# MASSES AND THE COUNT OF FORMS

Prerequisites. Hermitian forms over the Gaussian integers, at the level of definitions. The classification of positive definite unimodular $\mathbb{Z}$-lattices of rank at most 8 [CoS, Ch. 16]. Schur functors and the Weyl dimension formula for $\mathrm{GL}_n(\mathbb{C})$ [FH, Lecture 6, Theorem 6.3]. The values of Dirichlet $L$-functions at negative integers [Wa, Theorem 4.2], and the functional equations of $\zeta(s)$ and $L(s,\chi)$ [Dav, Ch. 8, 9]. Stirling's formula. No earlier chapter is used.

In the argument, the lattices $L_j$ of the chain of Chapter 8 (Section 8.4) lie in spaces of forms on unitary groups, and the maps $\varphi_j$ are theta lifts between them. We take definite unitary groups over $\mathbb{Z}[\mathrm{i}]$ as a model. A definite unitary group has finitely many lattice classes in a genus, and its automorphic forms are vectors attached to these classes. How many forms are there? The answer is the product of a mass and a Weyl dimension, and both factors carry $\prod_{j<n} j!$, which cancels. The theta lifts between these spaces are the subject of Chapter 23, *Definite theta lifts and their norms*; Chapter 22 first evaluates the genus averages they need.

## 21.1. Hermitian lattices over $\mathbb{Z}[\mathrm{i}]$

Let $\mathcal{O} := \mathbb{Z}[\mathrm{i}]$ and $K := \mathbb{Q}(\mathrm{i})$. A *Hermitian lattice* of rank $n$ is a free $\mathcal{O}$-module $L$ of rank $n$ with a positive definite Hermitian form $h\colon L \times L \to \mathcal{O}$, linear in the first variable. Its dual is $L^\sharp := \{y \in L \otimes_{\mathcal{O}} K : h(x,y) \in \mathcal{O} \text{ for all } x \in L\}$, and $L$ is *unimodular* if $L^\sharp = L$. The basic example is $I_n := \mathcal{O}^n$ with $h(x,y) = \sum_k x_k \bar{y}_k$. Let $\Gamma_L$ be the group of $\mathcal{O}$-linear automorphisms of $L$ preserving $h$; it is finite, since it permutes the finitely many vectors of bounded length that span $L$.

Two Hermitian lattices are in the same *genus* if they become isometric over $\mathcal{O} \otimes \mathbb{Z}_p$ for every prime $p$. Let $\mathrm{Cl}_n$ be the set of isometry classes in the genus of $I_n$. By Problem 21.1.1 below, a class gives a unimodular $\mathbb{Z}$-lattice of rank $2n$ together with an automorphism of it; there are finitely many such lattices up to isometry [Ca, Ch. 9], each with a finite automorphism group, so $\mathrm{Cl}_n$ is finite. It is the Shimura variety, of dimension zero, of the definite unitary group of $I_n$; we call it the *class set*. Its *mass* is

$$M_n := \sum_{L \in \mathrm{Cl}_n} \frac{1}{|\Gamma_L|}, \qquad M_0 := 1.$$

What do the class sets look like in small rank?

**Problem 21.1.1.** Let $L$ be a Hermitian lattice and $B(x,y) := \operatorname{Re} h(x,y)$. Then $B$ is a positive definite symmetric $\mathbb{Z}$-bilinear form with $B(x,x) = h(x,x)$, multiplication by $\mathrm{i}$ preserves $B$, and $h(x,y) = B(x,y) + \mathrm{i}B(x,\mathrm{i}y)$. The dual of $L$ for $B$ is $L^\sharp$; in particular $L$ is unimodular exactly when $(L,B)$ is.

**Problem 21.1.2.** $\Gamma_{I_n}$ is the group of monomial matrices whose nonzero entries lie in $\mu_4 = \{\pm 1, \pm \mathrm{i}\}$. Its order is $4^n n!$.

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**Problem 21.1.3.** Every lattice in the genus of $I_n$ contains a vector $x$ with $h(x,x)$ odd.

**Problem 21.1.4.** If $n \le 4$, then $\mathrm{Cl}_n = \{I_n\}$.

Combining Problems 21.1.2 and 21.1.4 gives $M_n = 1/(4^n n!)$ for $n \le 4$. In rank 5 a second class appears: $E \perp I_1$, where $E$ is $E_8$ with a Hermitian structure over $\mathcal{O}$, in which $h(x,x)$ is always even, and $|\Gamma_E| = 46080$ is the order of the reflection group $G_{31}$ of Shephard and Todd [ShT]. There are no others (Problem 21.2.3).

## 21.2. The mass of a genus

How does $M_n$ grow? Let $\chi$ be the non-trivial character modulo 4, and let $\chi^j$ be $\chi$ for odd $j$ and the trivial character for even $j$, so that $L(s,\chi^j) = \zeta(s)$ for even $j$. For the genus of $I_n$, the mass formula reads

$$M_n = \delta_n \prod_{j=1}^{n} \frac{|L(1-j,\chi^j)|}{2}, \tag{1}$$

where $\delta_n$ is a positive rational number, a local factor at the prime 2, which is ramified in $K$. This is the Hermitian analogue of Siegel's mass formula for quadratic forms [Si], with the local factors computed by Gan and Yu [GY] and, at the ramified prime 2, by Cho [Cho]; made explicit in [JZ, Corollary 4.3], it gives $\delta_n = 1$ for odd $n$ and $\delta_n = 2^n - 1$ for even $n \ge 2$. We set $\delta_0 := 1$, so that (1) also holds for $n = 0$, where $M_0 = 1$.

**Problem 21.2.1.** For $j = 1, 2, 3, 4, 5$, the numbers $|L(1-j,\chi^j)|/2$ are 1/4, 1/24, 1/4, 1/240, 5/4.

**Problem 21.2.2.** $\delta_1 = \delta_3 = 1$, $\delta_2 = 3$ and $\delta_4 = 15$.

In rank 5, with $\delta_5 = 1$, the formula gives $M_5 = 1/73728$. Two unimodular lattices of the same rank over $\mathcal{O}$, each containing a vector $x$ with $h(x,x)$ odd, lie in the same genus [JZ, Corollary 2.2]; thus $E \perp I_1 \in \mathrm{Cl}_5$.

**Problem 21.2.3.** The only vectors $x \in E \perp I_1$ with $h(x,x) = 1$ are the four multiples $ue$, $u \in \mu_4$, of a generator $e$ of $I_1$; hence $\Gamma_{E \perp I_1} = \Gamma_E \times \mu_4$. Moreover $\mathrm{Cl}_5 = \{I_5, E \perp I_1\}$.

The functional equation turns the values at $1-j$ into values at $j$, and in doing so brings out a factorial.

**Problem 21.2.4.** For $j \ge 1$,

$$\frac{|L(1-j,\chi^j)|}{2} = (j-1)!\,\pi^{-j}\beta_j, \qquad \beta_j := \begin{cases} 2^{j-1}L(j,\chi) & j \text{ odd}, \\ 2^{-j}\zeta(j) & j \text{ even}. \end{cases}$$

For $r \ge 1$ define $\kappa_r > 0$ by

$$\frac{M_r}{M_{r-1}} = (r-1)!\,\pi^{-r}\kappa_r. \tag{2}$$

By (1) and Problem 21.2.4, $\kappa_r = (\delta_r/\delta_{r-1})\beta_r$. Multiplying (2) over $r \le n$,

$$M_n = \pi^{-n(n+1)/2} \prod_{j=0}^{n-1} j! \prod_{j=1}^{n} \kappa_j. \tag{3}$$

So the mass is a product of factorials, up to a factor whose logarithm is linear in $n$ at each step.

**Problem 21.2.5.** For $r \le 4$, $\kappa_r = \pi^r/(4\cdot r!)$; thus $\kappa_1, \dots, \kappa_4$ are $\pi/4$, $\pi^2/8$, $\pi^3/24$, $\pi^4/96$. With $\delta_5 = 1$, $\kappa_5 = \pi^5/288$.

**Problem 21.2.6.** For every $r \ge 1$, $3/4 \le \kappa_r \le 2$; in particular $\log \kappa_r = O(1)$.

By (3), $\log M_n$ grows like $\frac{1}{2}n^2 \log n$; since every $\Gamma_L$ contains $\mu_4$, the class set has at least $4M_n$ elements, more than $8 \cdot 10^5$ for $n = 15$.

## 21.3. Forms and the Weyl dimension formula

On a definite group a form is a function on the class set, with values in a representation. How many are there?

For $L \in \mathrm{Cl}_n$ let $L_{\mathbb{C}} := L \otimes_{\mathcal{O}} \mathbb{C}$, using the inclusion $\mathcal{O} \subset \mathbb{C}$. The form $h$ extends to a Hermitian inner product on $L_{\mathbb{C}}$, and $\Gamma_L$ is a finite subgroup of its unitary group. A *weight* is a sequence of integers $\lambda = (\lambda_1 \ge \cdots \ge \lambda_n \ge 0)$; write $|\lambda| := \sum_k \lambda_k$. For a finite-dimensional Hermitian space $A$, let $\mathbb{S}_\lambda(A) \subset A^{\otimes|\lambda|}$ be the image of the Young symmetrizer [FH, §6.1], with the inner product induced from $A^{\otimes|\lambda|}$. A linear map $x\colon A \to B$ induces $\mathbb{S}_\lambda(x)\colon \mathbb{S}_\lambda(A) \to \mathbb{S}_\lambda(B)$, the restriction of $x^{\otimes|\lambda|}$. Then $\mathbb{S}_\lambda(xy) = \mathbb{S}_\lambda(x)\mathbb{S}_\lambda(y)$ and $\mathbb{S}_\lambda(x)^* = \mathbb{S}_\lambda(x^*)$, the latter because $(x^*)^{\otimes k}$ commutes with the symmetric group. If $\lambda$ has at most $\dim A$ nonzero entries, $\mathbb{S}_\lambda(A)$ is the irreducible representation of $\mathrm{GL}(A)$ of highest weight $\lambda$; otherwise it is zero [FH, Theorem 6.3]. For $\lambda = (a)$ it is $\mathrm{Sym}^a(A)$. There, for $u \in A$ we write $u^a := u \otimes \cdots \otimes u$; for a basis $e_1, \dots, e_n$ of $A$ and $|\nu| = a$, the vectors $e^\nu$ are defined by $(\sum_k t_k e_k)^a = \sum_\nu (a!/\nu!)\, t^\nu e^\nu$, and $e^\nu e^\mu := e^{\nu+\mu}$ is the product of the symmetric algebra.

Let $V_\lambda(L) := \mathbb{S}_\lambda(L_{\mathbb{C}})$. Weyl's dimension formula reads, with $z_k := \lambda_k + n - k$,

$$\dim V_\lambda(L) = D_n(z) := \frac{V_n(z)}{\prod_{j=0}^{n-1} j!}, \qquad V_n(z) := \prod_{1 \le k < l \le n} (z_k - z_l) \tag{4}$$

[FH, Theorem 6.3]. Fix one lattice in each class of $\mathrm{Cl}_n$. A *form of weight* $\lambda$ assigns to each $L \in \mathrm{Cl}_n$ a vector $\varphi(L) \in V_\lambda(L)$ fixed by $\Gamma_L$. The forms make up a space $\mathcal{A}_\lambda(n)$, with the norm

$$\|\varphi\|^2 := \sum_{L \in \mathrm{Cl}_n} \frac{1}{|\Gamma_L|}\, \|\varphi(L)\|^2.$$

These are the algebraic modular forms of Gross [Gr] on the unitary group of $I_n$, of level one.

**Problem 21.3.1.** $\dim \mathrm{Sym}^a(\mathbb{C}^n) = \binom{a+n-1}{n-1}$, and $\dim \mathbb{S}_{(3,2,0)}(\mathbb{C}^3) = 15$. The denominator in (4) is $\prod_{j=0}^{n-1} j! = V_n(n-1, \dots, 1, 0)$.

**Problem 21.3.2.** In $\mathrm{Sym}^a(A)$, $\langle u^a, v^a \rangle = \langle u, v \rangle^a$. If $e_1, \dots, e_n$ is orthonormal, the $e^\nu$ are orthogonal and $\|e^\nu\|^2 = \nu!/a!$.

**Problem 21.3.3.** $\mathcal{A}_\lambda(n) = 0$ unless $4 \mid |\lambda|$. The space $\mathcal{A}_{(a)}(1)$ is one-dimensional when $4 \mid a$.

**Problem 21.3.4.** $\dim \mathcal{A}_\lambda(n) = \sum_L |\Gamma_L|^{-1} \sum_{\gamma \in \Gamma_L} \mathrm{tr}(\gamma \mid V_\lambda(L))$. When $4 \mid |\lambda|$, the scalars $\gamma \in \mu_4$ contribute $4M_n \dim V_\lambda$ to this sum.

**Problem 21.3.5.** The $\Gamma_{I_2}$-invariants in the symmetric algebra of $\mathbb{C}^2$ form the polynomial algebra on $e_1^4 + e_2^4$ and $e_1^4 e_2^4$. Hence, for $4 \mid a$, $\dim \mathcal{A}_{(a)}(2) = \lfloor a/8 \rfloor + 1$, which is $(1 + o(1)) \cdot 4M_2 \dim V_{(a)}$ as $a \to \infty$.

Does $|\lambda| \to \infty$ suffice? Compare two weights of the same size.

**Problem 21.3.6.** Let $j \geq 1$, $\lambda := (8j, 0)$ and $\lambda' := (4j, 4j)$. Then $\dim \mathcal{A}_\lambda(2) = j+1$ and $\dim \mathcal{A}_{\lambda'}(2) = 1$, while the scalar contributions $4M_2 \dim V$ of Problem 21.3.4 are $j + \frac{1}{8}$ and $\frac{1}{8}$. In both cases the non-scalar elements of $\Gamma_{I_2}$ contribute exactly $\frac{7}{8}$.

The non-scalar elements contribute the same amount to both weights. Beside $\dim V_\lambda = 8j + 1$ it is negligible; for $\lambda'$, where $V_{\lambda'} = \det^{4j}$ is a line on which all of $\Gamma_{I_2}$ acts trivially, it is seven times the scalar term. What must grow is the gap $\lambda_1 - \lambda_2$, not $|\lambda|$.

The same holds in general. When $4 \mid |\lambda|$ and all the gaps $\lambda_k - \lambda_{k+1}$ tend to infinity, the Weyl character formula shows that each non-scalar $\gamma$ contributes $o(\dim V_\lambda)$, so $\dim \mathcal{A}_\lambda(n) \sim 4M_n \dim V_\lambda$. The count of forms is thus the product of the mass and the Weyl dimension. By (3) and (4) each factor contains $\prod_{j<n} j!$, once in the numerator and once in the denominator; §21.4 returns to this.

## 21.4. The count of forms

When $4 \mid |\lambda|$ and the gaps of $\lambda$ are large, the space of forms of weight $\lambda$ has dimension about $4M_n \dim V_\lambda$ (§21.3). The mass contains the product $\prod_{j<n} j!$, whose logarithm is about $\frac{1}{2}n^2 \log n$, and the Weyl dimension contains its inverse. The norm of a theta lift is controlled to within a term linear in the weight, so the count must be known to the same accuracy. Why is it?

**Problem 21.4.1.** $\sum_{j=0}^{n-1} \log j! = \frac{1}{2}n^2 \log n - \frac{3}{4}n^2 + O(n \log n)$.

**Problem 21.4.2.** Let $r_0 < r_1$, and let $u \in \mathbb{Z}^{r_0}$ and $w \in \mathbb{Z}^{r_1}$ be strictly decreasing. Then

$$\frac{M_{r_1} D_{r_1}(w)}{M_{r_0} D_{r_0}(u)} = \frac{V_{r_1}(w)}{V_{r_0}(u)} \, \pi^{-\sum_{r=r_0+1}^{r_1} r} \prod_{r=r_0+1}^{r_1} \kappa_r.$$

The extra $(r-1)!$ in the mass ratio (2) cancels the extra $(r-1)!$ in the denominator of (4). With $r_0 = 0$, where $M_0 = D_0 = V_0 = 1$ (empty products), Problem 21.4.2 reads as follows.

**Theorem 21.4.1** (The count identity)**.** *For every $r \geq 1$ and every strictly decreasing $z \in \mathbb{Z}^r$,*

$$M_r D_r(z) = V_r(z) \, \pi^{-r(r+1)/2} \prod_{j=1}^{r} \kappa_j.$$

No factorial survives: what remains is a product of differences of weights, a power of $\pi$ and the factors $\kappa_j$.

In the application one rank is added at a time, with one large new weight.

**Problem 21.4.3.** Let $u \in \mathbb{Z}^{r_0}$ be strictly decreasing, and $w = (w_1, u_1, \ldots, u_{r_0})$ with $w_1 > u_1$. Then

$$\log \frac{M_{r_0+1} D_{r_0+1}(w)}{M_{r_0} D_{r_0}(u)} = \sum_{i=1}^{r_0} \big( \log(w_1 - u_i) - \log \pi \big) - \log \pi + \log \kappa_{r_0+1}.$$

Thus each new rank contributes, for every existing index $i$, the quantity $\log(w_1 - u_i) - \log \pi$, and since $\log \kappa_r = O(1)$ (Problem 21.2.6) the remaining term is of lower order when $w_1$ is large.

## Hints

**§21.1.**

**21.1.1:** Compute $B(\mathrm{i}x, \mathrm{i}y)$ and $B(x, \mathrm{i}y)$. For the dual, replace $x$ by $\mathrm{i}x$ in the condition $B(x, y) \in \mathbb{Z}$.

**21.1.2:** An automorphism permutes the vectors with $h(x, x) = 1$. Which vectors of $I_n$ are these?

**21.1.3:** Over $\mathbb{Z}_2$ the lattice is isometric to $I_n \otimes \mathbb{Z}_2$. Approximate a vector of odd length 2-adically by a vector of $L$.

**21.1.4:** Use Problems 21.1.1 and 21.1.3 and the classification of odd unimodular lattices of rank at most 8; then look at the vectors of length one.

**§21.2.**

**21.2.1:** $B_{j,\chi} = 4^{j-1} \sum_{a=1}^{4} \chi(a) B_j(a/4)$, and $\zeta(1-j) = -B_j/j$.

**21.2.2:** Compare (1) with $M_n = 1/(4^n n!)$.

**21.2.4:** For $\zeta$ use $\zeta(1-s) = 2(2\pi)^{-s} \cos(\pi s/2) \Gamma(s) \zeta(s)$. For the odd character $\chi$ of conductor 4, whose Gauss sum is 2i, complete $L(s, \chi)$ by $(4/\pi)^{(s+1)/2} \Gamma(\frac{s+1}{2})$, and use the reflection and duplication formulas for $\Gamma$. Check the case $j = 1$: $L(1, \chi) = \pi/4$.

**21.2.3:** $E$ is even. For the second part, compare with the mass $M_5$.

**21.2.5:** Use $M_r/M_{r-1} = 1/(4r)$ for $r \le 4$, and $M_5 = 1/73728$.

**21.2.6:** Use $\kappa_r = (\delta_r/\delta_{r-1})\beta_r$ and the values of $\delta_n$, treating odd and even $r$ separately; $1 - 3^{-r} \le L(r, \chi) \le 1$. For $r = 1$ use Problem 21.2.5.

**§21.3.**

**21.3.1:** For $\lambda = (a)$, $z = (a + n - 1, n - 2, \ldots, 0)$.

**21.3.2:** Expand $\langle (\sum t_k e_k)^a, (\sum s_k e_k)^a \rangle$ in two ways.

**21.3.3:** The scalar $\mathrm{i} \in \Gamma_L$ acts on $V_\lambda(L)$ by $\mathrm{i}^{|\lambda|}$.

**21.3.4:** The dimension of the invariants of a finite group is the average of the trace.

**21.3.5:** An invariant must be invariant under $e_k \mapsto \mathrm{i}e_k$ and under the swap. Count the pairs $(i, j)$ with $4i + 8j = a$.

**21.3.6:** For $\lambda$ use Problem 21.3.5. For $\lambda'$, $\mathbb{S}_{(k,k)}(\mathbb{C}^2)$ is the $k$-th power of the determinant, and every element of $\Gamma_{I_2}$ has determinant in $\mu_4$. Use $M_2 = 1/32$; the non-scalar part is the dimension less the scalar contribution.

**§21.4.**

**21.4.1:** $\log j! = j \log j - j + O(\log(j+1))$; compare the sum with an integral.

**21.4.2:** Write the left side as a telescoping product, and use (2) and (4).

**21.4.3:** $V_{r_0+1}(w) = V_{r_0}(u) \prod_i (w_1 - u_i)$.

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