THE SLOPE INEQUALITY IN THE MEAN

Prerequisites. Chapter 6, *The slope method*, whose notation we keep: normed spaces over a number field K, their degrees and slopes, the heights h and h_∞ of a linear map, and Theorems 6.4.1 and 6.5.1. The spectral theorem and the Hilbert–Schmidt norm $\|A\|_{\mathrm{HS}}^2 = \mathrm{tr}(A^*A)$. Northcott's theorem: a projective space over K has only finitely many points of bounded height. The prime number theorem, also in arithmetic progressions, with its classical error term. Section 7.4 refers to Chapters 12, 14, 16 and 17, but proves nothing from them.

Theorem 6.5.1 bounds a sum over primes by $\mu_{\max}$ of the target. Which part of the target can the image of one map reach, and how do many maps reach its mean slope?

7.1. The Harder–Narasimhan filtration

Theorem 6.4.1 has $\mu_{\max}(F)$ on the right, not $\mu(F)$. This cannot be avoided.

Problem 7.1.1. Let $T > 0$. Let $L = \mathbb{Z}$ with $\|1\| = e^{-T}$, and let $M = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with $e_1 \perp e_2$, $\|e_1\| = e^{-T}$ and $\|e_2\| = e^{T}$. The map $\varphi(1) = e_1$ is injective with $h(\varphi) = 0$, while $\mu(L) = T$ and $\mu(M) = 0$.

The image of one map is a subspace, and only $\mu_{\max}$ bounds the slope of a subspace. To see which part of the target the image can reach, one filters the target by slopes. A nonzero normed space V is *semistable* if $\mu(U) \le \mu(V)$ for every nonzero subspace U. Set $\deg 0 := 0$.

Problem 7.1.2. For subspaces U, U' of V, $\deg(U \cap U') + \deg(U + U') \ge \deg U + \deg U'$.

Problem 7.1.3. The slopes of the nonzero subspaces of V have a largest value, and among the subspaces of that slope there is one that contains all the others.

Applying Problem 7.1.3 to the quotient by that subspace, and so on, one obtains a filtration.

Theorem 7.1.1 (Stuhler, Grayson)**.** *A nonzero normed space V has a unique chain of subspaces $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l = V$ whose successive quotients are semistable with $\mu(V_1/V_0) > \cdots > \mu(V_l/V_{l-1})$. Every nonzero subspace of V has slope at most $\mu(V_1)$, and every nonzero quotient of V has slope at least $\mu(V/V_{l-1})$.*

See [St], [Gr], and [Ga] for global fields; for function fields this is the theorem of Harder and Narasimhan [HN]. The *slopes* of V are the numbers $d_1 \ge \cdots \ge d_r$ in which $\mu(V_i/V_{i-1})$ occurs $\dim(V_i/V_{i-1})$ times. Their sum is $\deg V$. Write $\mu_{\max}(V) := d_1$ and $\mu_{\min}(V) := d_r$. For real λ let $V^{\ge\lambda}$ be the largest V_i all of whose slopes are at least λ. By uniqueness, the filtrations of V_i and of V/V_i are induced by that of V. Hence, when the spaces in question are nonzero,

$$\dim V/V^{\ge\lambda} = \#\{j : d_j < \lambda\}, \qquad \mu_{\min}(V^{\ge\lambda}) \ge \lambda, \qquad \mu_{\max}(V/V^{\ge\lambda}) < \lambda. \quad (1)$$

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Problem 7.1.4. Let τ lie in the usual fundamental domain, $|\tau| \geq 1$ and $|\mathrm{Re}\,\tau| \leq \frac{1}{2}$. Then $\mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$ is semistable if and only if $\mathrm{Im}\,\tau \leq 1$. Otherwise V_1 is the line spanned by 1.

Problem 7.1.5. Let $\dim V = D$, and let s and $H \geq 0$ be real with $\mu_{\min}(V) \geq s - H$. Then

$$s - \mu(V) \leq \frac{1}{D}\sum_j (s - d_j)_+ = \int_0^H \frac{\#\{j : d_j < s - t\}}{D}\, dt.$$

Problem 7.1.6. Let $\varphi\colon V \to F$ be a nonzero map of normed spaces. Then $\mu_{\min}(V) \leq \mu_{\max}(F) + h(\varphi)$. If $U \subseteq V$ is a nonzero subspace with $\mu_{\min}(U) \geq \lambda + h(\varphi)$, then $\varphi(U) \subseteq F^{\geq\lambda}$.

So a map sends the parts of large slope of its source into the parts of large slope of its target.

7.2. Several maps to the same target

Theorem 6.5.1 involves $\mu_{\max}(W^\vee)$: a map of small rank sees only the part of its target that contains its image. To reach the mean slope of a target F, one uses a family of maps into F. A line of large slope in F could contain the image of every member of the family, so the number of maps alone gives nothing. What is needed is a bound for the codimension of the span of the images that survives when the sources are cut down.

Let F be a normed space of dimension $D > 0$. Let $\varphi_i\colon V_i \to F$, for i in a finite nonempty set I, be linear maps from nonzero normed spaces, and let h_i be real numbers with $h(\varphi_i) \leq h_i$ whenever $\varphi_i \neq 0$. Let $r_i := \dim V_i$ and $R := \sum_i r_i$. The *adjusted slopes* of the family are the R numbers $d - h_i$, where d runs over the slopes of V_i. Let s be real, $u > 0$ and $H \geq 0$, with

$$\sum_i r_i\big(\mu(V_i) - h_i\big) \geq Rs, \qquad \max_i\big(\mu_{\max}(V_i) - h_i\big) \leq s + u, \qquad \mu_{\min}(F) \geq s - H. \tag{2}$$

Thus the adjusted slopes have mean at least s and are at most $s + u$, and no slope of F lies more than H below s. The family satisfies the *rank hypothesis* with constants $\theta, c \geq 0$ if, for all subspaces $U_i \subseteq V_i$,

$$\mathrm{codim}_F \sum_i \varphi_i(U_i) \leq \Big(\theta + \frac{c}{R}\sum_i \mathrm{codim}_{V_i} U_i\Big)D. \tag{3}$$

For $t > 0$ let $U_i(t) := V_i^{\geq s - t + h_i}$.

Problem 7.2.1. The number $k(t) := \sum_i \mathrm{codim}\, U_i(t)$ of adjusted slopes below $s - t$ satisfies $k(t) \leq Ru/(u + t)$.

Problem 7.2.2. $\varphi_i(U_i(t)) \subseteq F^{\geq s - t}$ for every i. Hence, under (3), the number of slopes of F below $s - t$ is at most $\big(\theta + cu/(u + t)\big)D$.

Inserting Problem 7.2.2 in Problem 7.1.5, and using $\int_0^H u/(u + t)\, dt = u\log(1 + H/u)$, proves the following.

Theorem 7.2.1. *Assume* (2) *and* (3). *Then*

$$\mu(F) \geq s - \theta H - c\,u\log\Big(1 + \frac{H}{u}\Big).$$

Statements of this kind are part of Alpöge's argument [Alp]. Theorem 7.2.1, with this proof, is [Zh, Prop. 4.3], which is stated with the constants θ and c found at the end of Section 7.3.

Problem 7.2.3. Let $\varphi\colon V \to F$ be bijective with V semistable, and take $h_1 = h(\varphi)$. Then (3) holds with $\theta = 0$ and $c = 1$, and Theorem 7.2.1 with $u \to 0$ gives $\mu(F) \geq \mu(V) - h(\varphi)$.

This is Theorem 6.4.1 for φ, with $U = F$. If V is not semistable, then $u \geq \mu_{\max}(V) - \mu(V)$, and the theorem is weaker by $u\log(1 + H/u)$: this is the price of using only the dimensions of the spans. The share $u/(u+t)$ of discarded directions in Problem 7.2.1 varies with the level, and the integration uses this.

Problem 7.2.4. Let $\theta = 0$. Using in Problem 7.1.5 the trivial bound D for $t < t_0$ and the bound of Problem 7.2.2 at t_0 for $t \geq t_0$, where $0 < t_0 \leq H$, gives the loss $t_0 + cu(H - t_0)/(u + t_0)$, which is at least $2\sqrt{cu(H+u)} - (1+c)u$.

So when the a priori bound H is much larger than u, a single threshold costs about $\sqrt{uH}$, whereas Theorem 7.2.1 costs $u\log(H/u)$.

Problem 7.2.5. Let D be even, let $F = \bigoplus_{k=1}^{D} \mathbb{Z}v_k$ with orthogonal v_k of lengths e^{-s_k}, where $s_k = 0$ for half of the k and $s_k = T > 0$ for the others. Let $V_k = \mathbb{Z}$ with $\|1\| = e^{-s_k}$, and $\varphi_k(1) := v_k$. Then (3) holds with $\theta = 0$ and $c = 1$, and Theorem 7.2.1 with $s = u = H = T/2$ gives $\mu(F) \geq (1 - \log 2)\, T/2$.

Here $\mu(F) = T/2$. The surjection $\bigoplus_k V_k \to F$, through Theorem 7.1.1, gives only $\mu(F) \geq \min_k s_k = 0$.

7.3. The rank hypothesis from Gram matrices

How can one verify (3) when R is much larger than D? By a second moment at one complex place. Let $\mathcal{H}$ be a complex inner product space of dimension $D' > 0$. Let $B_i\colon X_i \to \mathcal{H}$, $i \in I$, be linear maps from inner product spaces of dimensions r_i, with $R := \sum_i r_i > 0$, and let $\gamma_i > 0$. Let

$$K_i := \gamma_i B_i^* B_i, \qquad Q := \sum_i \gamma_i B_i B_i^*,$$

and suppose that

$$\sum_i \|K_i - 1\|_{\mathrm{HS}}^2 \leq eR, \qquad \operatorname{tr}(Q^2) \leq (1+\eta)\frac{(\operatorname{tr} Q)^2}{D'}, \tag{4}$$

with $0 \leq e \leq \frac{1}{4}$ and $\eta \geq 0$. The first condition says that each weighted map nearly preserves lengths.

Problem 7.3.1. If $Q \geq 0$ and $Q \neq 0$, then $\operatorname{rank} Q \geq (\operatorname{tr} Q)^2/\operatorname{tr} Q^2$, with equality exactly when Q is a multiple of a projection. Moreover $\|Q - \operatorname{tr} Q/D'\|_{\mathrm{HS}}^2 = \operatorname{tr} Q^2 - (\operatorname{tr} Q)^2/D'$.

So the second condition in (4) says that Q is nearly scalar. Let $v_1, \ldots, v_R$ be the images under the maps $\sqrt{\gamma_i}B_i$ of orthonormal bases of the X_i. Then $Q = \sum_k v_k v_k^*$ and $\operatorname{tr} Q^2 = \sum_{k,l} |\langle v_k, v_l\rangle|^2$, so Q is scalar exactly when the v_k form a tight frame of $\mathcal{H}$. Benedetto and Fickus characterize the tight frames of $R \geq D'$ unit vectors as the minimizers of $\sum_{k,l} |\langle v_k, v_l\rangle|^2$ among R-tuples of unit vectors [BF], and Zhang

compares (4) with tight frames [Zh, §4.4]. When the B_i^* are evaluations at points x_i, $\langle Qf, f\rangle = \sum_i \gamma_i |f(x_i)|^2$, and Q is scalar exactly when the weighted points form a cubature formula that is exact on the functions $|f|^2$, $f \in \mathcal{H}$, as a spherical design is exact on polynomials of bounded degree [DGS].

Problem 7.3.2. Let $\mathcal{H}$ be the space of polynomials in $\mathbb{C}[z]$ of degree less than D', with $\|f\|^2 := \int_0^1 |f(e^{2\pi i\theta})|^2\, d\theta$. Let $N \geq D'$ and $\zeta := e^{2\pi i/N}$. For $0 \leq j < N$ let $X_j = \mathbb{C}$, let $B_j(1)$ be the polynomial with $\langle f, B_j(1)\rangle = f(\zeta^j)$ for all f, and $\gamma_j := 1/D'$. Then (4) holds with $e = \eta = 0$, and $Q = N/D'$.

Now let $Y_i \subseteq X_i$ be subspaces with $\sum_i \operatorname{codim} Y_i \leq \alpha R$, where $0 \leq \alpha \leq 1$. Let Z be the orthogonal complement of $\sum_i B_i(Y_i)$, let $w := \dim Z$, let Π be the orthogonal projection onto Z, and let $\tau := \operatorname{tr} Q$.

Problem 7.3.3. $|\tau - R| \leq R\sqrt{e}$, so that $\tau \geq R/2$.

Problem 7.3.4. $\operatorname{tr}(\Pi Q) \leq R\big(\alpha + \sqrt{\alpha e}\big)$.

Problem 7.3.5. $\operatorname{tr}(\Pi Q) \geq \tau w/D' - \tau\sqrt{\eta w/D'}$.

Combining Problems 7.3.3–7.3.5, and using $\sqrt{\eta D' w} \leq (w + \eta D')/2$, gives the following.

Theorem 7.3.1. *Under* (4), $\operatorname{codim}_{\mathcal{H}} \sum_i B_i(Y_i) \leq \big(\eta + 4\alpha + 4\sqrt{e}\big) D'$.

This is [Zh, Lemma 4.2]. To apply it to maps $\varphi_i \colon V_i \to F$ of normed spaces, fix an embedding $\sigma \colon K \to \mathbb{C}$, and suppose that there are a surjection $\pi \colon \mathcal{H} \to F \otimes_\sigma \mathbb{C}$ and isomorphisms $\iota_i \colon X_i \to V_i \otimes_\sigma \mathbb{C}$ with $\pi B_i = (\varphi_i \otimes 1)\iota_i$. If $D \geq (1-\delta)D'$ with $0 \leq \delta < 1$, then the rank hypothesis (3) holds with $\theta = (\eta + 4\sqrt{e})/(1-\delta)$ and $c = 4/(1-\delta)$, since a surjection does not increase codimension. The inner products on $\mathcal{H}$ and X_i need not be related to the norms that define the slopes: only dimensions are transferred. But the positivity of Q is essential, and so is the complex place.

7.4. The inputs

The slope method is due to Bost [Bo1], [Bo2]; see [CL, §§1.3–1.5] for an account that compares it with the classical ingredients of a transcendence proof, and [CDT, §2.3] for a recent use with lattices of polynomials. Alpöge's argument for the Riemann hypothesis for Dirichlet L-functions [Alp], see also [Zh, §§3–4] and the reconstruction [Adr], can be organized around Theorems 6.5.1, 7.2.1 and 7.3.1; in the reconstruction the inputs are the seven inequalities of [Adr, Prop. 3.1, (3.1)–(3.7)]. The sources are spaces V_α of algebraic automorphic forms on unitary groups of hermitian spaces over a CM field E containing an imaginary quadratic field K_0, defined over the totally real subfield E^+ of E [Zh, §1.3], the target F is the dual of a space of test forms, and the maps come from periods computed through a chain of theta lifts. The status in [Adr] of each input below is recorded in Sections 58.5 and 58.7 of Chapter 58.

The finite places. At a prime p inert in K_0, a jet of a theta function of order a, and of order w at the conjugate place, is divisible by about p to the power $\sum_{e \text{ odd}} (a/p^e - w)_+$; the simplest cases are Theorems 16.4.1 and 17.3.1 of Chapters 16 and 17, and Section 14.3 of Chapter 14. The gains are recorded as in Problems 6.5.2, 6.5.3 and 6.3.7. Let χ be the quadratic character of K_0.

Problem 7.4.1. Let $a \geq w > 0$ and $x := a/w$. Then

$$2 \sum_{\chi(p)=-1} \log p \sum_{e \text{ odd}} \Big(\frac{a}{p^e} - w\Big)_+ + \sum_{\chi(p)=0} \log p \sum_{e \geq 1} \Big(\frac{a}{p^e} - w\Big)_+ \\ = w \sum_{n \leq x} \Lambda(n)\big(1 - \chi(n)\big)\Big(\frac{x}{n} - 1\Big).$$

So the exponents are ramps in p^e, as asked for in Section 6.1, and only the primes inert or ramified in K_0 occur.

Problem 7.4.2. $\sum_{n \leq x} \Lambda(n)\big(1-\chi(n)\big)(x/n-1) = x\big(\log x - 1 - \gamma + \frac{L'}{L}(1,\chi)\big) + o(x)$, where γ is Euler's constant.

The error term in Problem 7.4.2 carries the information on the zeros of $\zeta(s)$ and $L(s,\chi)$, which enter with opposite signs.

The infinite places. The bound for $h_\infty(\varphi_\alpha)$ is a Gaussian norm as in Problem 6.4.4, together with the ath power of a period, which brings in γ and $\frac{L'}{L}(1,\chi)$ by the Chowla–Selberg formula (Chapter 12, Theorem 12.3.1, for $K_0 = \mathbb{Q}(\mathrm{i})$; Chapter 20, Problem 20.4.6, in general).

The slopes of the sources. By (2), Theorem 7.2.1 needs two bounds on the sources: a lower bound s for the mean of the adjusted slopes, and an upper bound $s + u$ for their maximum, which enters the loss through u. The slope of a source is that of a base lattice raised by the gains at the finite places, so the gains enter the lower bound; for the base lattice see [Adr, (3.2), (3.7) and §15.7].

The target. Theorem 7.2.1 bounds $\mu(F)$ from below, so the argument also needs an upper bound for $\mu(F)$, that is, for the covolume of the lattice of test forms whose dual is F. The analogue for sections of a line bundle is the arithmetic Hilbert–Samuel theorem [GS2, Thm. 8], whose simplest case is Problem 6.3.5; here the bound is needed with explicit errors while the groups and the field grow [Adr, §15.4]. Theorem 7.2.1 also uses the a priori bound $\mu_{\min}(F) \geq s - H$, and by Problem 7.2.4 the logarithm in Theorem 7.2.1 makes a large H cheap. In the reconstruction these inputs enter through [Adr, (3.3)], which comes from its frame theorem, and the height inequality [Adr, (3.4)], which bounds the slope of the target together with the heights of the period maps.

The rank hypothesis. This is Theorem 7.3.1 for the family of periods, whose simplest case is Problem 7.3.2. In the argument it comes from near orthogonality of Poincaré series of large weight, not from equidistribution of CM points; see Chapter 11, Section 11.4.

The main term in Problem 7.4.2 has size $x \log x$, and to reach the exponent $\frac{1}{2}$ every error term must be $O(x^{1/2+\varepsilon})$ for each $\varepsilon > 0$ [Zh, §6 and Remark 6.2].

Hints

§7.1.

- **7.1.2:** The map $U/(U \cap U') \to (U + U')/U'$ is bijective and has norm at most 1 at every place. Use Theorem 6.4.1.
- **7.1.3:** A subspace U of dimension k is a point of $\mathbb{P}(\wedge^k V)$, and $-\deg U$ is its height for the induced norms, which differs from the standard height by a bounded amount; use Northcott's theorem. Then use Problem 7.1.2.
- **7.1.4:** In the fundamental domain, the shortest nonzero vector of $\mathbb{Z} + \mathbb{Z}\tau$ is 1.

7.1.5: The layer-cake formula.

7.1.6: Apply Theorem 6.4.1 to the bijection from $V/\ker\varphi$ to $\varphi(V)$. For the second statement, compose with $F \to F/F^{\geq\lambda}$ and use (1).

§7.2.

7.2.1: Markov's inequality.

7.2.2: Problem 7.1.6 with $\lambda = s - t$, and (1) for F.

7.2.3: Here $R = D$, and $\operatorname{codim}\varphi(U) = \operatorname{codim} U$. For a semistable V one may take $s = \mu(V) - h(\varphi)$.

7.2.4: The arithmetic–geometric mean inequality.

7.2.5: $h(\varphi_k) = 0$, and the adjusted slopes are the s_k. For $\mu_{\min}(F) \geq 0$, note that every quotient of F is spanned by images of the v_k, which have length at most 1.

§7.3.

7.3.1: Apply the Cauchy–Schwarz inequality to the nonzero eigenvalues.

7.3.2: Orthogonality of the characters of $\mathbb{Z}/N\mathbb{Z}$.

7.3.3: Express τ through the K_i, then use the Cauchy–Schwarz inequality.

7.3.4: $\operatorname{tr}(\Pi Q) = \sum_i \operatorname{tr}(\gamma_i B_i^* \Pi B_i)$, and $\gamma_i B_i^* \Pi B_i$ is at most K_i and vanishes on Y_i.

7.3.5: Write $Q = \tau/D' + (Q - \tau/D')$, and use Problem 7.3.1 and $\|\Pi\|_{\mathrm{HS}} = \sqrt{w}$.

§7.4.

7.4.1: Sort the prime powers by the splitting of p in K_0.

7.4.2: $\sum_{n\leq x} \Lambda(n)/n = \log x - \gamma + o(1)$ and $\sum_{n\leq x} \Lambda(n)\chi(n)/n = -\frac{L'}{L}(1,\chi) + o(1)$, by the prime number theorem with its classical error term.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 25 November 2026, commit `9eefc08`.

[Alp] L. Alpöge, untitled note announcing a proof of the Riemann hypothesis, 20 November 2026, `https://alpo.ge/remark.pdf`.

[Bo1] J.-B. Bost, *Périodes et isogénies des variétés abéliennes sur les corps de nombres (d'après D. Masser et G. Wüstholz)*, Séminaire Bourbaki, Exp. 795, Astérisque **237** (1996), 115–161.

[Bo2] J.-B. Bost, *Algebraic leaves of algebraic foliations over number fields*, Publ. Math. IHÉS **93** (2001), 161–221.

[CDT] F. Calegari, V. Dimitrov and Y. Tang, *Arithmetic holonomy bounds and effective Diophantine approximation*, preprint, 2025, arXiv:2510.04156.

[BF] J. J. Benedetto and M. Fickus, *Finite normalized tight frames*, Adv. Comput. Math. **18** (2003), 357–385.

[CL] A. Chambert-Loir, *Théorèmes d'algébricité en géométrie diophantienne (d'après J.-B. Bost, Y. André, D. & G. Chudnovsky)*, Séminaire Bourbaki, Exp. 886, Astérisque **282** (2002), 175–209.

[DGS] P. Delsarte, J.-M. Goethals and J. J. Seidel, *Spherical codes and designs*, Geom. Dedicata **6** (1977), 363–388.

[Ga] É. Gaudron, *Pentes des fibrés vectoriels adéliques sur un corps global*, Rend. Sem. Mat. Univ. Padova **119** (2008), 21–95.

[Gr] D. R. Grayson, *Reduction theory using semistability*, Comment. Math. Helv. **59** (1984), 600–634.

[GS2] H. Gillet and C. Soulé, *An arithmetic Riemann–Roch theorem*, Invent. Math. **110** (1992), 473–543.

[HN] G. Harder and M. S. Narasimhan, *On the cohomology groups of moduli spaces of vector bundles on curves*, Math. Ann. **212** (1975), 215–248.

[St] U. Stuhler, *Eine Bemerkung zur Reduktionstheorie quadratischer Formen*, Arch. Math. **27** (1976), 604–610.

[Zh] R. Zhang, *A preliminary note on Alpöge's proposed proof of the generalized Riemann hypothesis*, preprint, 2026, hexagon:2611.00809v2, `https://hexagonmath.org/2611.00809v2`.