COMPLEX TORI, THETA FUNCTIONS AND PERIODS

Prerequisites. Complex analysis: Liouville's theorem, the argument principle, Weierstrass products, and Euler's Gamma function with the Beta integral and the reflection formula. The Gaussian integers. Elliptic curves over $\mathbb{C}$ at the level of the statements of [Si1, Ch. VI]. No earlier chapter is used. Chapter 14, *Complex multiplication and reduction at an inert prime*, continues this one.

12.1. Complex tori

How does one compute on a curve of genus one over $\mathbb{C}$? One uniformizes it. Let $\Lambda \subset \mathbb{C}$ be a lattice, and for $k \geq 3$ let $G_k(\Lambda) := \sum'_{\omega\in\Lambda} \omega^{-k}$, where the prime means that $\omega = 0$ is omitted. The Weierstrass function

$$\wp(u) := \frac{1}{u^2} + \sum_{\omega\in\Lambda}{}' \Big(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2} \Big)$$

is elliptic, that is meromorphic and Λ-periodic. Let $g_2 := 60G_4(\Lambda)$ and $g_3 := 140G_6(\Lambda)$. The map $u \mapsto (\wp(u), \wp'(u))$ is an analytic isomorphism of groups from $\mathbb{C}/\Lambda$ onto the curve $y^2 = 4x^3 - g_2 x - g_3$, and every elliptic curve over $\mathbb{C}$ arises in this way [Si1, §§VI.3–VI.5].

Problem 12.1.1. $\wp(u) = u^{-2} + \sum_{k\geq 1} (2k+1) G_{2k+2}(\Lambda)\, u^{2k}$, and $\wp'^2 = 4\wp^3 - g_2\wp - g_3$.

Let

$$\zeta(u) := \frac{1}{u} + \sum_{\omega\in\Lambda}{}' \Big(\frac{1}{u-\omega} + \frac{1}{\omega} + \frac{u}{\omega^2} \Big), \qquad \sigma(u) := u \prod_{\omega\in\Lambda}{}' \Big(1 - \frac{u}{\omega}\Big) e^{u/\omega + u^2/2\omega^2}.$$

Then $\zeta' = -\wp$ and $\sigma'/\sigma = \zeta$. The function σ is entire and odd, with simple zeros exactly at the points of Λ.

Problem 12.1.2. There is a $\mathbb{Z}$-linear map $\eta\colon \Lambda \to \mathbb{C}$ with $\zeta(u+\omega) = \zeta(u) + \eta(\omega)$.

Problem 12.1.3. If ω_1, ω_2 is a basis of Λ with $\operatorname{Im}(\omega_2/\omega_1) > 0$, then $\eta(\omega_1)\omega_2 - \eta(\omega_2)\omega_1 = 2\pi\mathrm{i}$.

This is Legendre's relation.

Problem 12.1.4. $\sigma(u+\omega) = \psi(\omega)\, e^{\eta(\omega)(u+\omega/2)} \sigma(u)$ for $\omega \in \Lambda$, where $\psi(\omega) = 1$ if $\omega \in 2\Lambda$ and $\psi(\omega) = -1$ otherwise.

Problem 12.1.5. $\dfrac{\sigma(u+v)\,\sigma(u-v)}{\sigma(u)^2\,\sigma(v)^2} = \wp(v) - \wp(u)$.

Problem 12.1.6. $\zeta(u+v) - \zeta(u) - \zeta(v) = \dfrac{1}{2}\,\dfrac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}$. Consequently

$$\wp(u+v) = \frac{1}{4}\Big(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\Big)^2 - \wp(u) - \wp(v).$$

Date: November 24, 2026.

Now let Λ be *square*, that is $\mathrm{i}\Lambda = \Lambda$.

Problem 12.1.7. If $\mathrm{i}\Lambda = \Lambda$, then $\wp(\mathrm{i}u) = -\wp(u)$, $g_3 = 0$, and $\eta(\mathrm{i}\omega) = -\mathrm{i}\,\eta(\omega)$.

Problem 12.1.8. There is exactly one real $\lambda > 0$ with $g_2(\lambda\mathbb{Z}[\mathrm{i}]) = 4$.

Let $\varpi := 2\int_0^1 (1-t^4)^{-1/2}\,dt = 2.62205\ldots$ be the lemniscate constant.

Problem 12.1.9. The lattice $\varpi\mathbb{Z}[\mathrm{i}]$ has $g_2 = 4$ and $g_3 = 0$. Moreover $\varpi = \Gamma(\frac{1}{4})^2/(2\sqrt{2\pi})$.

From now on, *the lemniscatic lattice* is $\varpi\mathbb{Z}[\mathrm{i}]$, and $\wp$, ζ, σ without further mention are its Weierstrass functions. With $x = \wp$ and $y = \wp'/2$ it is the lattice of the curve $E\colon y^2 = x^3 - x$, and $dx/2y$ pulls back to du.

12.2. Theta functions

The function σ is not periodic, but by Problem 12.1.4 it changes by an exponential factor under the lattice. Entire functions with such a law are theta functions. How many are there, and how are they made?

Let τ lie in the upper half-plane, and let $\vartheta(z) := \sum_{n\in\mathbb{Z}} e^{\pi\mathrm{i}n^2\tau + 2\pi\mathrm{i}nz}$.

Problem 12.2.1. $\vartheta(z+1) = \vartheta(z)$, $\vartheta(z+\tau) = e^{-\pi\mathrm{i}\tau - 2\pi\mathrm{i}z}\vartheta(z)$, and $\vartheta(-z) = \vartheta(z)$.

Problem 12.2.2. In each period parallelogram of $\mathbb{Z} + \mathbb{Z}\tau$, ϑ has exactly one zero. It is simple, and it lies at $(1+\tau)/2$ modulo $\mathbb{Z} + \mathbb{Z}\tau$.

Problem 12.2.3. Let σ and η belong to $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. Then there is a constant c with

$$\sigma(u) = c\,e^{\eta(1)u^2/2 + \pi\mathrm{i}u}\,\vartheta\big(u + \tfrac{1+\tau}{2}\big).$$

The theta functions of Chapter 16 are built from the Gaussian integers:

$$\Theta(z) := \sum_{\lambda\in\mathbb{Z}[\mathrm{i}]} e^{-\pi|\lambda|^2 + \lambda z}, \qquad \theta(z) := \sum_{a\in\mathbb{Z}} e^{-\pi a^2 + az}.$$

Problem 12.2.4. $\Theta(z) = \theta(z)\,\theta(\mathrm{i}z)$, $\Theta(\mathrm{i}z) = \Theta(z)$, and for $\beta \in \mathbb{Z}[\mathrm{i}]$

$$\Theta(z + 2\pi\beta) = e_\beta(z)\,\Theta(z), \qquad e_\beta(z) := e^{\pi|\beta|^2 + \bar\beta z}.$$

Let T be the space of entire functions f with $f(z + 2\pi\beta) = e_\beta(z)f(z)$ for all $\beta \in \mathbb{Z}[\mathrm{i}]$.

Problem 12.2.5. $\dim T = 2$, and every $f \in T$ is even.

Problem 12.2.6. A nonzero $f \in T$ has exactly two zeros, counted with multiplicity, in each period parallelogram of $2\pi\mathbb{Z}[\mathrm{i}]$.

Problem 12.2.7. If $f \in T$, then $f(\mathrm{i}z) \in T$. The $f \in T$ with $f(\mathrm{i}z) = f(z)$ form the line $\mathbb{C}\,\Theta$.

The last fact about σ that we need is the theorem of the cube [Mu, §6, §10], which for σ takes the following elementary form.

Problem 12.2.8. The function

$$\Xi(x,y,w) := \frac{\sigma(x+y+w)\,\sigma(x)\,\sigma(y)\,\sigma(w)}{\sigma(x+y)\,\sigma(y+w)\,\sigma(w+x)}$$

is elliptic in each variable separately, and $1/\Xi = \zeta(x) + \zeta(y) + \zeta(w) - \zeta(x+y+w)$.

The Taylor coefficients of Θ at 0 are not rational. Chapter 16 expresses Θ through σ and $\wp$ (Problem 16.1.3); then the nth derivative at 0 is $\Theta(0)$ times a rational multiple of $(\varpi/2\pi)^n$. So one needs to know ϖ, and the rational numbers that $\wp$ produces.

12.3. Periods

By Problem 12.1.1, the Laurent coefficients of $\wp$ are the lattice sums G_k. For the lemniscatic lattice they are rational. Write μ for an element of $\mathbb{Z}[\mathrm{i}]$.

Problem 12.3.1. $G_k(\varpi\mathbb{Z}[\mathrm{i}]) \in \mathbb{Q}$ for every $k \geq 3$, and $G_k = 0$ unless $4 \mid k$. Define the Hurwitz numbers H_{4k} by

$$\sum_{\mu}{}' \mu^{-4k} = \frac{(2\varpi)^{4k}}{(4k)!}\, H_{4k}.$$

Then $H_4 = \frac{1}{10}$ and $H_8 = \frac{3}{10}$.

When $\bar{\mu}$ also appears, π enters. Summed first over m and then over n, Eisenstein's series $G_2(\tau) := \sum_n \sum_m{}' (m+n\tau)^{-2}$ satisfies $G_2(-1/\tau) = \tau^2 G_2(\tau) - 2\pi\mathrm{i}\tau$ [We].

Problem 12.3.2. $G_2(\mathrm{i}) = \pi$.

For integers $j \geq 0$ and $k > j+2$, let $S(j,k) := \sum'_\mu \bar{\mu}^j \mu^{-k}$.

Problem 12.3.3. $S(j,k) = 0$ unless $j + k \equiv 0 \pmod 4$, and $S(0,k) = \varpi^k G_k(\varpi\mathbb{Z}[\mathrm{i}])$.

Damerell proved that $S(j,k)$ is an algebraic multiple of $\varpi^{j+k}\pi^{-j}$ [Dam]; for instance $S(1,7) = \frac{2}{315}\varpi^8/\pi$.

How big is ϖ in terms of classical constants? Let χ be the non-trivial character modulo 4. In the rest of this section, $\eta(\tau) := e^{\pi\mathrm{i}\tau/12}\prod_{n\geq 1}(1-e^{2\pi\mathrm{i}n\tau})$ is Dedekind's function, and $\zeta(s)$, with argument s, is Riemann's.

Problem 12.3.4. For $\operatorname{Re} s > 1$, $\sum'_\mu |\mu|^{-2s} = 4\,\zeta(s)\,L(s,\chi)$.

Kronecker's first limit formula states that, for $\tau = x + \mathrm{i}y$ with $y > 0$,

$$\sum_{(m,n)\neq(0,0)} \frac{y^s}{|m+n\tau|^{2s}} = \frac{\pi}{s-1} + 2\pi\big(\gamma - \log 2 - \log(\sqrt{y}\,|\eta(\tau)|^2)\big) + O(s-1)$$

[La, Ch. 20]; Theorem 20.1.1 of Chapter 20 proves it.

Problem 12.3.5. $L'(1,\chi)/L(1,\chi) = \gamma - 2\log 2 - 4\log\eta(\mathrm{i})$.

The discriminant $\Delta(\Lambda) := g_2^3 - 27g_3^2$ satisfies $\Delta(\lambda\Lambda) = \lambda^{-12}\Delta(\Lambda)$ and $\Delta(\mathbb{Z} + \mathbb{Z}\tau) = (2\pi)^{12}\eta(\tau)^{24}$ [SeA, §VII.4.4].

Problem 12.3.6. $\eta(\mathrm{i})^2 = \varpi/(\sqrt{2}\,\pi)$, and so $\eta(\mathrm{i}) = \Gamma(\frac{1}{4})/(2\pi^{3/4})$.

Problems 12.1.9, 12.3.5 and 12.3.6 prove the following.

Theorem 12.3.1. $\dfrac{L'(1,\chi)}{L(1,\chi)} = \gamma + 2\log 2 + 3\log\pi - 4\log\Gamma(\frac{1}{4}) = 0.24560\ldots$

Chowla and Selberg found the general form [CS]. Chapter 20 proves it, together with Kronecker's limit formula, and states the period relation of Gross [Gr]: the periods of an elliptic curve over $\bar{\mathbb{Q}}$ with complex multiplication by $\mathcal{O}_K$ are algebraic multiples of an explicit product Ω_K of values of Γ. For $K = \mathbb{Q}(\mathrm{i})$ this product is $\sqrt{\pi}\,\Gamma(\frac{1}{4})/\Gamma(\frac{3}{4})$.

Problem 12.3.7. $\sqrt{\pi}\,\Gamma(\frac{1}{4})/\Gamma(\frac{3}{4}) = 2\varpi$. Moreover

$$\Omega := \frac{2\varpi^2}{\pi} = \frac{\Gamma(\frac{1}{4})^4}{4\pi^2} = 4\pi\,\eta(\mathrm{i})^4.$$

Hints

§12.1.

12.1.1: Expand $(u-\omega)^{-2} - \omega^{-2}$ in powers of u. The function $\wp'^2 - 4\wp^3 + g_2\wp + g_3$ is elliptic, has no poles, and vanishes at 0.
12.1.2: The derivative of $\zeta(u+\omega) - \zeta(u)$ is 0.
12.1.3: Integrate ζ around a period parallelogram whose boundary avoids Λ.
12.1.4: The logarithmic derivative of $\sigma(u+\omega)/\sigma(u)$ is $\eta(\omega)$. To find the constant for $\omega \notin 2\Lambda$, use that σ is odd; for $\omega \in 2\Lambda$, apply the law twice with $\omega/2$.
12.1.5: Both sides are elliptic in u with the same zeros and poles. Compare the leading terms at $u = 0$.
12.1.6: Take the logarithmic derivative of Problem 12.1.5 in u and in v, and add. For the second formula, differentiate the first in u.
12.1.7: $G_k(\mathrm{i}\Lambda) = \mathrm{i}^{-k} G_k(\Lambda)$.
12.1.8: $g_2(\lambda\Lambda) = \lambda^{-4} g_2(\Lambda)$, and the values of $\wp$ at the three half-periods of a rectangular lattice are real, distinct, and are the roots of $4x^3 - g_2 x - g_3$.
12.1.9: Let λ be as in Problem 12.1.8. Then $\wp(\mathrm{i}\lambda/2) = -\wp(\lambda/2)$ and $\wp((1+\mathrm{i})\lambda/2) = 0$. On $(0, \lambda/2]$, $\wp$ is real and decreasing, and $\wp'^2 = 4\wp^3 - 4\wp$. Substitute $x = t^{-2}$. For the Gamma value use the Beta integral and $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \pi\sqrt{2}$.

§12.2.

12.2.1: Replace n by $n+1$ and by $-n$.
12.2.2: Argument principle, as in Problem 12.1.3. At $z = (1+\tau)/2$, pair the terms n and $-1-n$.
12.2.3: Both sides have simple zeros exactly at Λ, and the same law under $u \mapsto u+1$ and $u \mapsto u+\tau$, by Problem 12.1.4 and Legendre's relation.
12.2.4: Write $\lambda = a + b\mathrm{i}$. For the law, substitute $\lambda = \mu + \bar{\beta}$ and note $\mathrm{Im}(\mu\beta) \in \mathbb{Z}$.
12.2.5: $e^{-z^2/4\pi} f(z)$ has period 2π. Expand it in a Fourier series $\sum_n a_n e^{\mathrm{i}nz}$, and use $\beta = \mathrm{i}$.
12.2.6: Integrate f'/f around a parallelogram with sides 2π and $2\pi\mathrm{i}$.
12.2.7: If $f(\mathrm{i}z) = f(z)$, only powers z^{4k} occur in the Taylor series of f. Use Problem 12.2.6 and $\Theta(0) > 0$.
12.2.8: Use Problem 12.1.4 in x. Both Ξ and $1/(\zeta(x) + \zeta(y) + \zeta(w) - \zeta(x+y+w))$ are elliptic in x with zeros at 0, $-y-w$ and poles at $-y$, $-w$.

§12.3.

12.3.1: $g_2 = 4$ and $g_3 = 0$. Differentiate $\wp'^2 = 4\wp^3 - 4\wp$ to get $\wp'' = 6\wp^2 - 2$, and compare coefficients in Problem 12.1.1.

12.3.3: Replace μ by $\mathrm{i}\mu$.

12.3.4: Count the μ of norm n by unique factorization in $\mathbb{Z}[\mathrm{i}]$.

12.3.5: Take $\tau = \mathrm{i}$, and compare with the Laurent expansion of $4\zeta(s)L(s,\chi)$ at $s = 1$. $L(1,\chi) = \pi/4$.

12.3.6: $\Delta(\varpi\mathbb{Z}[\mathrm{i}]) = 64$, and $\eta(\mathrm{i}) > 0$.

12.3.7: $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \pi\sqrt{2}$, and Problems 12.1.9 and 12.3.6.

References

[CS] A. Selberg and S. Chowla, *On Epstein's zeta-function*, J. reine angew. Math. **227** (1967), 86–110.

[Dam] R. M. Damerell, *L-functions of elliptic curves with complex multiplication. I*, Acta Arith. **17** (1970), 287–301.

[Gr] B. H. Gross, *On the periods of abelian integrals and a formula of Chowla and Selberg*, Invent. Math. **45** (1978), 193–211.

[La] S. Lang, *Elliptic Functions*, 2nd ed., Grad. Texts in Math. 112, Springer, 1987.

[Mu] D. Mumford, *Abelian Varieties*, Tata Inst. Fund. Res. Stud. Math. 5, Oxford Univ. Press, 1970.

[SeA] J.-P. Serre, *A Course in Arithmetic*, Grad. Texts in Math. 7, Springer, 1973.

[Si1] J. H. Silverman, *The Arithmetic of Elliptic Curves*, 2nd ed., Grad. Texts in Math. 106, Springer, 2009.

[We] A. Weil, *Elliptic Functions according to Eisenstein and Kronecker*, Ergeb. Math. Grenzgeb. 88, Springer, 1976.