REAL CHARACTERS OVER PRIME MODULI

Prerequisites. Chapter 2, *Inert primes and Estimate B*, Section 2.2: the coefficients $a_n(x) = \Lambda(n)\min(1, x/n)$, the sums $S_Z(x,\chi) = \sum_{n\le Z} a_n(x)\chi(n)$ and $t_\chi(Z) = \sum_{n>Z}\Lambda(n)\chi(n)/n$ and the identity $G_\chi(x) = -S_Z(x,\chi) - x\,t_\chi(Z)$, the bound in Problem 2.2.5, Problems 2.2.6 and 2.2.7, and the proof of Theorem 2.2.2; also $\Psi_\chi(t) = \sum_{n\le t}\Lambda(n)\chi(n)$ from Section 2.1. Chebyshev's bounds and the prime number theorem [Dav, Ch. 18]. From Davenport [Dav]: primitive real characters and Kronecker symbols (Ch. 5), the functional equation (Ch. 9), zero-free regions and the exceptional zero (Ch. 14), the number of zeros (Ch. 16), the explicit formula for $\Psi_\chi(x)$, there $\psi(x,\chi)$ (Ch. 19) and the prime number theorem for characters (Ch. 20).

Chapter 2 obtained a good modulus q by averaging $\chi_q(n) = \left(\frac{n}{q}\right)$ over primes $q \in (U, 2U]$ and letting $U \to \infty$ with everything else fixed. How large must the primes be, as a function of the window $[Y, Y^A]$? The average of $\left(\frac{s}{q}\right)$ over q is a sum over primes of a character of conductor about s, and it is small unless that character has a real zero near 1. By Page's theorem at most one s is affected. The tail $t_{\chi_q}(Z)$ needs a zero-free box for $L(s,\chi_q)$, and a density estimate gives one for all but few q. The answer is that $\log q$ of size $(\log Y)^3$ suffices.

Throughout, Q is large, $\mathcal{P}$ is the set of primes in $(Q, 2Q]$ (so Q and $\mathcal{P}$ are the U and $\mathcal{P}(U)$ of Chapter 2), and for squarefree $s \ge 1$

$$\Sigma(s) := \sum_{q\in\mathcal{P}} \Big(\frac{s}{q}\Big).$$

Constants c, C are positive and absolute unless their dependence is shown, and may change from line to line.

3.1. The symbol (s/q) as a function of q

For squarefree $s > 1$ let $D_s := s$ if $s \equiv 1 \pmod 4$ and $D_s := 4s$ otherwise; this is the discriminant of $\mathbb{Q}(\sqrt{s})$. Let $\omega_s(n) := \left(\frac{D_s}{n}\right)$ be the Kronecker symbol. It is a primitive real character of conductor D_s, and every primitive real character is a Kronecker symbol of a fundamental discriminant [Dav, Ch. 5].

Problem 3.1.1. For squarefree $s > 1$ and every odd prime $q \nmid s$, $\left(\frac{s}{q}\right) = \omega_s(q)$. The character ω_s is even, and $s \mapsto \omega_s$ is injective.

Problem 3.1.2. ω_2 has conductor 8 and $\omega_2(q) = 1$ exactly when $q \equiv \pm 1 \pmod 8$; ω_3 has conductor 12 and $\omega_3(q) = 1$ exactly when $q \equiv \pm 1 \pmod{12}$; and $\omega_5(q) = \left(\frac{q}{5}\right)$.

Two inputs are used. The first is the prime number theorem for characters [Dav, Ch. 20], [Dav, Ch. 14]: there are $c_1 \in (0, \frac12]$ and C_1 such that for every primitive

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character χ modulo $N \ge 3$ and every $u \ge 2$ with $\log N \le \sqrt{\log u}$,

$$\Big|\Psi_\chi(u) + \frac{u^\beta}{\beta}\Big| \le C_1\, u\, e^{-c_1\sqrt{\log u}}, \tag{1}$$

where the term with β is present only when χ is real and $L(s,\chi)$ has a real zero $\beta > 1 - c_1/\log N$; there is at most one such zero. The second is Page's theorem [Pa], [Dav, Ch. 14]: there is $c_2 > 0$ such that for every $R \ge 3$ at most one primitive real character of conductor at most R has a real zero in $(1 - c_2/\log R, 1)$. Call it the *exceptional character at level R*.

Fix $R \ge 3$ with $\log(4R) \le \sqrt{\log Q}$. If the exceptional character at level $4R$ exists and equals ω_s for some squarefree $s \le R$, let s^* be this s; by Problem 3.1.1 there is at most one. If there is none, a condition $s \ne s^*$ is empty.

Problem 3.1.3. Let $1 < s \le R$ be squarefree with $s \ne s^*$. Then $|\Psi_{\omega_s}(u)| \le Cu\, e^{-c\sqrt{\log Q}}$ for $Q \le u \le 2Q$.

Problem 3.1.4. For s as in Problem 3.1.3, $|\Sigma(s)| \le C|\mathcal{P}|\, e^{-c\sqrt{\log Q}}$.

Theorem 3.1.1. *There are $c_0, C_0 > 0$ such that, with $\eta_Q := C_0 e^{-c_0\sqrt{\log Q}}$, $|\Sigma(s)| \le \eta_Q|\mathcal{P}|$ for every squarefree s with $1 < s \le R$ and $s \ne s^*$.*

The exception is real. An exceptional zero very close to 1 makes almost every prime of $\mathcal{P}$ inert in $\mathbb{Q}(\sqrt{s^*})$:

Problem 3.1.5. Let s vary with Q, with $\log(4s) \le \sqrt{\log Q}$, and suppose $L(\sigma,\omega_s)$ has a real zero β with $(1-\beta)\log Q \to 0$. Then $\Sigma(s) = -|\mathcal{P}|\,(1+o(1))$ as $Q \to \infty$.

By Theorem 9.3.1 of Chapter 9 no such s exists, and s^* is not defined once c_2 is at most half the constant c there, since $D_s \le 4R$. We keep s^*, as an exceptional set of at most one element, so that this chapter does not depend on Chapter 9.

3.2. The mean square

Let $Z < Q$ and let c_n, for $1 \le n \le Z$, be complex numbers. Write $nn' = \square$ if nn' is a square, and $nn' \in s^*\square$ if nn'/s^* is a square; terms involving s^* are absent if s^* is not defined.

Problem 3.2.1. Suppose $Z^2 \le R$. Then

$$\frac{1}{|\mathcal{P}|}\sum_{q\in\mathcal{P}}\Big|\sum_{n\le Z} c_n\Big(\frac{n}{q}\Big)\Big|^2 \le \sum_{nn'=\square}|c_nc_{n'}| + \sum_{nn'\in s^*\square}|c_nc_{n'}| + \eta_Q\Big(\sum_{n\le Z}|c_n|\Big)^2.$$

We apply this to $c_n = \Lambda(n)\kappa(n)\min(1, y/n)$ with $|\kappa(n)| \le 1$ and $3 \le y \le Z$, so that $|c_n| \le a_n(y)$. The first sum is at most $Cy\log y$ by the bound in Problem 2.2.5.

Problem 3.2.2. For every squarefree $s^* > 1$,

$$\sum_{nn'\in s^*\square} a_n(y)\, a_{n'}(y) \le C\sqrt{y}\log y.$$

Problem 3.2.3. $\sum_{n\le Z} a_n(y) \le Cy\log Z$.

Problems 3.2.1–3.2.3 give the following.

Theorem 3.2.1. *Let $3 \le y \le Z$, take $R := Z^2$, and suppose $\log(4Z^2) \le \sqrt{\log Q}$ and $\eta_Q\, y \log^2 Z \le 1$. For all $\kappa(n)$ with $|\kappa(n)| \le 1$,*

$$\frac{1}{|\mathcal{P}|} \sum_{q \in \mathcal{P}} \Big| \sum_{n \le Z} \Lambda(n)\kappa(n) \min\Big(1, \frac{y}{n}\Big) \Big(\frac{n}{q}\Big) \Big|^2 \le C y \log y.$$

For $Q = 10^6$, $\kappa = 1$, $y = 100$ and $Z = 1000$ the left side is 988.3, and the square pairs $nn' = \square$ contribute 1034.1.

By Problem 2.2.6, $yD(y)$ is a sum of the same form with $\kappa(n) = 1$ for $n > y$ and $\kappa(n) = 0$ otherwise. The proof of Theorem 2.2.2, with Theorem 3.2.1 in place of Problem 2.2.5, therefore gives:

Theorem 3.2.2. *Let $0 < \tau \le \frac12$, $A \ge 1$, $Y \ge 3$ and $Z \ge 2Y^A$, take $R := Z^2$, and suppose $\log(4Z^2) \le \sqrt{\log Q}$ and $2\eta_Q\, Y^A \log^2 Z \le 1$. The number of $q \in \mathcal{P}$ with $|S_Z(y, \chi_q)| > \frac12 y^{1/2+\tau}$ for some $y \in [Y, Y^A]$ is at most $C_\tau |\mathcal{P}|\, Y^{-2\tau} \log Y$.*

Heath-Brown's large sieve for real characters [HB, Thm. 1] gives, with no information about zeros,

$$\sum_{m \le V}^{*} \Big| \sum_{n \le W}^{*} a_n \Big(\frac{n}{m}\Big) \Big|^2 \le C_\varepsilon (VW)^\varepsilon (V + W) \sum_n |a_n|^2,$$

the sums running over odd squarefree m and n. Here the loss $Q^\varepsilon = e^{\varepsilon(\log Y)^3}$ would be fatal, and Theorem 3.1.1 avoids it at the price of the exceptional s^*.

3.3. Zero-free boxes for most q

It remains to bound $x\, t_{\chi_q}(Z)$. For fixed χ the tail $t_\chi(Z)$ tends to 0 at a rate set by the zeros of $L(s, \chi)$ near $\operatorname{Re} s = 1$ with $|\operatorname{Im} s|$ up to a power of the conductor. For one q nothing better than (1) is known; for most q a density estimate does better. Gallagher's density estimate [Gal], in the form of [TZ2, (1.3)]: there are $c_3 \ge 1$ and C_3 such that for $T \ge 3$ and $0 \le \sigma \le 1$

$$\sum_{N \le T} \sum_{\chi \bmod N}^{*} \#\{\rho = \beta + i\gamma : L(\rho, \chi) = 0,\ \beta > \sigma,\ |\gamma| \le T\} \le C_3\, T^{c_3(1-\sigma)},$$

where $\sum^*$ runs over primitive characters and zeros are counted with multiplicity. For other density estimates of large sieve type see [Mon, Ch. 12]. Put $\delta_0 := 1/(2c_3)$, and call $q \in \mathcal{P}$ *good* if $L(s, \chi_q)$ has no zero with $\operatorname{Re} s > 1 - \delta_0$ and $|\operatorname{Im} s| \le 2Q$.

Problem 3.3.1. At most $C_3(2Q)^{1/2}$ primes $q \in \mathcal{P}$ are not good.

Problem 3.3.2. If q is good, every zero $\rho = \beta + i\gamma$ of $L(s, \chi_q)$ in $0 < \beta < 1$ with $|\gamma| \le 2Q$ has $\delta_0 \le \beta \le 1 - \delta_0$. In particular $|\rho| \ge \delta_0$, and for $Q \ge e^{c_1/\delta_0}$ there is no real zero in $(1 - c_1/\log q, 1)$.

The explicit formula [Dav, Chs. 16 and 19]: there is C_4 such that for every primitive χ modulo $N \ge 3$ the number of zeros with $|\gamma - t| \le 1$ is at most $C_4 \log(N(|t| + 2))$, and, if $L(s, \chi)$ has no real zero in $(1 - c_1/\log N, 1)$, then for $2 \le T \le u$

$$\Big| \Psi_\chi(u) + \sum_{|\gamma| < T} \frac{u^\rho}{\rho} \Big| \le C_4 \Big(\frac{u \log^2(Nu)}{T} + u^{1/4} \log u \Big).$$

In Problems 3.3.3–3.3.6, $q \in \mathcal{P}$ is good, $Q \ge e^{c_1/\delta_0}$, and $\lambda := \log q$.

Problem 3.3.3. For $2^{1/\delta_0} \le u \le e^{\lambda^2}$, $|\Psi_{\chi_q}(u)| \le C\lambda^4(u^{1-\delta_0} + u/Q)$.

Problem 3.3.4. For $u \ge e^{\lambda^2}$, $|\Psi_{\chi_q}(u)| \le 2u^{1-\delta_0} + C_1 u\, e^{-c_1\sqrt{\log u}}$.

Problem 3.3.5. For every primitive nonprincipal χ and $Z \ge 1$, $t_\chi(Z) = -\Psi_\chi(Z)/Z + \int_Z^\infty \Psi_\chi(u)\, u^{-2}\, du$.

Problem 3.3.6. For $2^{1/\delta_0} \le Z \le e^{\lambda^2}$, $|t_{\chi_q}(Z)| \le C\big(\lambda^6(Z^{-\delta_0} + Q^{-1}) + \lambda e^{-c_1\lambda}\big)$.

Now choose the parameters. Let $A \ge 1$ and $0 < \tau \le \frac12$, and put

$$X := Y^A, \qquad Z := X^{2/\delta_0}, \qquad \log Q := (\log Y)^3.$$

Problem 3.3.7. For $Y \ge Y_0(A)$: (i) the hypotheses of Theorem 3.2.2 hold, and $Q \ge e^{c_1/\delta_0}$; (ii) $y\,|t_{\chi_q}(Z)| \le 1$ for every good q and every $y \le X$.

Combining Theorem 3.2.2, Problems 3.3.1 and 3.3.7 and the identity $G_\chi = -S_Z - x\, t_\chi(Z)$ proves the following simple form of Estimate B.

Theorem 3.3.1. *Let $A \ge 1$ and $0 < \tau \le \frac12$. For $Y \ge Y_0(A,\tau)$ and $\log Q = (\log Y)^3$, all $q \in \mathcal{P}$ outside a set of at most $Y^{-\tau}|\mathcal{P}|$ elements satisfy: $L(s,\chi_q) \ne 0$ for $\operatorname{Re} s > 1 - \delta_0$, $|\operatorname{Im} s| \le 2Q$; and $|G_{\chi_q}(y)| \le y^{1/2+\tau}$ for every $y \in [Y, Y^A]$.*

The exponent 3 has room. Problem 3.1.3 needs $\log Q \ge (\log 4R)^2$, of order $(\log Y)^2$, and Problem 3.3.7 needs λ larger than a multiple of $\log Y$; any $\log Q = (\log Y)^{2+\varepsilon}$ with $\varepsilon > 0$ works. The range $\log N \le \sqrt{\log u}$ of (1) is what costs the square. At the other end, $\log Q$ may be as large as a small power of Y: besides the lower bounds just named, Problem 3.3.7 needs only $\lambda^6 \le cX$ with c small. Adriazola takes this upper end, $e^{z/2} < q \le e^z$ with $z = Y^{\varepsilon/5}$, where now $0 < \varepsilon \le \frac{1}{52}$ is the exponent of Chapter 58 [Adr, Thms. 5.10 and 5.12]; so does Chapter 58 (Section 58.5).

3.4. The second prime and Linnik's range

Estimate B also needs a second prime d, chosen after q, in a residue class modulo a multiple of q.

Problem 3.4.1. Let $q \equiv 3 \pmod 8$ be prime. The primes $d \equiv 7 \pmod{16}$ with $q \mid d+1$ are the primes in one class a modulo $M_q := 16q$, with $(a, M_q) = 1$, and each such d, with q, satisfies the congruences assumed in Problem 2.1.5.

Which s have a biased average over these d? For a modulus M, a class a prime to M and a primitive character χ of conductor D, put $L := \operatorname{lcm}(M, D)$; the *fiber* of a is the set of classes r mod L with $(r, L) = 1$ and $r \equiv a \pmod M$.

Problem 3.4.2. The average of χ over the fiber of a is $\chi(a)$ if $D \mid M$, and 0 otherwise.

Problem 3.4.3. Take $q = 11$, so that $M_q = 176$ and $a = 87$. The averages of ω_2, ω_{11} and ω_3 over the fiber of a are 1, 1 and 0. In general, for $q \equiv 3 \pmod 8$ and a as in Problem 3.4.1, the average of ω_s over the fiber of a, for squarefree $s > 1$, is 1 if $s \in \{2, q, 2q\}$ and 0 otherwise.

So conditioning on the class a moves exactly one term of the mean square. In the analogue of Problem 3.2.1 for the primes $d \equiv a \pmod{M_q}$, with $n, n' \le Z < q$, the values $s = q, 2q$ cannot occur, but the pairs with $nn' \in 2\square$ no longer average to

zero: they join the square pairs as a main term $\sum_{nn'\in 2\square}|c_nc_{n'}|$, at most $C\sqrt{y}\log y$ for $|c_n|\le a_n(y)$, by Problem 3.2.2 with 2 in place of s^*. The class plays the part of an exceptional s^* that is always present and costs no more.

The average of $(\frac{s}{d})$ over these d is a sum over primes of the characters $\xi\omega_s$, ξ modulo M_q, whose conductors divide $\operatorname{lcm}(M_q, D_s)$. With (1) this needs $\log d \ge (\log M_q)^2$. One can do with less: apart from the effect of an exceptional zero, the prime number theorem in progressions modulo k holds with a relative error $e^{-c\log x/\log k}$ as soon as $\log x$ is a large multiple of $\log k$. This is the range of Linnik's theorem on the least prime [Lin]; the statement we use is Gallagher's.

Let $k\ge 3$. By a theorem of McCurley, as stated in [TZ1, §1], $\prod_{\chi \bmod k} L(s,\chi)$ has at most one zero in $\operatorname{Re} s \ge 1 - 1/(13\log(k(|\operatorname{Im} s|+3)))$; if it exists it is real and simple, call it β_1, and $L(\beta_1,\chi_1)=0$ for a unique real nonprincipal χ_1 modulo k. For $(a,k)=1$ and $h\le x$ put $\lambda_a := h^{-1}\int_{x-h}^{x}(1-\chi_1(a)t^{\beta_1-1})\,dt$ if β_1 exists and $\lambda_a := 1$ otherwise. Gallagher's Theorem 7 [Gal, Thm. 7] gives, in the form of [TZ1, (3.3)–(3.4)]: there are $c_4, c_5 > 0$ and C_5 such that if $e^{\sqrt{\log x}} \le k \le x^{c_4}$ and $x/k \le h \le x$, then

$$\Bigl|1 - \frac{\varphi(k)}{\lambda_a h}\sum_{\substack{x-h<p\le x\\ p\equiv a \ (k)}}\log p\Bigr| \le C_5\,\mu\, e^{-c_5\log x/\log k},$$

with $\mu := (1-\beta_1)\log k$ if β_1 exists and $\mu := 1$ otherwise.

In Problems 3.4.4 and 3.4.5, $x := k^B$ and $h := x/2$, with B and then k large.

Problem 3.4.4. If β_1 does not exist, then $\sum_{x/2<p\le x,\, p\equiv a \ (k)}\log p \ge x/(4\varphi(k))$.

Problem 3.4.5. If β_1 exists, then $\lambda_a \ge \frac12\min\bigl(1,(1-\beta_1)\log(x/2)\bigr)$, and the sum of Problem 3.4.4 is at least $\lambda_a x/(4\varphi(k)) > 0$.

Problems 3.4.4 and 3.4.5 prove Linnik's theorem [Lin] for large k; for the other k it is Dirichlet's theorem, with a suitable C.

Theorem 3.4.1. *There is an absolute B_0 such that for every $k\ge 2$ and every a prime to k there is a prime $p\equiv a \pmod k$ with $p \le Ck^{B_0}$.*

The factor $\mu = (1-\beta_1)\log k$ is the Deuring–Heilbronn phenomenon, which Gallagher's proof uses through [Gal, (27)]: an exceptional zero pushes the other zeros away from 1. Without it, Problem 3.4.5 would need $e^{-c_5B} < \lambda_a$; the classical effective lower bound $1-\beta_1 \ge ck^{-1/2}(\log k)^{-2}$ [Dav, Ch. 14] would force B of order $\log k$, that is, $\log x$ of order $(\log k)^2$, the range of (1) again. Theorem 9.3.1 gives instead $1-\beta_1 \ge c/\log(2k)$, so that μ is bounded below; as with s^*, we do not use it here. For Estimate B the class a is the one of Problem 3.4.1, and the analogues of Sections 3.1–3.3 for d run with the main term λ_a in place of 1; to make η_Q smaller than Y^{-A} one takes $\log d$ of order $(\log M_q)(\log Y)^2$. We do not carry this out here. Adriazola [Adr, §5] does, from the inputs of this chapter; there d is also taken $\equiv -1$ modulo q and modulo every odd prime up to z, with q and z as at the end of Section 3.3, and the exceptional set of Section 3.1 is replaced by a larger one.

Hints

§3.1.

3.1.1: On odd primes not dividing D the Kronecker symbol $(\frac{D}{q})$ is a Legendre symbol, and $(\frac{4s}{q}) = (\frac{s}{q})$. For the parity, $\chi(-1)$ is the sign of D [Dav, Ch. 5].

3.1.2: The supplementary law for 2 and quadratic reciprocity.
3.1.3: Apply (1) with $N = D_s \le 4R$, and bound a possible β by Page's theorem at level $4R$.
3.1.4: Remove the prime powers with Chebyshev's bound, then pass from $\sum \omega_s(q)\log q$ to $\sum \omega_s(q)$ by partial summation.
3.1.5: Keep the term with β in (1); note $u^{\beta-1} \to 1$ uniformly on $[Q, 2Q]$.

§3.2.

3.2.1: Expand the square and write $nn' = sm^2$ with s squarefree; $q > Z$ does not divide nn'.
3.2.2: Since n and n' are prime powers, the sum is empty unless s^* has at most two prime factors. Bound $\sum_{a\ge 1} \log p \min(1, y/p^a)$ by $3\log y$, and the sum of $a_n(y)$ over squares n by $C\sqrt{y}$.
3.2.3: Split at $n = y$ and use Chebyshev's bound with partial summation.

§3.3.

3.3.1: χ_q is primitive of conductor $q \le 2Q$, and distinct q give distinct characters. Take $T = 2Q$ and $\sigma = 1 - \delta_0$.
3.3.2: For a real primitive χ the functional equation makes the zeros symmetric under $\rho \mapsto 1 - \bar{\rho}$ [Dav, Ch. 9].
3.3.3: Take $T = \min(u^{\delta_0}, Q)$ in the explicit formula; use Problem 3.3.2 and count zeros in intervals of length 1.
3.3.4: Apply (1) with $N = q$; by Problem 3.3.2 it has no term with β.
3.3.5: Partial summation, and $\Psi_\chi(V)/V \to 0$.
3.3.6: Split the integral of Problem 3.3.5 at e^{λ^2}.
3.3.7: Compare sizes: $\log Z$ is a multiple of $\log Y$, $\eta_Q = C_0 e^{-c_0(\log Y)^{3/2}}$, and $\lambda > (\log Y)^3$.

§3.4.

3.4.2: The fiber is a coset of the kernel of reduction $(\mathbb{Z}/L\mathbb{Z})^\times \to (\mathbb{Z}/M\mathbb{Z})^\times$. Use orthogonality on that kernel; if χ is trivial on it, then $\gcd(M, D)$ is an induced modulus of χ.
3.4.3: Apply Problem 3.4.2: list the fundamental discriminants D_s dividing $16q$, and evaluate as in Problem 2.1.5.
3.4.4: Check the hypotheses $e^{\sqrt{\log x}} \le k \le x^{c_4}$ and $x/k \le h$.
3.4.5: $1 - e^{-v} \ge \frac{1}{2}\min(1, v)$ for $v \ge 0$, and the definition of β_1 bounds μ.

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