THE SIGN-ONLY PROOF OF THE RIEMANN HYPOTHESIS FOR CURVES

Prerequisites. Finite fields and their additive characters. The formalism of ℓ-adic cohomology with compact supports for schemes of finite type over a finite field, as summarized in [SGA4$\frac{1}{2}$, Rapport]: the Grothendieck–Lefschetz trace formula [SGA4$\frac{1}{2}$, Rapport, Théorème 3.2], the Künneth formula, Poincaré duality, Tate twists, Artin's vanishing theorem, and the long exact sequence of an open subscheme and its closed complement; see also [Mil]. Noether normalization. The Artin–Schreier sheaf, whose trace function is the additive character e of §S1.1 [SGA4$\frac{1}{2}$, Sommes trig.]. One theorem of weak Lefschetz type, stated in §S1.3. No earlier chapter is used. The sequel, *Deligne's bound by the sign-only method*, extends the argument to all dimensions.

S1.1. The identity $qN = M^k + R$

How large is a weighted count of the zeros of a quadratic form, and which part of it has square-root size? Alpöge's note sketches a "$qN = M^k + R$ sign-only proof of Weil and Deligne" using the form $x_1x_2 + \cdots + x_{k-1}x_k$ [Alp, p. 2 and footnote 3], without defining t, M, N or R. We follow the reading of [Dup, §§2.8–2.9] and the proof written out in [Ald]. That text reports that its proof was produced by an AI model from Alpöge's sketch and checked by two others, and that no human has checked it. D. Zhang has since written out the same proof for lisse sheaves on the affine line [Zh]. His note is the primary source for §S1.3 wherever it applies, and the text says where [Ald] goes further.

Let $q = p^f$, and let $e(y) := \exp\bigl(2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(y)/p\bigr)$ for $y \in \mathbb{F}_q$. For $h \in \mathbb{F}_q$ let K_h be the $q \times q$ matrix with entries $K_h(x, y) := e(hxy)$.

Problem S1.1.1. Every additive character of $\mathbb{F}_q$ is $y \mapsto e(ay)$ for a unique $a \in \mathbb{F}_q$, and $\sum_{h \in \mathbb{F}_q} e(hy)$ equals q if $y = 0$ and 0 otherwise.

Problem S1.1.2. If $h \neq 0$, then $K_h K_h^* = q$. So K_h has operator norm $\sqrt{q}$.

Problem S1.1.3. Let $t(x) := e(x)$ for $x \neq 0$ and $t(0) := 0$. Then $\sum_x t(x) = -1$ and $\sum_x t(x)\, e(x\xi) = q\, \delta_{\xi,-1} - 1$.

Let $k \geq 2$, let $Q(x) := x_1x_2 + x_2x_3 + \cdots + x_{k-1}x_k$ be the quadratic form of the path with vertices $1, \ldots, k$, and let $Z \subset \mathbb{A}^k$ be the quadric $Q = 0$. For $t\colon \mathbb{F}_q \to \mathbb{C}$ let

$$N := \sum_{x \in Z(\mathbb{F}_q)} t(x_1) \cdots t(x_k), \qquad M := \sum_{x \in \mathbb{F}_q} t(x),$$

$$R := \sum_{h \neq 0} \sum_{x \in \mathbb{F}_q^k} t(x_1) \cdots t(x_k)\, e\bigl(hQ(x)\bigr).$$

Problem S1.1.4. $qN = M^k + R$.

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The identity holds for every t. Its content is that R is controlled by Fourier analysis, and N and M^k by cohomology. Let D_t be the diagonal matrix with entries $t(x)$, and regard t as a column vector.

Problem S1.1.5. For every h,

$$\sum_{x\in\mathbb{F}_q^k} t(x_1)\cdots t(x_k)\, e(hQ(x)) = t^{\mathsf{T}} K_h D_t K_h D_t \cdots D_t K_h\, t,$$

with $k-1$ factors K_h and $k-2$ factors D_t.

Problem S1.1.6. If $|t| \le T$ and $h \neq 0$, the sum in Problem S1.1.5 has absolute value at most $T^k q^{(k+1)/2}$. Hence $|R| \le T^k q^{(k+3)/2}$.

The trivial bound is $|R| \le T^k q^{k+1}$. If M has the hoped-for size $\sqrt{q}$, then M^k has size $q^{k/2}$, and Problem S1.1.6 exceeds it by $q^{3/2}$ for every k. After a k-th root the loss $q^{3/(2k)}$ tends to 1. In §S1.3 the weights vary with the variable and the field.

Problem S1.1.7. Let E be a finite field with r elements, ψ a nontrivial additive character of E, and $t_1,\dots,t_k\colon E\to\mathbb{C}$ functions bounded by T. For $h\in E^\times$ and $c\in E$, $\bigl|\sum_{x\in E^k} t_1(x_1)\cdots t_k(x_k)\,\psi\bigl(hQ(x)+hc\bigr)\bigr| \le T^k r^{(k+1)/2}$.

This is [Zh, Lemma 2.6] and [Ald, Lemma 3]. For $k=2$ it is a finite-field form of Vinogradov's bilinear bound [Zh, Remark 2.7]. The path is what matters. For odd q and $t(x) := e(-x^2/2)$, completing the square gives $|\sum_{x,y} t(x)t(y)\,e(xy)| = q^{3/2}$. So with k even and the disjoint pairs $x_1x_2 + x_3x_4 + \cdots + x_{k-1}x_k$ in place of Q, the sum can reach $q^{3k/4}$, and the loss after a k-th root does not tend to 1 [Zh, Remark 2.7].

Problem S1.1.8. Let t be as in Problem S1.1.3. Then $N = 1-q$ for $k=3$, and $N = q-2$ for $k=4$.

S1.2. The sign under a vanishing hypothesis

Problem S1.1.6 bounds R, but qN and M^k may each be much larger than R. Why can they not cancel? The note's answer is that their Frobenius eigenvalues sit in adjacent cohomological degrees, "one off" [Alp, footnote 3].

Let $U\subset \mathbb{A}^1_{\mathbb{F}_q}$ be a nonempty open subset with $0\notin U$, and $\mathcal{F}$ a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U of rank ρ, pointwise pure of weight 0 for a fixed embedding $\iota\colon \overline{\mathbb{Q}}_\ell\to\mathbb{C}$. Let $t(x)$ be the trace of Frobenius on the stalk at $x\in U(\mathbb{F}_q)$, and $t(x):=0$ for $x\notin U$; then $|t|\le\rho$. Over $\mathbb{F}_{q^n}$ write M_n, N_n, R_n for the sums of §S1.1. Cohomology is taken over $\overline{\mathbb{F}}_q$, and Fr is the geometric Frobenius. Let $\mathcal{F}_k := \mathcal{F}^{\boxtimes k}$ on U^k and $Z_k := Z\cap U^k$. By the trace formula $M_n = \sum_i (-1)^i \operatorname{Tr}(\mathrm{Fr}^n \mid H^i_c(U,\mathcal{F}))$ and $N_n = \sum_i (-1)^i \operatorname{Tr}(\mathrm{Fr}^n \mid H^i_c(Z_k,\mathcal{F}_k))$. Following [Dup, §2.10], assume:

(H1) $H^i_c(U,\mathcal{F}) = 0$ for $i\neq 1$;
(H2) for infinitely many k, $H^i_c(Z_k,\mathcal{F}_k)=0$ for $i\neq k-1$.

Let $\alpha_1,\dots,\alpha_d$ be the eigenvalues of Fr on $H^1_c(U,\mathcal{F})$, and $\alpha_I := \alpha_{i_1}\cdots\alpha_{i_k}$ for $I=(i_1,\dots,i_k)$. Under (H1), $M_n = -\sum_j \alpha_j^n$.

Problem S1.2.1. If $\mathcal{F}$ has rank one and is geometrically nonconstant, then (H1) holds.

Problem S1.2.2. $H^i_c(U^k,\mathcal{F}_k) = 0$ for $i<k$, and $H^k_c(U^k,\mathcal{F}_k) = H^1_c(U,\mathcal{F})^{\otimes k}$. Under (H1) it vanishes also for $i>k$, and $M_n^k = (-1)^k \sum_I \alpha_I^n$.

Problem S1.2.3. Z_k is smooth, of pure dimension $k-1$ or empty. Hence $H^i_c(Z_k, \mathcal{F}_k) = 0$ for $i < k-1$.

Let $\beta_1, \dots, \beta_e$ be the eigenvalues of Fr on $H^{k-1}_c(Z_k, \mathcal{F}_k)$.

Problem S1.2.4. Under (H1), and (H2) for the given k, $R_n = (-1)^{k-1}\big(\sum_j (q\beta_j)^n + \sum_I \alpha_I^n\big)$ for all $n \geq 1$.

Every eigenvalue enters R_n with the same sign: M^k comes from degree k, and qN from degree $k-1$.

Problem S1.2.5. Let $\gamma_1, \dots, \gamma_s \in \mathbb{C}$ be distinct and nonzero, and $m_1, \dots, m_s \in \mathbb{C}$ nonzero. If $|\sum_i m_i \gamma_i^n| \leq A\, C^n$ for all $n \geq 1$, then $|\gamma_i| \leq C$ for all i.

In Problem S1.2.4 the coefficients are positive integers after equal eigenvalues are collected, so no α_I is cancelled by a $q\beta_j$. Problems S1.1.6, S1.2.4 and S1.2.5 with $I = (j, \dots, j)$ give $|\alpha_j|^k \leq q^{(k+3)/2}$, and letting $k \to \infty$ proves:

Theorem S1.2.1 ([Dup, Theorem 2.10.2])**.** *Under (H1) and (H2), every eigenvalue α of* Fr *on $H^1_c(U, \mathcal{F})$ satisfies $|\alpha| \leq \sqrt{q}$, and $|\sum_{x \in \mathbb{F}_q} t(x)| \leq d\sqrt{q}$.*

Hypothesis (H1) is harmless: as in Problem S1.2.1, it holds whenever $\mathcal{F}$ has no nonzero geometrically constant quotient. Hypothesis (H2) is not, and [Dup] does not prove it.

Problem S1.2.6. Let $U = \mathbb{A}^1 \smallsetminus \{0\}$ and $\mathcal{F}$ the Artin–Schreier sheaf, with trace function t as in Problem S1.1.3. Then $\mathcal{F}$ is geometrically nonconstant, (H1) holds with $d = 1$ and $\alpha_1 = 1$, and the vanishing in (H2) fails for $k = 3$ and for $k = 4$.

Replacing Q by $\sum_i a_i x_i x_{i+1}$ with $a_i \neq 0$ dilates the weight in each variable, by different factors. For a given sheaf this can matter: in Problem S1.2.6 with $k = 3$, the cohomology of $a_1x_1x_2 + a_2x_2x_3 = 0$ in U^3 is concentrated in degree 2 when $a_1 \neq a_2$. But (H2) asks for infinitely many k, and the proof below does not rest on a choice of coefficients. What is wanted is a hypersurface in U^k whose cohomology above the middle degree is forced to match that of U^k. This is a weak Lefschetz property, and it holds for a generic hyperplane section, which has linear terms as well as quadratic ones.

S1.3. Weak Lefschetz in place of (H2)

The argument of [Ald] and [Zh] keeps the identity and the estimate, and replaces (H2) by a weak Lefschetz theorem. The higher cohomology of the hypersurface cancels against that of U^k, and what survives is the trace on an actual vector space.

Duality and the Gysin map. Let X be smooth of pure dimension D over a finite field $\mathbb{F}_r$, and $\mathcal{G}$ lisse on X. Poincaré duality pairs $H^i_c(X, \mathcal{G})$ perfectly with $H^{2D-i}(X, \mathcal{G}^\vee)(D)$, compatibly with Frobenius, and the twist (-1) multiplies Frobenius eigenvalues by r. If X is affine, Artin's theorem and duality give $H^i_c(X, \mathcal{G}) = 0$ for $i < D$. Let $Y \subset X$ be smooth and closed of pure dimension $D-1$, or empty. The transpose of restriction $H^a(X, \mathcal{G}^\vee) \to H^a(Y, \mathcal{G}^\vee)$ is the Gysin map $H^j_c(Y, \mathcal{G})(-1) \to H^{j+2}_c(X, \mathcal{G})$, $j = 2D - 2 - a$. We say that Y has property (WL) for $\mathcal{G}$ if this restriction is bijective for $a < D-1$ and injective for $a = D-1$.

Problem S1.3.1. If Y has property (WL) for $\mathcal{G}$, the Gysin map $H^j_c(Y, \mathcal{G})(-1) \to H^{j+2}_c(X, \mathcal{G})$ is bijective for $j > D-1$ and surjective for $j = D-1$.

Now let U be a smooth affine curve over $\mathbb{F}_r$, $\mathcal{F}$ lisse on U, $X := U^k$, $D = k$, and $\mathcal{F}_k := \mathcal{F}^{\boxtimes k}$. Let $Y \subset X$ be as above, with property (WL) for $\mathcal{F}_k$. Let M_n and N_n be the sums of the traces of Frobenius on the stalks of $\mathcal{F}$ over $U(\mathbb{F}_{r^n})$ and of $\mathcal{F}_k$ over $Y(\mathbb{F}_{r^n})$, let $R_n := r^n N_n - M_n^k$, and let K be the kernel of the Gysin map $H_c^{k-1}(Y, \mathcal{F}_k)(-1) \to H_c^{k+1}(X, \mathcal{F}_k)$.

Problem S1.3.2. For all $n \geq 1$, $(-1)^{k-1} R_n = \mathrm{Tr}\big(\mathrm{Fr}^n \mid H_c^1(U, \mathcal{F})^{\otimes k}\big) + \mathrm{Tr}\big(\mathrm{Fr}^n \mid K\big)$.

This is [Ald, Proposition 4]. In [Zh, Lemma 2.2 and Definition 2.3] the right side is the trace on the single group $H_c^k(X, Rj_*j^*\mathcal{F}_k)$, where j is the inclusion of $X \smallsetminus Y$. That group is an extension of K by $H_c^k(X, \mathcal{F}_k)$, and property (WL) says that it is the only nonzero one. Up to one global sign, R_n is the trace of Frobenius on an actual vector space, so Problem S1.2.5 applies with positive integer coefficients. Nothing is assumed about the higher cohomology of U or of Y.

Problem S1.3.3. Let $U = \mathbb{A}^1$, $\mathcal{F} = \overline{\mathbb{Q}}_\ell$, $k = 2$, and Y the conic $xy + ax + by + c = 0$ with $ab \neq c$. Then $N_n = r^n - 1$ and $-R_n = r^n$; here $H_c^1(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell) = 0$, and K has dimension one and eigenvalue r.

The following theorem supplies hypersurfaces with property (WL).

Theorem S1.3.1 (Deligne; [Kat93, Appendix, Corollary A.5])**.** *Let V be smooth of pure dimension D over $\overline{\mathbb{F}}_q$, let $f\colon V \to \mathbb{P}^m$ be quasi-finite, and let $\mathcal{G}$ be lisse on V. For H in a dense open subset of the space of hyperplanes of $\mathbb{P}^m$, restriction $H^a(V, \mathcal{G}) \to H^a(f^{-1}H, \mathcal{G})$ is bijective for $a < D - 1$ and injective for $a = D - 1$.*

Katz states this for a perverse sheaf $\mathcal{K}$, in degrees $a < -1$ and $a = -1$. The case above is $\mathcal{K} = \mathcal{G}[D]$, which is perverse because V is smooth of pure dimension D [Kat93, proof of Corollary 3.4.1]. The proof uses generic base change, duality and Artin's theorem, and no weights.

Problem S1.3.4. Let V be smooth and affine of pure dimension D over $\mathbb{F}_q$, let $\Gamma = (y_1, \ldots, y_m)\colon V \to \mathbb{A}^m$ be a closed immersion, and let $\mathcal{G}$ be lisse on V. For (λ, μ) in a dense open subset of the affine space over $\overline{\mathbb{F}}_q$ of parameters with $\lambda_m = 1$, let $Y := \{\sum_i \lambda_i y_i + \mu = 0\}$. (a) Restriction $H^a(V, \mathcal{G}) \to H^a(Y, \mathcal{G})$ is bijective for $a < D - 1$ and injective for $a = D - 1$. (b) Y is smooth, of pure dimension $D - 1$ or empty.

This is the form used below. It is [Ald, Lemma 8], and for $V = \mathbb{A}^n$ it is [Zh, Theorem 2.4 and Construction 2.5].

Let U be a smooth affine curve over $\mathbb{F}_q$, and $\mathcal{F}$ lisse on U with $|\iota\,\mathrm{Tr}(\mathrm{Fr}_u \mid \mathcal{F}_{\bar{u}})| \leq C$ for all n and $u \in U(\mathbb{F}_{q^n})$. Choose a finite morphism $\pi\colon U \to \mathbb{A}^1$ of degree B, and generators $z_1, \ldots, z_s$ of the $\mathbb{F}_q$-algebra $\mathcal{O}(U)$. For $a = (a_{ij}) \in \mathbb{A}^{ks}$ and $c \in \mathbb{A}^1$ let $Y_{a,c} \subset U^k$ be the zero set of

$$P_{a,c}(u_1, \ldots, u_k) := \sum_{i=1}^{k-1} \pi(u_i)\,\pi(u_{i+1}) + \sum_{i=1}^{k} \sum_{j=1}^{s} a_{ij}\, z_j(u_i) + c.$$

Problem S1.3.5. The map $u \mapsto \big(z(u_1), \ldots, z(u_k), \sum_i \pi(u_i)\pi(u_{i+1})\big)$ is a closed immersion $U^k \to \mathbb{A}^{ks+1}$, and the $Y_{a,c}$ are its hyperplane sections with last coefficient 1.

By Problem S1.3.4 with $\mathcal{G} = \mathcal{F}_k^\vee$, there are $e \geq 1$ and (a, c) with coordinates in $\mathbb{F}_r$, $r := q^e$, for which $Y := Y_{a,c}$ has property (WL) for $\mathcal{F}_k$. Over the field E with r^n elements, let ψ be a nontrivial additive character and $\tau(u)$ the trace of Frobenius at $u \in U(E)$, and for $h, x \in E$ let

$$t_{i,h}(x) := \sum_{u \in U(E),\ \pi(u)=x} \tau(u)\, \psi\Big(h \sum_j a_{ij} z_j(u)\Big).$$

Problem S1.3.6. $r^n N_n = \sum_{h \in E} \psi(hc) \sum_{x \in E^k} t_{1,h}(x_1) \cdots t_{k,h}(x_k)\, \psi\big(hQ(x)\big)$, and the term $h = 0$ is M_n^k.

Problem S1.3.7. $|t_{i,h}| \leq BC$, and $|R_n| \leq (BC)^k\, r^{n(k+3)/2}$.

So the weights of §S1.1 reappear, twisted by additive characters that depend on h. It remains to let k and the field vary.

Problem S1.3.8. Let L be a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space with an automorphism Φ, and a real. Suppose that for infinitely many k there are $e_k \geq 1$, a finite-dimensional space K_k with an automorphism Φ_k, a sign ε_k, and constants $C_k > 0$, $c_k \geq 0$ with $c_k/k \to 0$, such that for all $n \geq 1$

$$\varepsilon_k R_{k,n} = \iota \operatorname{Tr}\big(\Phi^{e_k n} \mid L^{\otimes k}\big) + \iota \operatorname{Tr}\big(\Phi_k^n \mid K_k\big), \qquad |R_{k,n}| \leq C_k\, q^{e_k n(ka + c_k)}.$$

Then every eigenvalue α of Φ satisfies $|\iota\alpha| \leq q^a$.

This is [Ald, Lemma 2]. Problems S1.3.2, S1.3.5, S1.3.6, S1.3.7 and S1.3.8, with $a = 1/2$ and $c_k = 3/2$, together with Problem S1.3.4, prove:

Theorem S1.3.2 ([Ald, Theorem 9], $d = 1$; for U open in $\mathbb{A}^1$, [Zh, Theorem 1.1 and Remark 1.2])**.** *Let U be a smooth affine curve over $\mathbb{F}_q$, and $\mathcal{F}$ a lisse sheaf on U with bounded traces as above. Then every eigenvalue α of* Fr *on $H_c^1(U, \mathcal{F})$ satisfies $|\iota\alpha| \leq \sqrt{q}$.*

[Zh] treats a lisse sheaf on an open subset of $\mathbb{A}^1$, with weights at most w in place of bounded traces. That case is enough for curves. Let C be a smooth projective geometrically connected curve of genus g over $\mathbb{F}_q$. Choose $x \in \mathbb{F}_q(C)$ such that $\mathbb{F}_q(C)$ is finite and separable over $\mathbb{F}_q(x)$, let $V \subset \mathbb{A}^1$ be a nonempty open subset over which x is finite and étale, and let $U := x^{-1}(V) \subset C$.

Problem S1.3.9. $H^1(C, \overline{\mathbb{Q}}_\ell)$ is a quotient of $H_c^1(U, \overline{\mathbb{Q}}_\ell)$, compatibly with Fr. Moreover $H_c^1(U, \overline{\mathbb{Q}}_\ell) = H_c^1(V, x_*\overline{\mathbb{Q}}_\ell)$, and $x_*\overline{\mathbb{Q}}_\ell$ is lisse on V with traces of Frobenius between 0 and the degree of x.

Problem S1.3.10. If α is an eigenvalue of Fr on $H^1(C, \overline{\mathbb{Q}}_\ell)$, so is q/α.

Theorem S1.3.2 for $x_*\overline{\mathbb{Q}}_\ell$ on V, and Problems S1.3.9 and S1.3.10, give the Riemann hypothesis for C: every eigenvalue of Fr on $H^1(C, \overline{\mathbb{Q}}_\ell)$ has absolute value $\sqrt{q}$. This is Weil's theorem [Wei48]. The route is that of [Ald], after [Alp], and it uses Theorem S1.3.2 only in the case of [Zh, Theorem 1.1 and Remark 1.2].

Problem S1.3.11. $|\#C(\mathbb{F}_{q^n}) - q^n - 1| \leq 2g\, q^{n/2}$.

Nothing in this section used $\dim U = 1$ except the degrees of the cohomology groups, the finite map π to $\mathbb{A}^1$ and the shape of the coupling $\pi(u_i)\pi(u_{i+1})$. Replacing it by a nondegenerate bilinear form on $\mathbb{A}^d$ is the note's "vectorize to go up dimensions", and leads to Deligne's bound. We do not know for which sheaves and which k hypothesis (H2) holds for the plain quadric $Q = 0$ [Dup, Question 2.10.10]; it is no longer needed.

Hints

§S1.1.

S1.1.1: For $y \neq 0$, $h \mapsto e(hy)$ is a nontrivial character, because the trace is onto $\mathbb{F}_p$. Count the characters.
S1.1.2: The (x, x') entry of $K_h K_h^*$ is a sum over y of a character of y.
S1.1.3: Problem S1.1.1, applied to $\sum_{x \in \mathbb{F}_q} e(x(1+\xi))$.
S1.1.4: Sum over h first, using Problem S1.1.1. Then split off $h = 0$.
S1.1.5: Sum over x_1, then x_2, and so on.
S1.1.6: The operator norm of K_h is $\sqrt{q}$, that of D_t at most T, and $\|t\| \leq T\sqrt{q}$.
S1.1.7: As in Problem S1.1.6, with the matrices $\psi(hxy)$ and the diagonal matrices of the t_i.
S1.1.8: For $k = 4$, sum over x_1 and x_4 first, and use Problem S1.1.3.

§S1.2.

S1.2.1: U is geometrically connected. On an affine curve H_c^0 of a lisse sheaf vanishes; $H_c^2(U, \mathcal{F})$ is the space of coinvariants of the geometric monodromy, twisted by (-1).
S1.2.2: Apply the Künneth formula.
S1.2.3: The gradient of Q has first coordinate x_2. For the vanishing, combine Poincaré duality with Artin's theorem on the affine scheme Z_k.
S1.2.4: Write $q^n N_n$ and M_n^k by the trace formula and Problems S1.2.2 and S1.2.3.
S1.2.5: Compare $\sum_n (\sum_i m_i \gamma_i^n) z^n$ with $\sum_i m_i \gamma_i z/(1 - \gamma_i z)$ in $|z| < 1/C$.
S1.2.6: Problems S1.2.1 and S1.1.8 over $\mathbb{F}_{q^n}$. Compare with the form that (H2) prescribes; by Problem S1.2.5, applied to the difference of two power sums, the coefficients of a power sum are unique.

§S1.3.

S1.3.1: Dualize the hypothesis on restriction, using Poincaré duality on X and on Y.
S1.3.2: Write $r^n N_n$ and M_n^k by the trace formula on Y and X. The first statement of Problem S1.2.2 holds for every smooth affine curve. Then use Problem S1.3.1.
S1.3.3: Translate x and y to factor the equation of Y.
S1.3.4: (a) Theorem S1.3.1 for $V \to \mathbb{A}^m \subset \mathbb{P}^m$; the hyperplanes with $\lambda_m \neq 0$ can be scaled to $\lambda_m = 1$. (b) Bound the dimension of the set of pairs (v, H) with H containing the tangent space of $\Gamma(V)$ at $\Gamma(v)$.
S1.3.5: The first ks coordinates already give a closed immersion.
S1.3.6: Problem S1.1.1, then group the u_i by $x_i = \pi(u_i)$.
S1.3.7: A fibre of π over a point of E is the spectrum of an algebra of dimension at most B. Then apply Problem S1.1.7.
S1.3.8: Problem S1.2.5.
S1.3.9: Use the exact sequence of the open subscheme U and its finite complement. For the rest, x is finite, and the stalk of $x_* \overline{\mathbb{Q}}_\ell$ at a geometric point is the sum over its fibre.
S1.3.10: Use Poincaré duality on C.
S1.3.11: The trace formula on C, and $\dim H^1(C, \overline{\mathbb{Q}}_\ell) = 2g$.

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