}\big(P_{t,w_u}(a)-M_{w_u}(a)\big)$$
$$=\frac{1}{2n}\sum_{\rho\ in\ row\ 1}w(\rho)R_{q,d}\Big(\frac{a}{w(\rho)}\Big)+\frac{1}{2n}\sum_{\rho\ in\ row\ 0}w(\rho)R'_{q,d}\Big(\frac{a}{w(\rho)}\Big).$$

Averaging is the projection onto the characters of $\mathrm{Gal}\big(\mathbb{Q}(\sqrt{-q},\sqrt{-d})/\mathbb{Q}\big)$; the column characters vanish exactly, for every weight. The survivors other than 1 have conductors q, d, qd, within reach of Estimate B, and G has coefficient $+1$. In the example the four excesses $P_{t,w_u}(a)-M_{w_u}(a)$ are 237.64, -300.26, -176.73, 229.13, with mean -2.5545; Theorem 4.3.1 writes it as $\frac{5}{4}R_{11,1319}(1000)=-2.7034$ plus 0.1488 from the cells at the scale 200000.

4.4. Thick and thin weights, and two scales

In [Dup, Def. 6.2.1] the weights are not translates of one function, and they are of two kinds; from here on w_u, $u \in H$, is any family of weights $\Phi \to (0,\infty)$. Fix a nonempty $T_2 \subset H$, $\delta := |T_2|/n$, and let Θ_u be the cells in row 1 with column in uT_2. Let $x \geq 2$, $|a/b - x| \leq 1$, $\Delta \geq 0$, $X > x+1$, $\vartheta \geq 0$, and suppose that for every u

$$b \leq w_u(\rho) \leq b + \Delta \quad (\rho \in \Theta_u,\ \text{thick}), \qquad a/w_u(\rho) \geq X \quad (\rho \notin \Theta_u,\ \text{thin}), \tag{2}$$

and that $|G_\theta(y)| \leq \vartheta y$ for all θ and $y \geq X$ (the thin-scale bound). Thick cells test primes near x; thin ones test primes beyond X. There, for a small fixed $\varepsilon > 0$, $b = \exp(x^\varepsilon)$ and $X = \frac{1}{3}\exp(x^{\varepsilon/3})$.

Problem 4.4.1. The thin cells contribute at most $4na\vartheta$ to the right side of (1) for w_u.

Problem 4.4.2. Replacing w_u by b on Θ_u changes $P_{t,w_u}(a) - M_{w_u}(a)$ by at most

$$\Delta\Big(2\sum_{r \leq x+1} \Lambda(r) + x + 1\Big).$$

Problem 4.4.3. With weight b on Θ_u, the average over $u \in H$ of the thick part of (1) is $\frac{\delta b}{2} R_{q,d}(a/b)$.

Problem 4.4.4. For $|y - x| \leq 1$, $|R_{q,d}(y) - R_{q,d}(x)| \leq \mathcal{K}(x)$, where

$$\mathcal{K}(x) := 4\sum_{r \leq x+1} \frac{\Lambda(r)}{r} + \log(x+1) + 1 + |c_{\chi_q}| + |c_{\chi_d}| + |c_\eta|.$$

Theorem 4.4.1. *Under* (2) *and the thin-scale bound, for every* $t \in H$,

$$R_{q,d}(x) \leq \frac{2}{\delta b} \cdot \frac{1}{n} \sum_{u \in H} \big(P_{t,w_u}(a) - M_{w_u}(a)\big) + \mathcal{K}(x) + \frac{8n(x+1)\vartheta}{\delta} + \frac{2\Delta}{\delta b}\Big(2\sum_{r \leq x+1} \Lambda(r) + x + 1\Big).$$

At the scale x only the quadratic characters appear; at the scale X all $4n$ do, but only a bound of prime number theorem type is needed there. The cost conditions of [Dup, Def. 6.2.3(2)] make the last two terms $O(1/(\delta x))$. The thin-scale bound follows from the classical zero-free region [Dav, Ch. 14], since $\log X$ is large compared with the logarithms of the conductors, and from the exclusion of exceptional zeros: by Theorem 9.3.1 of Chapter 9, or by the choice of q, d [Dup, Def. 5.4.2], [Adr, §6]. We do not carry out the computation here.

4.5. Averaging over a subgroup, and top places

For $L(s,\psi)$, averaging over all of H would remove ψ. In the simplest case ψ is a character of H (in general E is enlarged, [Dup, Def. 5.1.3]). Let o be its order and $A_0 := \ker\psi$; in the example $\psi := \xi^2$, $\chi_\psi = \chi_8$, $o = 2$, $A_0 = \{[1],[7]\}$. Now u runs over A_0, $T_2 \subset A_0$, thick weights equal b, and the shift t runs over a nonempty set $T \subset H$ of top places [Dup, §6.3].

Problem 4.5.1. The characters of Γ trivial on A_0 are the $4o$ characters $\theta_{\xi,j,l}$ with $\xi \in \langle\psi\rangle$; in the example the conductors of their χ_θ are 1, 11, 1319, 14509, 8, 88, 10552, 116072.

Problem 4.5.2. The average over $u \in A_0$ of the thick part of (1) is

$$\frac{\delta b}{2} \sum_{\xi \in \langle \psi \rangle} \xi(t) R_\xi(a/b), \qquad R_\xi := G_\xi + G_{\xi\chi_q} - G_{\xi\chi_d} - G_{\xi\eta}$$

(primitive characters), with $R_1 = R_{q,d}$.

Problem 4.5.3. If a proportion λ of T lies in A_0 and λ' in each other coset, then the mean of $\xi(t)$ over $t \in T$ is 1 for $\xi = 1$ and $\kappa := \lambda - \lambda'$ for $\xi \in \langle \psi \rangle \smallsetminus 1$.

Combining Problems 4.5.2 and 4.5.3: the average over $u \in A_0$ and $t \in T$ of the thick part is $\frac{\delta b}{2}\big(R_{q,d} + \kappa \sum_{\xi \neq 1} R_\xi\big)(a/b)$. Its untwisted part $G + \kappa \sum_{\xi \neq 1} G_\xi$ is the G^ψ of Section 1.4 for the shares $\lambda = 4/(3o+1)$, $\lambda' = 3/(3o+1)$ of [Dup, Def. 5.3.3], where T has these shares up to $x^{-1+\varepsilon}$. The characters $\xi \notin \langle \psi \rangle$, of small conductor, are gone; the rest are twists of large conductor.

Problem 4.5.4. For odd r, the coefficient of $\Lambda(r)(y/r-1)_+$ in $G + \kappa \sum_{\xi \neq 1} G_\xi$ is o times the proportion of T in the coset of $[r]$; for $o = 2$, $\lambda = 4/7$, it is $8/7$ or $6/7$, as in Problem 1.4.1.

The upper bound for the mean of $P - M$ that this averaging consumes is the main inequality of the argument, not treated here [Dup, Thm. 6.3.4].

Hints

§4.1.

4.1.1: The three generating fields are ramified only at 2, q, d respectively. Frob_p multiplies $\sqrt{-q}$ by $\left(\frac{-q}{p}\right) = \left(\frac{p}{q}\right)$.
4.1.2: Complex conjugation after σ_0 is $\sigma_0 \circ c$, and $c \notin \Phi$.
4.1.3: Problem 2.1.1.
4.1.4: For instance 13 is not a square modulo 11, and $[13] = [3]$.
4.1.5: $I_2 = H \times 1 \times 1$, $I_q = 1 \times \{\pm 1\} \times 1$, $I_d = 1 \times 1 \times \{\pm 1\}$; on the other coordinates F_p acts by $\chi_q(p)$, $\chi_d(p)$ or $[p]$. Use Problem 2.1.5 and $\chi_q(d) = -1$; for $p = 2$, $-q \equiv 5$ and $-d \equiv 1 \pmod 8$.
4.1.6: Problems 4.1.3 and 4.1.5.

§4.2.

4.2.1: $\xi([9]) = -1$ with $9 \equiv 1 \pmod 8$, and $\xi^2([5]) = -1$.
4.2.2: Write $\chi_\theta(p^h) = |I_p|^{-1} \sum_{g \in F(p^h)} \theta(g)$ (p divides the conductor of χ_θ exactly when θ is nontrivial on I_p, [Wa, Ch. 3]), apply orthogonality, and use Problem 4.1.6.
4.2.3: Φ has index 2 and $\theta_K(c) = -1$.
4.2.4: Exchange the sums and use Problem 4.2.2.
4.2.5: Problem 4.2.3.

§4.3.

4.3.1: Orthogonality on Φ and Problem 4.2.3.
4.3.2: Substitute $\rho = u\rho'$ in (1), and use orthogonality on H.
4.3.3: Evaluate each $\theta_{1,j,l}$ at ρc.

§4.4.

4.4.1: Apply the thin-scale bound to each G_θ in $R_{t,\rho}$.

4.4.2: $w \mapsto (z-w)_+$ is 1-Lipschitz, and only $r < a/b$ are affected; $\log w - \log b \le \Delta/b$.

4.4.3: Problem 4.3.2 restricted to $\rho \in \Theta_u$, then Problem 4.3.3.

4.4.4: Differentiate Problem 2.1.7 in y; $(1+\chi_q)(1-\chi_d) \le 4$.

§4.5.

4.5.1: The characters of H/A_0 are the powers of ψ; Problem 4.2.1.

4.5.2: As for Problem 4.4.3, with $\sum_{u \in A_0} \xi(u) = |A_0| \mathbf{1}[\xi \in \langle\psi\rangle]$ and $\theta = 1$ on T_2.

4.5.3: The values of $\xi \ne 1$ on the cosets other than A_0 add up to -1.

4.5.4: Orthogonality of the characters in $\langle\psi\rangle$, and $\lambda + (o-1)\lambda' = 1$.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

[Alp] L. Alpöge, untitled note announcing a proof of the Riemann hypothesis, 20 November 2026, `https://alpo.ge/remark.pdf`.

[Dav] H. Davenport, *Multiplicative Number Theory*, 3rd ed., revised by H. L. Montgomery, Grad. Texts in Math. **74**, Springer, 2000.

[Dup] *Notes on Alpöge's proof of the Riemann hypothesis*, printed by T. Dupuy, AI generated, 21 November 2026, `https://tdupu.github.io/riemann-public/riemann-book.pdf`.

[Ne] J. Neukirch, *Algebraic Number Theory*, Grundlehren Math. Wiss. **322**, Springer, 1999.

[SeA] J.-P. Serre, *A Course in Arithmetic*, Grad. Texts in Math. **7**, Springer, 1973.

[Wa] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed., Grad. Texts in Math. **83**, Springer, 1997.