DEFINITE THETA LIFTS AND THEIR NORMS

Prerequisites. Chapter 21, *Masses and the count of forms*: the class sets Cl_n, the groups Γ_L, the representations $V_\lambda(L) = \mathbb{S}_\lambda(L_\mathbb{C})$, the notation e^ν in Sym^a, and the spaces of forms $\mathcal{A}_\lambda(n)$ with their norm (Sections 21.1 and 21.3). From Chapter 12: Theorem 12.3.1 and Problem 12.3.7. From Chapter 16: Section 16.1 and Problem 16.1.3. Stirling's formula.

Since a definite unitary group has a finite class set, a theta lift between two such groups is a finite matrix. What are its entries, and how large is it? We compute both in the smallest cases.

23.1. The theta kernel

How does one map forms on one class set to forms on another? The idea is to sum over all homomorphisms between the two lattices, with a Gaussian weight.

Let L and L' be Hermitian lattices of ranks n and n'. For $x \in \mathrm{Hom}_\mathcal{O}(L, L')$ let $x^* \in \mathrm{Hom}(L'_\mathbb{C}, L_\mathbb{C})$ be its adjoint, and let $h(x, y) := \mathrm{tr}(y^* x)$. Let

$$\Theta_{L,L'}(Z) := \sum_{x \in \mathrm{Hom}_\mathcal{O}(L,L')} e^{-\pi h(x,x) + \mathrm{tr}(Zx)}, \qquad Z \in \mathrm{Hom}(L'_\mathbb{C}, L_\mathbb{C}).$$

For a polynomial P in the entries of x, let $P(\partial)$ be the differential operator obtained by replacing $x_{\beta\alpha}$ with $\partial/\partial Z_{\alpha\beta}$. For a weight λ define

$$\vartheta_\lambda(L, L') := \sum_{x \in \mathrm{Hom}_\mathcal{O}(L,L')} \mathbb{S}_\lambda(x)\, e^{-\pi h(x,x)} \colon V_\lambda(L) \to V_\lambda(L'),$$

and, for the class sets Cl_n and $\mathrm{Cl}_{n'}$, the *lift* $\Theta_\lambda \colon \mathcal{A}_\lambda(n) \to \mathcal{A}_\lambda(n')$,

$$(\Theta_\lambda \varphi)(L') := \sum_{L \in \mathrm{Cl}_n} \frac{1}{|\Gamma_L|}\, \vartheta_\lambda(L, L')\, \varphi(L). \tag{1}$$

In bases, Θ_λ is a finite matrix whose blocks are the $\vartheta_\lambda(L, L')$, weighted by $1/|\Gamma_L|$. For a matrix coefficient P of $\mathbb{S}_\lambda$, the series $\sum_x P(x)\, q^{h(x,x)}$ is a classical theta series with polynomial coefficients, and the corresponding entry of ϑ_λ is its value at $q = e^{-\pi}$.

Thus $\Theta_{L,L'}$ is a theta function of the Hermitian lattice $\mathrm{Hom}_\mathcal{O}(L, L')$, which is unimodular of rank nn' by Problem 23.1.1 below. Chapter 13, *The Fock model and theta intertwiners*, treats such functions in one variable, for $\mathbb{Z}[\mathrm{i}]$ and its neighbours, and shows how a theta function of one lattice is carried to one of a commensurable lattice by an intertwiner; composing two intertwiners costs a root of unity given by a Gauss sum. The problems of this chapter use only the series itself.

Problem 23.1.1. If L and L' are unimodular, then with $h(x, y) = \mathrm{tr}(y^* x)$, $\mathrm{Hom}_\mathcal{O}(L, L')$ is a unimodular Hermitian lattice of rank nn', and $\mathrm{Hom}_\mathcal{O}(I_n, I_{n'}) \cong I_{nn'}$.

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Problem 23.1.2. $\Theta_{L,L'}$ is entire, and $P(\partial)\Theta_{L,L'}(0) = \sum_x P(x)\, e^{-\pi h(x,x)}$. If $\Theta_{L,L'}(Z) = \sum_\nu c_\nu Z^\nu$ and $P(x) = \sum_\nu P_\nu x^\nu$, with the matching indexing of the entries of Z and of x, this equals $\sum_\nu P_\nu\, \nu!\, c_\nu$.

Problem 23.1.3. In bases of $V_\lambda(L)$ and $V_\lambda(L')$ built from $\mathcal{O}$-bases of L and L', each entry of $\vartheta_\lambda(L,L')$ is $P(\partial)\Theta_{L,L'}(0)$ for a polynomial P, homogeneous of degree $|\lambda|$, with rational coefficients. Moreover $\vartheta_\lambda(L,L') = 0$ unless $4 \mid |\lambda|$.

Problem 23.1.4. For $\gamma \in \Gamma_L$ and $\gamma' \in \Gamma_{L'}$, $\mathbb{S}_\lambda(\gamma')\,\vartheta_\lambda(L,L') = \vartheta_\lambda(L,L') = \vartheta_\lambda(L,L')\,\mathbb{S}_\lambda(\gamma)$. Hence (1) is a form on $\mathrm{Cl}_{n'}$, and it does not depend on the lattices chosen in the classes.

Problem 23.1.5. $\vartheta_\lambda(L,L')^* = \vartheta_\lambda(L',L)$. Hence the lift $\Theta'_\lambda\colon \mathcal{A}_\lambda(n') \to \mathcal{A}_\lambda(n)$ in the other direction is the adjoint of Θ_λ: $\langle \Theta_\lambda\varphi, \psi\rangle = \langle \varphi, \Theta'_\lambda\psi\rangle$.

23.2. The smallest examples

What are these matrices for $\mathrm{U}(1)\times\mathrm{U}(1)$ and $\mathrm{U}(1)\times\mathrm{U}(2)$? By Problem 21.1.4 each class set has one element, and the lift is a single block. Everything is expressed through the Gaussian moments

$$c_a := \sum_{\alpha\in\mathcal{O}} \alpha^a e^{-\pi|\alpha|^2} \qquad (a \ge 0).$$

Let ϖ, ϑ_0 and $F = \sigma^2\wp$ be as in §16.1.

Problem 23.2.1. c_a is real, $c_a = 0$ unless $4 \mid a$, and $c_a = \vartheta_0\,(\varpi/2\pi)^a F^{(a)}(0)$.

Problem 23.2.2. For $a = 0, 4, 8, 12, 16$ the rational numbers $F^{(a)}(0)$ are 1, 4, 272, -7104, 3203328.

Problem 23.2.3. For $n = n' = 1$ and $4 \mid a$, $\Theta_{(a)}$ is multiplication by $c_a/4$ on the line $\mathcal{A}_{(a)}(1)$.

For $n = 1$ and $n' = 2$, $\mathrm{Hom}_{\mathcal{O}}(I_1, I_2) = \mathcal{O}^2$ is the space of columns $x = (x_1, x_2)$, and $\mathrm{Sym}^a(x)$ sends the basis vector $1 \in \mathrm{Sym}^a(\mathbb{C})$ to $(x_1e_1 + x_2e_2)^a$.

Problem 23.2.4. For $\varphi \in \mathcal{A}_{(a)}(1)$,

$$(\Theta_{(a)}\varphi)(I_2) = \frac{\varphi(I_1)}{4} \sum_j \binom{a}{j} c_j c_{a-j}\, e_1^j e_2^{a-j}.$$

For $a = 8$ this is $4\varphi(I_1)\,\vartheta_0^2(\varpi/2\pi)^8\big(17e_1^8 + 70e_1^4e_2^4 + 17e_2^8\big)$.

Problem 23.2.5. The lift $\Theta'_{(a)}\colon \mathcal{A}_{(a)}(2) \to \mathcal{A}_{(a)}(1)$ is

$$(\Theta'_{(a)}\psi)(I_1) = \frac{1}{32} \sum_j c_j c_{a-j}\, \psi_j, \qquad \text{where } \psi(I_2) = \sum_j \psi_j\, e_1^j e_2^{a-j}.$$

How large are these entries? The terms of c_a are largest near $|\alpha|^2 = a/2\pi$.

Problem 23.2.6. $\log c_a^2 \le a\log a - a(1 + \log 2\pi) + O(\log a)$, where $\log 0 := -\infty$.

Numerically the bound is attained up to $O(\log a)$: for $a = 40, 80, \ldots, 240$ the left side exceeds the right side without the $O(\log a)$ by between 1.7 and 4.8. The scale 2π belongs to the Gaussian; the rational part $F^{(a)}(0)$ of c_a sees instead the period.

By Problem 12.3.7, $\varpi^2/2\pi = \Omega/4$ with $\Omega = 2\varpi^2/\pi$, and by Theorem 12.3.1, $\log\Omega = \gamma + \log\pi - L'(1,\chi)/L(1,\chi)$. Combining these with Problems 23.2.1 and 23.2.6 gives

$$\log F^{(a)}(0)^2 \le a\log a - a\Big(1+\gamma+\log\frac{\pi}{4} - \frac{L'(1,\chi)}{L(1,\chi)}\Big) + O(\log a). \tag{2}$$

This is how Euler's constant and $L'(1,\chi)/L(1,\chi)$ enter the linear term of a size bound.

23.3. The Gaussian norm and the norm of the lift

The size of an entry is one thing; the size of the lift is measured by its operator norm. What is $\|\Theta_{(a)}\|$ in the smallest case?

Let $\mathcal{F}_m$ be the space of entire functions on $\mathbb{C}^m$ with

$$\|f\|^2 := \pi^{-m}\int_{\mathbb{C}^m} |f(z)|^2 e^{-|z|^2}\,dA(z) < \infty.$$

For $m=1$ this is the norm of Chapter 6, in which the monomials are orthogonal and $\|z^a\|^2 = a!$ (Problem 6.4.4).

Problem 23.3.1. The monomials z^ν are orthogonal in $\mathcal{F}_m$, and $\|z^\nu\|^2 = \nu!$.

By Problems 21.3.2 and 23.3.1, sending e^ν to z^ν identifies $\mathrm{Sym}^a(\mathbb{C}^m)$ with the homogeneous polynomials of degree a in $\mathcal{F}_m$, the squared norm being divided by $a!$. In this normalization the size of a derivative is controlled as follows.

Problem 23.3.2. If $\Theta(Z) = \sum_\nu c_\nu Z^\nu$ and $P = \sum_\nu P_\nu x^\nu$, then

$$|P(\partial)\Theta(0)|^2 \le \|P\|_{\mathcal{F}}^2 \sum_{\nu\in\operatorname{supp} P} \nu!\,|c_\nu|^2.$$

The norm of the lift itself is computed by composing it with its adjoint.

Problem 23.3.3. For $\Theta_{(a)}\colon \mathcal{A}_{(a)}(1) \to \mathcal{A}_{(a)}(2)$ and $\varphi \ne 0$,

$$\frac{\|\Theta_{(a)}\varphi\|^2}{\|\varphi\|^2} = \frac{1}{128}\sum_{x,y\in\mathcal{O}^2} h(x,y)^a e^{-\pi(|x|^2+|y|^2)} = \frac{1}{128}\sum_j \binom{a}{j} c_j^2 c_{a-j}^2.$$

For $a=4$ and $a=8$ it equals $\frac14\vartheta_0^4(\varpi/2\pi)^8$ and $1296\,\vartheta_0^4(\varpi/2\pi)^{16}$.

The middle expression is a theta sum over $\mathrm{Hom}_{\mathcal{O}}(I_1\oplus I_1, I_2)$, the lattice of pairs (x,y). This is the structure of the Rallis inner product formula [Ra]. Write $\Theta'_\lambda\Theta_\lambda\varphi(L_1) = \sum_L |\Gamma_L|^{-1} K(L,L_1)\,\varphi(L)$, in the shape of (1). By Problem 23.1.5, the diagonal entry $K(L,L)$ of this kernel is

$$\sum_{L'\in \mathrm{Cl}_{n'}} \frac{1}{|\Gamma_{L'}|} \sum_{x,y\in\mathrm{Hom}(L,L')} \mathbb{S}_\lambda(y^*x)\, e^{-\pi(h(x,x)+h(y,y))},$$

an average over the genus of L' of theta series of the doubled lattice $\mathrm{Hom}(L\oplus L, L')$. By the Siegel–Weil formula [We65], in its range of convergence such averages are Eisenstein series, and on a Hecke eigenform Rallis's formula then gives the eigenvalue of $\Theta'_\lambda\Theta_\lambda$ as an L-value times local factors. In the smallest case the source is a line, and Problem 23.3.3 computes the scalar directly.

Problems 23.2.3, 23.2.6 and 23.3.3, together with $\binom{a}{j}\,j!\,(a-j)! = a!$, prove the following.

Theorem 23.3.1. *For the lifts* $\Theta_{(a)}\colon \mathcal{A}_{(a)}(1) \to \mathcal{A}_{(a)}(n')$ *with* $n' = 1, 2$*, one has* $\|\Theta_{(a)}\|^2 = c_a^2/16$ *and* $\|\Theta_{(a)}\|^2 = \frac{1}{128}\sum_j \binom{a}{j} c_j^2 c_{a-j}^2$ *respectively, and in both cases*

$$\log\|\Theta_{(a)}\|^2 \le a\log a - a(1+\log 2\pi) + O(\log a).$$

So the squared norm of the lift has the size of one factorial, $\|z^a\|^2 = a!$, times $(2\pi)^{-a}$. Numerically, for $n' = 2$ and $a = 40, 80, \dots, 240$, the left side exceeds the right side without the $O(\log a)$ by an amount growing from 1.5 to 5.1, roughly like $2\log a$.

23.4. The seesaw

A seesaw computes one period in two ways: lift to a large group and restrict to a subgroup, or restrict first and lift between smaller groups [Ku]. For definite groups the seesaw comes from an orthogonal decomposition $L' = L_1 \perp L_2$, and $\mathrm{Hom}(L, L_1 \perp L_2) = \mathrm{Hom}(L, L_1) \perp \mathrm{Hom}(L, L_2)$.

Problem 23.4.1. $\Theta_{L, L_1 \perp L_2}(Z_1, Z_2) = \Theta_{L,L_1}(Z_1)\,\Theta_{L,L_2}(Z_2)$.

Here is the smallest case. Write $I_2 = Y \perp \mathcal{O}z$ with $Y = \mathcal{O}e_1$ and $z = e_2$, and let $4 \mid j$ and $4 \mid a - j$. For $\psi \in \mathcal{A}_{(a)}(2)$ and $\xi \in \mathcal{A}_{(j)}(1)$, viewed as a form on the class set of Y, the *period* of ψ along Y is

$$P(\psi, \xi) := \frac{1}{|\Gamma_Y|}\big\langle \psi(Y \perp \mathcal{O}z),\ \xi(Y)\, e_2^{a-j}\big\rangle.$$

On the other side let $\theta_z := \vartheta_{(a-j)}(\mathcal{O}z, I_1)\, e_2^{a-j} \in \mathrm{Sym}^{a-j}(\mathbb{C})$, the vector obtained from $\mathcal{O}z$ alone.

Problem 23.4.2. For $\varphi \in \mathcal{A}_{(a)}(1)$ and $\xi \in \mathcal{A}_{(j)}(1)$,

$$P(\Theta_{(a)}\varphi, \xi) = \big\langle \varphi,\ (\Theta_{(j)}\xi)\,\theta_z \big\rangle,$$

where the product on the right is that of $\mathrm{Sym}^j(\mathbb{C}) \otimes \mathrm{Sym}^{a-j}(\mathbb{C}) \to \mathrm{Sym}^a(\mathbb{C})$. Both sides equal $\varphi(I_1)\overline{\xi(I_1)}\, c_j c_{a-j}/16$.

The left side lifts φ to $\mathrm{U}(2)$ and integrates along $\mathrm{U}(1)$. The right side lifts ξ to $\mathrm{U}(1)$ and pairs with φ there; it depends on ξ only through $\Theta_{(j)}\xi$. In a chain of lifts the same identity, with exact constants, makes the value of the chain at a form depend only on the image of that form.

23.5. Chains

The argument composes many lifts. What survives composition?

Problem 23.5.1. The composite $\mathcal{A}_\lambda(n) \to \mathcal{A}_\lambda(n') \to \mathcal{A}_\lambda(n'')$ of two lifts is $\varphi \mapsto \big(L'' \mapsto \sum_{L \in \mathrm{Cl}_n} |\Gamma_L|^{-1} K(L, L'')\,\varphi(L)\big)$, with the kernel

$$K(L, L'') = \sum_{L' \in \mathrm{Cl}_{n'}} \frac{1}{|\Gamma_{L'}|}\, \vartheta_\lambda(L', L'')\, \vartheta_\lambda(L, L').$$

Problem 23.5.2. In the bases e^μ of $\mathrm{Sym}^a(\mathbb{C}^n)$ and $\mathrm{Sym}^a(\mathbb{C}^{n'})$, every entry of $\vartheta_{(a)}(I_n, I_{n'})$ lies in $\vartheta_0^{nn'}(\varpi/2\pi)^a\,\mathbb{Q}$.

So for the standard lattices the transcendental factor of a lift is uniform, a power of the period fixed by the degree; and through a chain of m lifts of degree a it is $(\varpi/2\pi)^{ma}$ times a power of ϑ_0. The weights $1/|\Gamma_L|$ are the only other denominators.

Problem 23.5.3. Let $S \subset R$ be commutative rings and $D_1, \dots, D_m \in R$. Suppose that in a composite of m maps $(T_j f)(i') = \sum_i h_j(i)\, K_j(i,i')\, f(i)$, with finite sums and $h_j(i) \in R$, all $K_j(i,i')$ and all starting values lie in S, and that $D_j h_j(i) \in S$ for all i. Then $D_1 \cdots D_m$ times every final value lies in S.

Beyond the smallest cases a chain needs more: in large rank Cl_n is large (Section 21.2), and the norm of a lift on a Hecke eigenform comes from the Rallis inner product formula [Ra] instead of Problem 23.3.3; the rational parts of the entries, here $F^{(a)}(0)$, must be divisible by high powers of inert primes uniformly along the chain (for the one-variable entry $F^{(a)}(0)$ Problem 16.5.3 is an obstruction, and Chapter 17 obtains the divisibility for the two-variable moments of Section 16.1); and the seesaw, masses and periods must be compared with exact constants, since an error of a constant factor per derivative changes the linear term.

23.6. Questions

We do not know whether $c_a \neq 0$ for every a divisible by 4; equivalently, whether $F^{(a)}(0) \neq 0$. We also do not know whether the bound of Problem 23.2.6 is attained up to $O(\log a)$ for every such a, as the computations of §23.2 suggest.

Hints

§23.1.

23.1.1: In bases, $h(x,y) = \sum_{\alpha,\beta} x_{\beta\alpha} \bar{y}_{\beta\alpha}$ when L and L' are standard. In general, if G and G' are the Gram matrices of L and L', so that $h(u,v) = \bar{v}^t G u$, then $y^* = G^{-1} \bar{y}^t G'$; compute the Gram matrix of $\mathrm{Hom}(L, L')$ and its determinant.

23.1.2: Differentiate under the sum; $\partial/\partial Z_{\alpha\beta}$ brings down $x_{\beta\alpha}$.

23.1.3: The entries of $x^{\otimes k}$ are monomials in the entries of x. For the vanishing, replace x by $\mathrm{i}x$.

23.1.4: Substitute $x \mapsto \gamma' x$ and $x \mapsto x\gamma$; both preserve $\mathrm{Hom}_{\mathcal{O}}(L, L')$ and $h(x,x)$.

23.1.5: $\mathbb{S}_\lambda(x)^* = \mathbb{S}_\lambda(x^*)$. What does $x \mapsto x^*$ do to $\mathrm{Hom}(L, L')$ when L and L' are unimodular?

§23.2.

23.2.1: For reality, substitute $\alpha \mapsto \bar{\alpha}$; for the vanishing, $\alpha \mapsto \mathrm{i}\alpha$. Differentiate Problem 16.1.3 a times at $z = 0$.

23.2.2: Expand σ and $\wp$ with $g_2 = 4$, $g_3 = 0$ further than in Problem 16.1.2.

23.2.3: $\Gamma_{I_1} = \mu_4$, and $\mathrm{Sym}^a(x)$ is multiplication by x^a.

23.2.4: Expand by the binomial theorem; the sum over $\mathcal{O}^2$ factors.

23.2.5: $\mathrm{Hom}(I_2, I_1)$ consists of rows y, and $\mathrm{Sym}^a(y)\, e_1^j e_2^{a-j} = y_1^j y_2^{a-j}$. Alternatively use Problems 23.1.5 and 21.3.2.

23.2.6: $|c_a| \leq \sum_\alpha f(\alpha)$ with $f(z) := |z|^a e^{-\pi|z|^2}$. Bound the $O(a)$ terms with $|\alpha|^2 \leq a$ by $\max f$, attained at $|z|^2 = a/2\pi$, and the others by a geometric series.

§23.3.

23.3.1: The norm is a product over the variables; apply Problem 6.4.4 in each.

23.3.2: Problem 23.1.2, and the Cauchy–Schwarz inequality.

23.3.3: $\|\Theta\varphi\|^2 = \langle \Theta'\Theta\varphi, \varphi\rangle$; use Problems 23.2.4 and 23.2.5. For the middle expression use $\langle u^a, v^a\rangle = \langle u, v\rangle^a$.

§23.4.

23.4.1: For $x = (x_1, x_2)$, $h(x,x) = h(x_1,x_1) + h(x_2,x_2)$ and $\mathrm{tr}(Zx) = \mathrm{tr}(Z_1x_1) + \mathrm{tr}(Z_2x_2)$.

23.4.2: Problems 23.2.3, 23.2.4 and 21.3.2.

§23.5.

23.5.1: Substitute (1) into itself.

23.5.2: Expand $\mathrm{Sym}^a(x)\, e^{\mu}$; the coefficients are rational polynomials in the entries of x, and the sum over $x \in \mathcal{O}^{n'n}$ of a monomial factors into moments c_k. Then use Problem 23.2.1; every $F^{(k)}(0)$ is rational, since the Taylor coefficients of σ and $\wp$ lie in $\mathbb{Q}[g_2, g_3]$.

23.5.3: Induction on m: move D_1 inside the first sum.

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