FROM ZEROS TO A ONE-SIDED RAMP SUM

Prerequisites. The zeta function and Dirichlet L-functions as in [Dav]: the Hadamard product and the constant B (Ch. 12), the explicit formula for Chebyshev's function $\psi(x)$ and its contour shift (Ch. 17), and their analogues for $L(s,\chi)$ (Ch. 12 and 19), with $L(1,\chi)\neq 0$ (Ch. 4). Chebyshev's bound $\psi(x)\ll x$ [MV, §2.2]. Mellin transforms. The theorems of Landau and of Kaczorowski–Pintz are stated in §1.2. Chapter 17 is used only for motivation, in §1.3.

A comparison of divisibility with size, as in Chebyshev's method, gives upper bounds for weighted sums over primes, and nothing else. Can an upper bound for a sum over primes prove the Riemann hypothesis? This chapter shows that it can, for a suitable weight, following the reduction used in Alpöge's argument [Alp]. The upper bound itself is not treated here.

1.1. The ramp sum

Write $v_+ := \max(v,0)$ for real v, and let Λ be the von Mangoldt function. The *ramp* at x is the function $n\mapsto (x/n-1)_+$: it is large for small n, decreases, and vanishes for $n\geq x$. Let

$$S(x) := \sum_n \Lambda(n)\Big(\frac{x}{n}-1\Big)_+, \qquad G(x) := S(x) - x\log x + (1+\gamma)x, \tag{1}$$

where γ is Euler's constant, and let $\psi(x) := \sum_{n\leq x}\Lambda(n)$. The two subtracted terms are the main term of S, as the problems show.

Problem 1.1.1. For $n\geq 1$ and $\operatorname{Re} s>1$, $\int_1^\infty (x/n-1)_+\, x^{-s-1}\,dx = n^{-s}/\big(s(s-1)\big)$.

Problem 1.1.2. $S(x) = x\int_1^x \psi(t)\,t^{-2}\,dt$. Hence $G(x)\ll x\log x$ for $x\geq 2$.

Problem 1.1.3. For $\operatorname{Re} s>1$,

$$F(s) := \int_1^\infty G(x)\,x^{-s-1}\,dx = -\frac{\zeta'}{\zeta}(s)\,\frac{1}{s(s-1)} - \frac{1}{(s-1)^2} + \frac{1+\gamma}{s-1}. \tag{2}$$

Problem 1.1.4. F is meromorphic in $\mathbb{C}$ and regular at $s=1$. Its poles in $\operatorname{Re} s>0$ are the zeros ρ of ζ in the critical strip $0\leq \operatorname{Re} s\leq 1$, and a zero of multiplicity m gives residue $-m/\big(\rho(\rho-1)\big)$.

Problem 1.1.5. The residue of $F(s)x^s$ at $s=0$ is $\log 2\pi$.

Problems 1.1.3–1.1.5 and a contour shift as in [Dav, Ch. 17], applied to $-\frac{\zeta'}{\zeta}(s)\,x^s/(s(s-1))$, the part of $F(s)x^s$ that comes from S, where now every integral converges absolutely because of the factor $1/(s(s-1))$, give the following.

Theorem 1.1.1. *For $x>1$,*

$$G(x) = \log 2\pi - \sum_\rho \frac{x^\rho}{\rho(\rho-1)} + O(x^{-2}),$$

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where ρ runs over the zeros of ζ in the critical strip with multiplicity, and the sum converges absolutely.

Problem 1.1.6. $\sum_\rho 1/\big(\rho(1-\rho)\big) = 2+\gamma-\log 4\pi = 0.0461\ldots$, with multiplicity.

Problem 1.1.7. If the Riemann hypothesis holds, then $|G(x)-\log 2\pi| \le (2+\gamma-\log 4\pi)\sqrt{x}+O(x^{-2})$.

For instance $G(10^4)=1.749$, $G(10^5)=3.671$, $G(10^6)=11.040$ and $G(2\cdot 10^6)=-12.268$, to be compared with $\log 2\pi = 1.838$ and $0.0462\sqrt{10^6}=46.2$.

Problem 1.1.8. Let $\rho=\beta+i\gamma_0$ with $\gamma_0\ne 0$, and let $c:=-1/\big(\rho(\rho-1)\big)$. The joint term $cx^\rho+\bar c x^{\bar\rho}$ of ρ and $\bar\rho$ in Theorem 1.1.1 exceeds $|c|x^\beta$ for all x in a set whose image under log has density $1/3$ in $[0,\infty)$, and it is less than $-|c|x^\beta$ on another such set.

A zero with $\beta>\frac12$ thus contributes to G terms of both signs, larger than $\sqrt{x}$. A comparison of divisibility with size sees only one sign. The next section shows that one is enough.

1.2. One side suffices

The following theorem is due to Landau [Lan]; see [MV, Lemma 15.1]. Let $A\colon [1,\infty)\to\mathbb{R}$ be measurable and bounded on compact sets, with $A(x)\ge 0$ for $x\ge X_0$. Let σ_c be the infimum of the real σ with $\int_1^\infty |A(x)|x^{-\sigma-1}\,dx<\infty$, and suppose that $-\infty<\sigma_c<\infty$. Then $H(s):=\int_1^\infty A(x)x^{-s-1}\,dx$ is holomorphic in $\operatorname{Re} s>\sigma_c$, and it has no holomorphic continuation to a neighborhood of σ_c.

Problem 1.2.1. In this situation $H(\sigma)$ is bounded below for real $\sigma\in(\sigma_c,\sigma_c+1]$.

Fix $0<\theta<1$, suppose that $G(x)\le x^\theta$ for all large x, and let $A(x):=x^\theta-G(x)$.

Problem 1.2.2. $-\infty<\sigma_c\le 1$, and $H(s)=1/(s-\theta)-F(s)$ for $\operatorname{Re} s>1$.

Problem 1.2.3. $\zeta(\sigma)<0$ for $0<\sigma<1$.

Problem 1.2.4. If $\sigma_c>\theta$, then σ_c is a real zero of ζ, and $H(\sigma)\to-\infty$ as $\sigma\downarrow\sigma_c$. Hence $\sigma_c\le\theta$.

By Problem 1.2.3 the real zeros do not exist, but Problem 1.2.4 does not use this, and in §1.4 the same argument will be applied where real zeros cannot be excluded in advance. By Problem 1.2.4, $F=1/(s-\theta)-H$ is holomorphic in $\operatorname{Re} s>\theta$, and by Problem 1.1.4 this excludes zeros there.

Theorem 1.2.1. *Let $0<\theta<1$. If $G(x)\le x^\theta$ for all large x, then $\zeta(s)\ne 0$ for $\operatorname{Re} s>\theta$.*

An upper bound obtained arithmetically will hold only for x in certain long intervals. The following theorem of Kaczorowski and Pintz [KP, Theorem 1 and Corollary 1], a quantitative form of Landau's theorem, handles this.

Theorem 1.2.2. *Let $f\colon(0,\infty)\to\mathbb{R}$ be locally integrable, and suppose that $\Phi(s):=\int_0^\infty f(x)x^{-s-1}\,dx$ converges absolutely for $\operatorname{Re} s$ large. Suppose that Φ is regular in $\operatorname{Re} s>\Theta$ but in no half-plane $\operatorname{Re} s>\Theta-\eta$ with $\eta>0$, that Φ is meromorphic in $\operatorname{Re} s\ge\Theta-c_0$ for some $c_0>0$, and that Φ is regular at the real point Θ. Then for every $\alpha>0$ and every $T\ge T_0(f,\alpha)$ there are points of $[T^{1-\alpha},T]$ at which f is nonzero and of opposite signs.*

The last hypothesis is needed: $f(x) = x^{\Theta}$ for $x \ge 1$ and $f(x) = 0$ for $x < 1$ satisfy the others, with $\Phi(s) = 1/(s - \Theta)$.

Problem 1.2.5. Let $0 < \theta < 1$ and suppose that ζ has a zero with real part greater than θ. Let Θ be the supremum of the real parts of the zeros of ζ. Let $f(x) := G(x) - 2x^{\theta}$ for $x \ge 1$ and $f(x) := 0$ for $x < 1$. Then f and Θ satisfy the hypotheses of Theorem 1.2.2.

If moreover $G(x) \le x^{\theta}$ on $[Y, Y^{3/2}]$, then $f(x) \le -x^{\theta} < 0$ there. Theorem 1.2.2 with $\alpha = 1/4$ and $T = Y^{3/2}$ finds a positive value of f in $[Y^{9/8}, Y^{3/2}]$ once Y is large. This proves the following; the exponent $3/2$ may be replaced by any number greater than 1.

Theorem 1.2.3. *Let $0 < \theta < 1$. If there are arbitrarily large Y such that $G(x) \le x^{\theta}$ for all $x \in [Y, Y^{3/2}]$, then $\zeta(s) \ne 0$ for $\operatorname{Re} s > \theta$.*

Problem 1.2.6. If for every $\varepsilon > 0$ there are arbitrarily large Y with $G(x) \le x^{1/2+\varepsilon}$ for all $x \in [Y, Y^{3/2}]$, then the Riemann hypothesis holds.

1.3. Smoothing, and the triangle weight

Why the ramp, and not another weight? Let $\phi\colon (0,\infty) \to \mathbb{R}$ be measurable, vanish on $(1,\infty)$, be bounded on compact subsets of $(0,1]$, and satisfy $\phi(t) = O(1/t)$ as $t \to 0$. Let

$$\psi_\phi(x) := \sum_n \Lambda(n)\phi(n/x), \qquad \tilde\phi(s) := \int_0^\infty \phi(t) t^{s-1}\, dt \quad (\operatorname{Re} s > 1).$$

We write $\tilde\phi(s)$ wherever the integral converges absolutely. The sharp cutoff $\phi = \mathbf{1}_{(0,1]}$ gives $\psi_\phi = \psi$ and $\tilde\phi(s) = 1/s$. The ramp is $\phi(t) = (1/t - 1)_+$, with $\tilde\phi(s) = 1/(s(s-1))$ by Problem 1.1.1. The *triangle weight* is $\phi(t) = (1-t)_+$, with $\psi_\phi(x) = x^{-1}\int_0^x \psi(t)\, dt$.

Problem 1.3.1. For $\operatorname{Re} s > 1$, $\int_0^\infty \psi_\phi(x) x^{-s-1}\, dx = -\frac{\zeta'}{\zeta}(s)\, \tilde\phi(s)$.

Problem 1.3.2. For the triangle weight $\tilde\phi(s) = 1/(s(s+1))$. If $\phi_1(t) := \phi(t)/t$, then $\tilde\phi_1(s) = \tilde\phi(s-1)$; the ramp is the triangle weight divided by t.

The following theorem is proved as Theorem 1.1.1: by Problem 1.3.1 with $\tilde\phi(s) = 1/(s(s+1))$, shift the line of integration to $\operatorname{Re} s = -3/2$ as in [Dav, Ch. 17], along horizontal segments at heights T where $\zeta'/\zeta \ll \log^2 T$. The factor $1/(s(s+1))$ makes every integral converge absolutely, and the poles of $-\frac{\zeta'}{\zeta}(s)\, x^s/(s(s+1))$ to the right of $-3/2$ are at 1, 0, -1 and the zeros ρ.

Theorem 1.3.1. *For $x > 1$,*

$$\sum_{n \le x} \Lambda(n)\Big(1 - \frac{n}{x}\Big) = \frac{x}{2} - \log 2\pi - \sum_\rho \frac{x^\rho}{\rho(\rho+1)} + O\Big(\frac{1}{x}\Big).$$

As a check, the left side minus $x/2 - \log 2\pi$ is 0.106, 7.975 and -12.115 at $x = 10^4$, 10^6 and $2 \cdot 10^6$. For the ramp and the triangle weight $|\tilde\phi(\rho)| \ll |\rho|^{-2}$, so the sums over zeros converge absolutely. For the sharp cutoff they do not, since $\sum_\rho 1/|\rho|$ diverges.

Problem 1.3.3. Let $0 < \theta < 1$. If $\sum_{n \le x} \Lambda(n)(1 - n/x) \le x/2 + x^\theta$ for all large x, then $\zeta(s) \neq 0$ for $\operatorname{Re} s > \theta$.

The two smooth weights differ in one respect. A real zero $\beta \in (0,1)$ contributes $-x^\beta/(\beta(\beta-1)) > 0$ to $G(x)$, but $-x^\beta/(\beta(\beta+1)) < 0$ to the sum of Theorem 1.3.1, so an upper bound for the latter does not see it. For ζ this is harmless by Problem 1.2.3; for Dirichlet L-functions it is not (§1.4).

The ramp itself is forced by the arithmetic. Theorem 17.3.1 gives, at primes $p \equiv 3 \pmod 8$, lower bounds for p-adic valuations of the form $\sum_{i \text{ odd}} (\lfloor a/p^i \rfloor - w)_+$. Up to an error of 1 for each odd i with $p^i \le a$, and with $a = wx$, the summand is $(a/p^i - w)_+ = w(x/p^i - 1)_+$, which is w times the ramp at x. Bounds of this form at every inert prime, summed with weights $\log p$, give the ramp sum over odd powers of primes inert in an imaginary quadratic field; this is the subject of the sequel, *Inert primes and Estimate B*.

1.4. Dirichlet L-functions

What replaces G for a Dirichlet L-function? For a primitive nonprincipal Dirichlet character χ let $c_\chi := \frac{L'}{L}(1, \chi)$ and

$$G_\chi(x) := \sum_{n \le x} \Lambda(n)\chi(n)\Big(\frac{x}{n} - 1\Big) + c_\chi x. \tag{3}$$

Let ψ be a primitive Dirichlet character of conductor M and order o; from here on ψ is this character, not Chebyshev's function. For $1 \le j < o$ let ψ_j be the primitive character inducing ψ^j. Let

$$G^\psi(y) := G(y) + \frac{1}{3o+1} \sum_{j=1}^{o-1} \operatorname{Re} G_{\psi_j}(y).$$

The weight $1/(3o+1)$ is the one that arises from averaging over a subgroup in *Fourier analysis on the Galois group* (Problem 4.5.4); the positivity of the coefficients in Problems 1.4.1 and 1.4.2 only uses that it lies between 0 and 1.

Problem 1.4.1. For n prime to M, the coefficient of $\Lambda(n)(y/n - 1)_+$ in $G^\psi(y)$ is $4o/(3o+1)$ if $\psi(n) = 1$, and $3o/(3o+1)$ otherwise. For a character of order 4 modulo 5 these are $16/13$ and $12/13$.

Problem 1.4.2. For $n = p^h$ with $p \mid M$ the coefficient is at least $3o/(3o+1)$. In particular every coefficient is positive.

Problem 1.4.3. $(3o+1)$ times the Mellin transform of G^ψ has, in $\operatorname{Re} s > 0$, simple poles only, at the zeros of $E(s) := \zeta(s)^{3o+1} \prod_{j=1}^{o-1} L(s, \psi_j)$ in the critical strip. At a zero ρ of E of multiplicity m the residue is $-m/\big(\rho(\rho-1)\big)$.

The multiplicities of the factors of E add, so no zero of ζ or of an $L(s, \psi_j)$ is cancelled. A real zero $\beta \in (0,1)$ of some $L(s, \psi_j)$ cannot be excluded in advance, but its residue $m/(\beta(1-\beta))$ is positive, and Problem 1.2.4 applies to it unchanged. As in Problem 1.2.2, $\sigma_c > -\infty$ because the nonreal zeros of ζ are poles. The argument of §1.2 therefore gives:

Theorem 1.4.1. *Let $0 < \theta < 1$. If $G^\psi(y) \le y^\theta$ for all large y, then $\zeta(s)$ and every $L(s, \psi_j)$, $1 \le j < o$, are nonzero for $\operatorname{Re} s > \theta$.*

For $o = 1$ this is Theorem 1.2.1. The version on intervals follows from Theorem 1.2.2 as before, unless the supremum Θ of the real parts of the zeros of E is itself a real zero in $(\theta, 1)$; then the last hypothesis of Theorem 1.2.2 fails. The next section treats that case.

1.5. A real zero on windows

Suppose that $G^\psi(y) \le y^\theta$ for all $y \in [Y, Y^{3/2}]$, for arbitrarily large Y, and that $\Theta = \beta \in (\theta, 1)$ is a real zero of E, of multiplicity m. Let $\mathcal{A}(s) := \int_1^\infty G^\psi(y) y^{-s-1}\, dy$, and for $\sigma_1 > 2$ and integers $k \ge 0$ let

$$T_k := \frac{(-1)^k}{k!}\mathcal{A}^{(k)}(\sigma_1), \qquad r := \sigma_1 - \beta, \qquad a := \sigma_1 - 2.$$

We follow [Adr, Thm. 2.4 and Cor. 2.5], where the nearest pole may also be a pair of complex zeros, and [Rub, §I.10 and Part II, §1].

Problem 1.5.1. (i) $|G^\psi(y)| \le By^2$ for $y \ge 1$, with B a constant. (ii) $T_k = \int_0^\infty G^\psi(e^u)\, e^{-\sigma_1 u} u^k/k!\, du$. (iii) For $p \ne \sigma_1$, the signed Taylor coefficient $\frac{(-1)^k}{k!}\frac{d^k}{ds^k}(s-p)^{-1}$ at $s = \sigma_1$ is $(\sigma_1 - p)^{-k-1}$.

Problem 1.5.2. For every real $\sigma_1 > 1$, β is the only zero of E in the closed disc $|s - \sigma_1| \le \sigma_1 - \beta$. The residue of $\mathcal{A}$ at β is $R := m/\big((3o+1)\beta(1-\beta)\big) > 0$.

Problem 1.5.3. There are $r'' > r$ and C with $|T_k - Rr^{-k-1}| \le Cr''^{-k}$ for all k. Hence $T_k \ge \frac{1}{2}Rr^{-k-1}$ for all large k.

So the nearest real pole contributes a positive term to *every* signed Taylor coefficient. A conjugate pair $p, \bar p$ would contribute $2|c|r^{-k-1}\cos(\alpha + (k+1)\varphi)$, of both signs, where $c = |c|e^{i\alpha}$ is the residue and $\sigma_1 - p = re^{-i\varphi}$; compare Problem 1.1.8. An upper bound for G^ψ bounds every T_k from above, because the kernel $e^{-\sigma_1 u}u^k/k!$ is positive. Only the window $[L, \frac{3}{2}L]$, $L := \log Y$, is controlled; but for k near $\frac{6}{5}aL$ the kernel $e^{-au}u^k/k!$ is concentrated near $u = k/a$, inside the window.

Problem 1.5.4. Let $U \ge 0$ and $a > 0$. If $b > a$, then $\int_0^U e^{-au}u^k/k!\, du \le e^{(b-a)U}b^{-k-1}$. If $0 < b < a$, then $\int_U^\infty e^{-au}u^k/k!\, du \le e^{-(a-b)U}b^{-k-1}$.

Problem 1.5.5. Let $\sigma_1 \ge 302$, let $L = \log Y \ge 1$ for one of the Y above, and let $k + 1 := \lceil \frac{6}{5}aL \rceil$. Then

$$T_k \le (\sigma_1 - \theta)^{-k-1} + 2Be^{-L}r^{-k-1}.$$

Problem 1.5.6. Problems 1.5.3 and 1.5.5 contradict each other as $Y \to \infty$. So the window bound excludes a real zero $\beta = \Theta$ in $(\theta, 1)$.

With Theorem 1.2.2 for the other case, this proves the version on intervals of Theorem 1.4.1.

Theorem 1.5.1. *Let $0 < \theta < 1$. If there are arbitrarily large Y such that $G^\psi(y) \le y^\theta$ for all $y \in [Y, Y^{3/2}]$, then $\zeta(s)$ and every $L(s, \psi_j)$, $1 \le j < o$, are nonzero for $\operatorname{Re} s > \theta$.*

Hints

§1.1.

1.1.2: Integrate each step of ψ. For the bound use Chebyshev's estimate.

1.1.3: Use Problem 1.1.1 termwise, and compute the transforms of $x\log x$ and x.

1.1.4: Use $-\zeta'/\zeta(s) = 1/(s-1) - \gamma + O(s-1)$ and expand $1/(s(s-1))$ at $s=1$.

1.1.5: The functional equation gives $\zeta'/\zeta(0) = \log 2\pi$.

1.1.6: Write $1/(\rho(1-\rho)) = 1/\rho + 1/(1-\rho)$ and use the value of B in [Dav, Ch. 12].

§1.2.

1.2.1: Split the integral at X_0.

1.2.2: Use Problem 1.1.2 for $\sigma_c \le 1$ and Problem 1.1.3 for the formula for H. For $\sigma_c > -\infty$ use Problem 1.1.4: ζ has infinitely many zeros in the critical strip [Dav, Ch. 12], so it has nonreal ones.

1.2.3: Consider $(1-2^{1-\sigma})\zeta(\sigma) = \sum_{n\ge1}(-1)^{n-1}n^{-\sigma}$.

1.2.4: By Landau's theorem σ_c is a singularity of H. Of the real points in $(\theta,1]$, only real zeros of ζ can be singular (Problem 1.1.4). Find the sign of the residue of F at a real zero $\beta\in(0,1)$, and compare with Problem 1.2.1.

1.2.5: $\Phi(s) = F(s) - 2/(s-\theta)$. For regularity at Θ use Problem 1.2.3, and remember $s=1$.

1.2.6: Use Theorem 1.2.3 and the symmetry $\rho\mapsto 1-\rho$.

§1.3.

1.3.1: Substitute $x = n/t$ in each term.

1.3.3: Repeat §1.2 with $\tilde\phi(s) = 1/(s(s+1))$, including the argument for $\sigma_c > -\infty$ in Problem 1.2.2, and compare the sign of the residue at a real zero with that in Problem 1.2.4; conclude with Problem 1.2.3.

§1.4.

1.4.1: $\sum_{j=0}^{o-1}\psi(n)^j$ is o or 0.

1.4.2: Factor ψ into its p-component and the rest, and decide for which j the value $\psi_j(p)$ is nonzero.

1.4.3: Since $\psi_{o-j} = \overline{\psi_j}$, the real parts may be dropped. Then the coefficient of $\Lambda(n)(y/n-1)_+$ in $(3o+1)G^\psi$ is that of $\Lambda(n)n^{-s}$ in $-E'/E$; handle the pole at $s=1$ as in Problem 1.1.4.

§1.5.

1.5.1: (i) Problem 1.1.2 and Chebyshev's bound. (ii) Substitute $y = e^u$ and differentiate under the integral.

1.5.2: Compare $|\sigma_1-\rho|^2$ with $(\sigma_1-\beta)^2$ for $\operatorname{Re}\rho\le\beta$; then Problem 1.4.3.

1.5.3: Subtract the principal part at β and use Cauchy's estimate on a slightly larger disc; Problem 1.5.1(iii).

1.5.4: Compare e^{-au} with e^{-bu} on the range, and use $\int_0^\infty e^{-bu}u^k/k!\,du = b^{-k-1}$.

1.5.5: Split at L and $\frac32 L$. On the outer ranges use Problem 1.5.1(i) and Problem 1.5.4 with $b=\frac54 r$ and $b=\frac34 a$, and $a<r<a+2$.

1.5.6: $\theta<\beta$, and $k\to\infty$ with L.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

[Alp] L. Alpöge, untitled note announcing a proof of the Riemann hypothesis, 20 November 2026, `https://alpo.ge/remark.pdf`.

[Dav] H. Davenport, *Multiplicative Number Theory*, 3rd ed., revised by H. L. Montgomery, Grad. Texts in Math. **74**, Springer, 2000.

[KP] J. Kaczorowski and J. Pintz, *Oscillatory properties of arithmetical functions. I*, Acta Math. Hungar. **48** (1986), 173–185.

[Lan] E. Landau, *Über einen Satz von Tschebyschef*, Math. Ann. **61** (1905), 527–550.

[MV] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge Stud. Adv. Math. **97**, Cambridge University Press, 2007.

[Rub] *The Alpöge–Claude proof of the Riemann hypothesis explained for the Riemann zeta function*, written by AI models at the behest of M. O. Rubinstein, working draft, 21 November 2026, `https://www.math.uwaterloo.ca/~mrubinst/alpoge_claude_grh/zeta-paper.pdf`.