DELIGNE'S BOUND BY THE SIGN-ONLY METHOD

Prerequisites. Chapter S1, *The sign-only proof of the Riemann hypothesis for curves*, whose notation, background and results are used throughout. Constructible ℓ-adic sheaves [SGA4$\frac{1}{2}$, Rapport], [Mil]. Noether normalization, and generic smoothness of a reduced scheme of finite type over a perfect field.

S2.1. The lowest degree

Nothing in §S1.3 used $\dim U = 1$ except the degrees of the cohomology groups, the finite map π to $\mathbb{A}^1$ and the shape of the coupling $\pi(u_i)\pi(u_{i+1})$. What does the sign-only method give for U of dimension d? Fix $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. A constructible sheaf $\mathcal{F}$ on X has *weights at most* w if at every closed point x all eigenvalues of the local Frobenius have absolute value at most $|\kappa(x)|^{w/2}$. It is *pure of weight* w if all these absolute values equal $|\kappa(x)|^{w/2}$.

Theorem S2.1.1 (Deligne)**.** *Let X be separated of finite type over $\mathbb{F}_q$, and $\mathcal{F}$ a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on X.*

(i) *If $\mathcal{F}$ has weights at most w, every eigenvalue of* Fr *on $H^i_c(X,\mathcal{F})$ has absolute value at most $q^{(w+i)/2}$.*

(ii) *If X is smooth and proper and $\mathcal{F}$ is lisse and pure of weight w, every eigenvalue of* Fr *on $H^i(X,\mathcal{F})$ has absolute value $q^{(w+i)/2}$.*

These are theorems of Deligne [Del74], [Del80, Théorème 3.3.1]; there (i) is stated for mixed sheaves, those with a filtration whose quotients are pure. By a result of Laumon and Deligne's dévissage, (i) follows from the case of a lisse sheaf on an open subset of $\mathbb{A}^1$ [Zh, Remark 1.3], and that case is written out by the sign-only method in [Zh]. The proof below instead treats every dimension directly. It is that of [Ald], and as recorded in Section S1.1, no human has checked that text. Zhang states that the argument adapts to this setting, but he does not write it out [Zh, Remark 1.3].

Problem S2.1.1. For $X = \mathbb{A}^d$ and $\mathcal{F} = \overline{\mathbb{Q}}_\ell$, $H^i_c(X,\overline{\mathbb{Q}}_\ell) = 0$ for $i \neq 2d$, and Fr acts on the line $H^{2d}_c(X,\overline{\mathbb{Q}}_\ell)$ by q^d. So (i) is attained for $w = 0$, $i = 2d$.

Let U be smooth and affine of pure dimension $d \geq 1$ over $\mathbb{F}_q$, and $\mathcal{F}$ lisse on U with $|\iota \operatorname{Tr}(\mathrm{Fr}_u \mid \mathcal{F}_{\bar{u}})| \leq C q^{nw/2}$ for all n and $u \in U(\mathbb{F}_{q^n})$. Let $\pi\colon U \to \mathbb{A}^d$ be finite (Noether normalization), B the number of generators of $\mathcal{O}(U)$ as an $\mathcal{O}(\mathbb{A}^d)$-module, and $z_1,\dots,z_s$ generators of the $\mathbb{F}_q$-algebra $\mathcal{O}(U)$. Let b be a nondegenerate bilinear form on $\mathbb{A}^d$, and for $k \geq 2$, $a = (a_{ij}) \in \mathbb{A}^{ks}$ and $c \in \mathbb{A}^1$ let $Y_{a,c} \subset U^k$ be the zero set of

$$\sum_{i=1}^{k-1} b\big(\pi(u_i),\pi(u_{i+1})\big) + \sum_{i=1}^{k}\sum_{j=1}^{s} a_{ij}\, z_j(u_i) + c.$$

This is the note's "vectorize to go up dimensions" [Alp, footnote 3]. Let $X := U^k$, $D := kd$, $\mathcal{F}_k := \mathcal{F}^{\boxtimes k}$, and, when (a,c) has coordinates in $\mathbb{F}_r$, $r = q^e$, define M_n,

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N_n, R_n and K for $Y = Y_{a,c}$ as in §S1.3, with sums over the points with values in $\mathbb{F}_{r^n}$, and with H_c^{D-1} and H_c^{D+1} in the definition of K.

Problem S2.1.2. Let E be a finite field with r elements, ψ a nontrivial additive character of E, $b_1, \dots, b_{k-1}$ nondegenerate bilinear forms on E^d, and $t_1, \dots, t_k \colon E^d \to \mathbb{C}$ bounded by T. For $h \in E^\times$ and $c \in E$,

$$\Big| \sum_{x \in (E^d)^k} t_1(x_1) \cdots t_k(x_k)\, \psi\Big(h \sum_{i=1}^{k-1} b_i(x_i, x_{i+1}) + hc\Big) \Big| \le T^k r^{d(k+1)/2}.$$

This is [Ald, Lemma 3].

Problem S2.1.3. The map $u \mapsto \big(z(u_1), \dots, z(u_k), \sum_i b(\pi(u_i), \pi(u_{i+1}))\big)$ is a closed immersion $U^k \to \mathbb{A}^{ks+1}$, and the $Y_{a,c}$ are its hyperplane sections with last coefficient 1.

Problem S2.1.4. Over every finite extension of $\mathbb{F}_q$, each fibre of π has at most B rational points.

Problem S2.1.5. $H_c^j(X, \mathcal{F}_k) = 0$ for $j < D$, and $H_c^D(X, \mathcal{F}_k) = H_c^d(U, \mathcal{F})^{\otimes k}$.

By Problems S1.3.4 and S2.1.3 with $\mathcal{G} = \mathcal{F}_k^\vee$, there are $e \ge 1$ and (a, c) over $\mathbb{F}_r$, $r := q^e$, for which $Y := Y_{a,c}$ has property (WL) for $\mathcal{F}_k$.

Problem S2.1.6. For all $n \ge 1$, $(-1)^{D-1} R_n = \mathrm{Tr}\big(\mathrm{Fr}^{en} \mid H_c^d(U, \mathcal{F})^{\otimes k}\big) + \mathrm{Tr}\big(\mathrm{Fr}^{en} \mid K\big)$, where Fr^e is the geometric Frobenius over $\mathbb{F}_r$.

Problem S2.1.7. $|R_n| \le (BC)^k\, r^{n(k(w+d)/2+d/2+1)}$.

Problems S2.1.6, S2.1.7 and S1.3.8, the latter with its a equal to $(w+d)/2$ and $c_k = d/2 + 1$, prove:

Theorem S2.1.2 ([Ald, Theorem 9])**.** *Every eigenvalue of* Fr *on* $H_c^d(U, \mathcal{F})$ *has absolute value at most* $q^{(w+d)/2}$.

Below degree d there is nothing to prove, since $H_c^i(U, \mathcal{F}) = 0$ for $i < d$. The degrees above d are reached by induction on the dimension.

S2.2. Induction on the dimension

Problem S2.2.1. $H_c^1(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ is a line on which Fr acts by 1, and $H_c^2(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ a line on which it acts by q. So the inequality in (i) can be strict.

Problem S2.2.2. Theorem S2.1.1(i) holds when X has dimension 0.

Problem S2.2.3. A bound $|\alpha| \le q^{(w+i)/2}$ for the eigenvalues of Fr on $H_c^i(X, \mathcal{F})$ may be proved after replacing $\mathbb{F}_q$ by $\mathbb{F}_{q^e}$, with Fr^e and q^e in place of Fr and q; weights at most w are preserved.

Problem S2.2.4. Let $U \subset X$ be open with closed complement Z. Every eigenvalue of Fr on $H_c^i(X, \mathcal{F})$ is an eigenvalue on $H_c^i(U, \mathcal{F})$ or on $H_c^i(Z, \mathcal{F})$.

Problem S2.2.5. Every X of dimension d has a dense open subscheme U such that U_{red} is a finite disjoint union of smooth affine irreducible schemes and $\mathcal{F}$ is lisse on U.

Problem S2.2.6. If $\mathcal{F}$ is lisse of rank ρ on U with weights at most w, then $|\iota\, \mathrm{Tr}(\mathrm{Fr}_u \mid \mathcal{F}_{\bar{u}})| \le \rho\, q^{nw/2}$ for all n and $u \in U(\mathbb{F}_{q^n})$.

Problem S2.2.7. Let X be smooth and affine of pure dimension $d \geq 1$, $\mathcal{F}$ lisse, and $i > d$. Let $Y \subset X$ be a hyperplane section with property (WL) for $\mathcal{F}$, defined over $\mathbb{F}_q$ (after replacing $\mathbb{F}_q$ by a finite extension). If the eigenvalues of Fr on $H_c^{i-2}(Y, \mathcal{F})$ have absolute value at most $q^{(w+i-2)/2}$, then those on $H_c^i(X, \mathcal{F})$ have absolute value at most $q^{(w+i)/2}$.

Problems S2.2.2–S2.2.7 and Theorem S2.1.2, with Problem S1.3.4 applied to an embedding of X and to $\mathcal{F}^\vee$ to produce the hyperplane sections of Problem S2.2.7, prove Theorem S2.1.1(i) by induction on $\dim X$ [Ald, §8].

Problem S2.2.8. Theorem S2.1.1(i) implies Theorem S2.1.1(ii).

Hints

§S2.1.

S2.1.1: The Künneth formula, and $H_c^*(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell(-1)$ in degree 2.

S2.1.2: As in Problem S1.1.6; each matrix $\big(\psi(h\, b_i(x,y))\big)_{x,y \in E^d}$ times its adjoint is r^d, as in Problem S1.1.2.

S2.1.3: As in Problem S1.3.5.

S2.1.4: A fibre is the spectrum of an algebra spanned by B elements.

S2.1.5: The Künneth formula and the vanishing of $H_c^j(U, \mathcal{F})$ for $j < d$.

S2.1.6: As in Problem S1.3.2, with Problems S1.3.1 and S2.1.5.

S2.1.7: As in Problems S1.3.6 and S1.3.7, with Problems S2.1.2 and S2.1.4 and $T = BC\, r^{nw/2}$.

§S2.2.

S2.2.1: The exact sequence of $\mathbb{G}_m \subset \mathbb{A}^1$ and the point 0.

S2.2.2: Frobenius permutes cyclically the geometric points over a closed point of degree m, and its m-th power acts on each stalk as the local Frobenius.

S2.2.3: Compare the eigenvalues of Fr and of Fr^e.

S2.2.4: The long exact sequence of $0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$.

S2.2.5: Cohomology does not see nilpotents. Remove the intersections of the irreducible components, then use constructibility and generic smoothness.

S2.2.6: A point $u \in U(\mathbb{F}_{q^n})$ lies over a closed point of degree $m \mid n$, and Fr_u is the (n/m)-th power of its local Frobenius.

S2.2.7: Problem S1.3.1.

S2.2.8: Apply (i) to $\mathcal{F}^\vee$, which is pure of weight $-w$, and use Poincaré duality.

References

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