INERT PRIMES AND ESTIMATE B

Prerequisites. Chapter 1, *From zeros to a one-sided ramp sum* (Sections 1.1–1.4): the functions G and G_χ of its equations (1) and (3), the argument of Problem 1.2.4, and Theorem 1.2.3. Quadratic reciprocity with its supplements [SeA, Ch. I, §3], and the splitting of primes in an imaginary quadratic field [Ne, Ch. I, §8]. Orthogonality of Dirichlet characters and the prime number theorem in progressions to a fixed modulus [Dav, Ch. 4 and 20].

The arithmetic gives upper bounds for ramp sums over odd powers of primes inert in an imaginary quadratic field, not for the full ramp sum S. How far is such a sum from G, and what must be known about the quadratic characters involved to recover a bound for G?

2.1. Inert primes and a quadratic character

Let $d \equiv 3 \pmod 4$ be a prime, let $K := \mathbb{Q}(\sqrt{-d})$, and let $\chi_d(n) := \left(\frac{n}{d}\right)$ be the Legendre symbol.

Problem 2.1.1. (i) For every prime p, $\chi_d(p)$ is 1, -1 or 0 according as p splits, is inert, or ramifies in K. (ii) For a prime power $r = p^h$, $1 - \chi_d(r)$ is 2 if p is inert and h is odd, 1 if $p = d$, and 0 otherwise.

The ramp sum over odd powers of inert primes, with weight 2, and over powers of d, with weight 1, is therefore

$$S_d(x) := \sum_r \Lambda(r)\big(1 - \chi_d(r)\big)\Big(\frac{x}{r} - 1\Big)_+ .$$

For a primitive nonprincipal character χ let $\Psi_\chi(t) := \sum_{n \le t} \Lambda(n)\chi(n)$, and let c_χ and G_χ be as in (3).

Problem 2.1.2. $G_\chi(x) = -\sum_n \Lambda(n)\chi(n)\min(1, x/n) = -x\int_x^\infty \Psi_\chi(t)\, t^{-2}\, dt$.

Problem 2.1.3. $S_d(x) - x\log x + (1 + \gamma - c_{\chi_d})x = G(x) - G_{\chi_d}(x)$.

The arithmetic thus bounds $G - G_{\chi_d}$, not G. How much does such a bound say about the zeros of ζ?

Problem 2.1.4. Let $0 < \theta < 1$. Suppose that $L(s, \chi_d)$ has no real zero in $(\theta, 1)$, and that $G(x) - G_{\chi_d}(x) \le x^\theta$ for all large x. Then every zero of $\zeta(s)$ or of $L(s, \chi_d)$ in $\operatorname{Re} s > \theta$ is a zero of both, with the same multiplicity.

A one-sided bound for $G - G_{\chi_d}$ thus cannot exclude a zero of ζ that $L(s, \chi_d)$ shares, and the argument needs a separate hypothesis on real zeros of $L(s, \chi_d)$. The way out is to bound $G_{\chi_d}(x)$ from both sides, for one well-chosen d and for x in a given range only (§2.2).

A second quadratic character enters through a table of residue classes. Let $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod 8$ be primes with $q \mid d + 1$, and let $\chi_q(n) := \left(\frac{n}{q}\right)$.

Date: November 25, 2026.

Problem 2.1.5. $\chi_d(2) = \chi_d(q) = 1$. Hence $1 - \chi_d(r)$ vanishes at every power of 2 and of q.

Let n be a power of 2, let $N := 4n$, and let $H := (\mathbb{Z}/N\mathbb{Z})^\times/\{\pm 1\}$, a group of order n; write $[r]$ for the class of an odd integer r. Assign an odd prime power r prime to q to the *cell* $(v(r), [r])$, where $v(r) := \frac{1}{2}(1 + \chi_q(r))$ is the *row*. There are $2n$ cells. To each cell i attach a weight $w_i > 0$, and for $a > 0$ consider the sum over these r

$$P := \sum_r \Lambda(r)\big(1 - \chi_d(r)\big)\Big(\frac{a}{r} - w_{\mathrm{cell}(r)}\Big)_+.$$

As in Section 1.3, cell i contributes w_i times its own ramp sum at a/w_i. Problem 4.1.4 of Chapter 4 fills in the table for $q = 11$, $d = 1319$ and $N = 16$.

Let $b > 0$ and $T_2 \subset H$, and let $\delta := |T_2|/n$. For $u \in H$, call the cells $(1, s)$ with $s \in uT_2$ *distinguished*, give them weight b, and give the others weights so small that a/w_i is enormous. Averaging over the translates u removes the column condition:

Problem 2.1.6. For every odd r prime to q, $\frac{1}{n}\sum_{u\in H} \mathbf{1}\big[v(r) = 1,\ [r] \in uT_2\big] = \frac{\delta}{2}\big(1 + \chi_q(r)\big)$.

So the average over u of the distinguished part of P is $\frac{\delta b}{2} S_{q,d}(a/b)$, where

$$S_{q,d}(x) := \sum_r \Lambda(r)\big(1 + \chi_q(r)\big)\big(1 - \chi_d(r)\big)\Big(\frac{x}{r} - 1\Big)_+.$$

Problem 2.1.7. Let $\eta := \chi_q\chi_d$. Then

$$S_{q,d}(x) - x\log x + (1 + \gamma + c_{\chi_q} - c_{\chi_d} - c_\eta)x = G(x) + G_{\chi_q}(x) - G_{\chi_d}(x) - G_\eta(x).$$

The function G appears with coefficient $+1$, so an upper bound for the left side and two-sided bounds for G_{χ_q}, G_{χ_d} and G_η give an upper bound for G. The other cells test primes at the enormous scales a/w_i. There the remainder of each cell must be small with a saving better than any power of x, and for this one expands the cell in characters. By Problem 4.2.2 of Chapter 4, for r prime to $2qd$ the function $r \mapsto \big(1 - \chi_d(r)\big)\mathbf{1}[\mathrm{cell}(r) = (v, s)]$ is a linear combination $\sum_\chi f_\chi\chi(r)$ of the $4n$ characters $\xi\chi_q^i\chi_d^j$, ξ a character of H and $i, j \in \{0, 1\}$, with $\sum_\chi |f_\chi| = 2$.

Hence, apart from explicit terms from the powers of 2, q and d, the remainder of a cell is a combination, with total mass 2, of the functions G_{χ^*}, where χ^* is the primitive character inducing χ and $G_{\chi^*} := G$ for χ principal; and a uniform bound $|G_{\chi^*}(y)| \le \vartheta y$, with $\vartheta > 0$ small, for these characters at the large scales makes the other cells negligible. Such a bound comes from a zero-free region for the $L(s, \chi)$ [Dav, Ch. 14], which is effective here, even if there is an exceptional real zero [Dav, Ch. 14, (12)], because the large scales may be taken with $\log y > qdN$.

2.2. Estimate B in a simple form

To use Problem 2.1.7 one needs two-sided bounds $|G_\chi(x)| \le x^{1/2+\varepsilon}$ for $\chi = \chi_q$, χ_d and $\chi_q\chi_d$. For one fixed character and all large x this is as hard as the problem we started from:

Problem 2.2.1. Let χ be primitive and nonprincipal, and $0 < \theta < 1$. If $|G_\chi(x)| \le x^\theta$ for all large x, then $L(s, \chi) \ne 0$ for $\mathrm{Re}\, s > \theta$.

What is needed is weaker: the bound for x in one interval $[Y, Y^{3/2}]$, for some q and d that may be chosen after Y. One takes the moduli enormous, with $\log q$ and $\log d$ powers of Y. For most of them a character sum of length far below the modulus behaves like a random sum, of size about $\sqrt{x}$, and one good choice suffices.

For $1 \le x \le Z$ let $a_n(x) := \Lambda(n)\min(1, x/n)$, and for a Dirichlet character χ let

$$S_Z(x,\chi) := \sum_{n\le Z} a_n(x)\chi(n), \qquad t_\chi(Z) := \sum_{n>Z} \frac{\Lambda(n)\chi(n)}{n},$$

the second for χ nonprincipal. By Problem 2.1.2, $G_\chi(x) = -S_Z(x,\chi) - x\,t_\chi(Z)$ for primitive nonprincipal χ.

Problem 2.2.2. $\sum_{n\le Z} a_n(x)^2 \le Cx\log x$ for $2 \le x \le Z$, with C absolute.

Problem 2.2.3. Let $Q > Z$ be prime, and let b_n be complex numbers. Then $\frac{1}{Q-1}\sum_{\chi \bmod Q}\bigl|\sum_{n\le Z} b_n\chi(n)\bigr|^2 = \sum_{n\le Z}|b_n|^2$. Hence, for $2 \le x \le Z$, at most $C(Q-1)x^{-2\varepsilon}\log x$ characters χ modulo Q satisfy $|S_Z(x,\chi)| > x^{1/2+\varepsilon}$.

The characters of the argument are quadratic. For an odd prime q let $\chi_q(n) := \left(\frac{n}{q}\right)$, and let $\mathcal{P}(U)$ be the set of primes in $(U, 2U]$.

Problem 2.2.4. Let $m \ge 1$ be an integer that is not a square. There is a nonprincipal Dirichlet character ω_m modulo $4m$ with $\left(\frac{m}{q}\right) = \omega_m(q)$ for every odd prime $q \nmid m$.

Problem 2.2.5. Fix $2 \le x \le Z$ and real numbers b_n with $|b_n| \le a_n(x)$. As $U \to \infty$,

$$\frac{1}{|\mathcal{P}(U)|}\sum_{q\in\mathcal{P}(U)}\Bigl(\sum_{n\le Z} b_n\chi_q(n)\Bigr)^2 \;\longrightarrow\; \sum_{\substack{n,n'\le Z\\ nn'=\square}} b_n b_{n'} \;\le\; Cx\log x,$$

the sum being over the pairs with nn' a square.

Problem 2.2.5 and Chebyshev's inequality give the following.

Theorem 2.2.1. *Let $2 \le x \le Z$ and $\varepsilon > 0$. For all U large in terms of Z, the proportion of $q \in \mathcal{P}(U)$ with $|S_Z(x,\chi_q)| > x^{1/2+\varepsilon}$ is at most $Cx^{-2\varepsilon}\log x$, with C absolute.*

A good modulus at one point x is not yet enough; one wants the bound at all x of an interval simultaneously.

Problem 2.2.6. As a function of $y \in [1, Z]$, $S_Z(y,\chi)$ is continuous and piecewise linear, and off the integers its derivative $D(y)$ satisfies $yD(y) = \sum_{y<n\le Z} a_n(y)\chi(n)$.

Problem 2.2.7. For $1 \le W \le V \le \min(2W, Z)$ and $y \in [W, V]$, $|S_Z(y,\chi)|^2 \le 2|S_Z(W,\chi)|^2 + 2W\int_W^V |D(t)|^2\,dt$.

By Problem 2.2.6, Problem 2.2.5 applies to $yD(y)$ as well. Applying Problems 2.2.5 and 2.2.7 on the blocks $[2^kY, \min(2^{k+1}Y, Y^A)]$ and summing over k gives:

Theorem 2.2.2. *Let $Y \ge 2$, $A \ge 1$, $Z \ge Y^A$ and $\varepsilon > 0$. For all U large in terms of Z, the proportion of $q \in \mathcal{P}(U)$ for which $|S_Z(y,\chi_q)| > y^{1/2+\varepsilon}$ for some $y \in [Y, Y^A]$ is $O_\varepsilon(Y^{-\varepsilon})$.*

It remains to make $x\,t_{\chi_q}(Z)$ small. For a fixed character $t_\chi(Z) \to 0$, but at a rate governed by the zeros of $L(s,\chi)$ near $\operatorname{Re} s = 1$, and here q is far larger than Z. One needs a box $\operatorname{Re} s > 1 - \delta_0$, $|\operatorname{Im} s| \le T$ free of zeros of $L(s,\chi_q)$ for all but a small proportion of $q \in \mathcal{P}(U)$; zero-density estimates near $\operatorname{Re} s = 1$ give such boxes for most moduli [Mon, Theorem 12.2]. To have $\log q$ a small power of Y, instead of letting $U \to \infty$, one needs primes in progressions to large moduli [Gal, Thm. 7], and Page's theorem that at most one primitive real character of conductor at most R has a real zero within $c/\log R$ of 1 [Pa], [Dav, Ch. 14].

The statement wanted, which we call *Estimate B*, is this: for every small $\varepsilon > 0$ and every large Y there are primes $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$ with $q \mid d+1$ such that $|G_\chi(x)| \le x^{1/2+\varepsilon}$ for $\chi = \chi_q$, χ_d, $\chi_q\chi_d$ and all $x \in [Y, Y^{3/2}]$. One chooses q first and then d, and the mean squares over d involve the twists by χ_q. We do not carry out this combination here, and we do not know a published proof of Estimate B. Adriazola [Adr, §5, Thms. 5.10 and 5.12] proves it, for most q and then most d in suitable ranges with $\log q$ of size $Y^{\varepsilon/5}$ and $\log d$ of size $Y^{\varepsilon/5}\log^2 Y$, from the inputs named above; there d is also taken $\equiv -1$ modulo every odd prime up to $Y^{\varepsilon/5}$. There is no circularity: the route sketched above uses only mean squares and zero-free boxes near $\operatorname{Re} s = 1$ for most q, and nothing about zeros near the critical line. For the function G^ψ of Section 1.4, Estimate B is needed for all the twists $\xi\chi_q^i\chi_d^j$ with ξ a character modulo M and $(i,j) \ne (0,0)$.

Hints

§2.1.

- **2.1.1:** For odd $p \ne d$ compare $\left(\frac{-d}{p}\right)$ with $\left(\frac{p}{d}\right)$ by reciprocity. For $p = 2$ use $\mathcal{O}_K = \mathbb{Z}[(1+\sqrt{-d})/2]$.
- **2.1.2:** Use $\sum_n \Lambda(n)\chi(n)/n = -c_\chi$, convergent by the prime number theorem for χ.
- **2.1.3:** Write S_d as the ramp sum with coefficient 1 minus the one with coefficient χ_d, and use (1) and (3).
- **2.1.4:** The Mellin transform of $G - G_{\chi_d}$ is $F(s) + \frac{L'}{L}(s,\chi_d)/(s(s-1)) - c_{\chi_d}/(s-1)$, with F as in (2). Follow Problem 1.2.4: a real zero of ζ gives the residue sign of that problem, but a real zero of $L(s,\chi_d)$ gives the opposite sign.
- **2.1.5:** $d \equiv 7 \pmod 8$ settles 2. For q use reciprocity with $q, d \equiv 3 \pmod 4$ and $d \equiv -1 \pmod q$.
- **2.1.7:** Expand $(1+\chi_q)(1-\chi_d)$; the character η is primitive modulo qd.

§2.2.

- **2.2.1:** The bound gives absolute convergence of the Mellin transform of G_χ in $\operatorname{Re} s > \theta$. Compute that transform as in Problem 1.1.3.
- **2.2.2:** Split at $n = x$ and use Chebyshev's bound and partial summation on each part.
- **2.2.3:** Expand the square; since $Q > Z$, distinct $n, n' \le Z$ are distinct modulo Q.
- **2.2.4:** Write $m = 2^e m'$ with m' odd and use the Jacobi symbol. For nontriviality, separate the cases where m' is a square and where it is not.
- **2.2.5:** Expand the square and apply Problem 2.2.4 with $m = nn'$ and the prime number theorem modulo $4nn'$. For the bound, n and n' are prime

powers; the pairs with $n = n'$ are covered by Problem 2.2.2. Group the pairs by prime, and treat the even powers separately.

2.2.7: Use Problem 2.2.6.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

[Dav] H. Davenport, *Multiplicative Number Theory*, 3rd ed., revised by H. L. Montgomery, Grad. Texts in Math. **74**, Springer, 2000.

[Gal] P. X. Gallagher, *A large sieve density estimate near* $\sigma = 1$, Invent. Math. **11** (1970), 329–339.

[Mon] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Lecture Notes in Math. **227**, Springer, 1971.

[Ne] J. Neukirch, *Algebraic Number Theory*, Grundlehren Math. Wiss. **322**, Springer, 1999.

[Pa] A. Page, *On the number of primes in an arithmetic progression*, Proc. London Math. Soc. (2) **39** (1935), 116–141.

[SeA] J.-P. Serre, *A Course in Arithmetic*, Grad. Texts in Math. **7**, Springer, 1973.