# THE OPEN CELL: SIEGEL'S FORMULA AND POISSON SUMMATION

Prerequisites. From Chapter 36: the hermitian space $Y_0 = \bigoplus_a Ev_a \oplus X_0 \oplus Ez \oplus \bigoplus_a Ev'_a$ with $h(z,z) = -N$, the unipotent group $\mathcal{U}$ (Section 36.1), and the covolume $\nu$ (Definition 36.4.1). From Chapter 37: the degrees $d_\tau$ of Theorem 37.3.1, $\deg = \prod_\tau d_\tau$, and the sizes $D_{\mathrm{id}}$, $D_{\mathrm{main}} = D_X^2/(\deg \cdot \mathrm{mass}_r)$ of Definition 37.3.2. From Chapter 38: the frame and the leading minors $\langle\gamma\rangle_j$ (Definition 38.2.1), disc (Definition 38.2.2), links, open reductions, fixers and link second moments (Definitions 38.2.3, 38.2.6, 38.2.7, 38.3.1). From Chapter 22: the densities $\alpha_\infty$, $\alpha_p$ of Section 22.2 and Braun's formula (3). Problem 21.1.4; Problem 10.4.5. Fourier analysis on finite abelian groups; Hensel's lemma; Schur orthogonality for square-integrable representations [Kn, Ch. IX]. Hermitian forms are linear in the first variable, and $|\cdot|^2$ is the product with the complex conjugate. In this sheet $j$, $l$ are integers or residues, $y$ is a vector, $\omega$ is an isometry, $\mathrm{e}(\cdot) := e^{2\pi \mathrm{i} \cdot}$ on $\mathbb{R}$, and $K'$ is a level (not the field $K$).

The second moment of the Gram operators of Chapter 36 is bounded in Theorem 40.3.1 by splitting the reductions $s \in \bar{G}$ into the open ones and the rest. This sheet treats the open ones. Over the open reductions that fix chosen vectors of the span of the frame at $\#\mathcal{D}$ chosen places, the link second moment of the terms with disc $\neq 0$ is at most $(1+b^{-4})\, b^{-6k\#\mathcal{D}}\, D_{\mathrm{main}}\, \nu^2$ (Theorem 39.3.3). Where does $D_{\mathrm{main}}$ come from? A term over an open reduction is a head, a last entry and a residual isometry. The residual isometries with a given last entry are counted by vectors of a given norm, and Siegel's formula counts these exactly. The heads are summed by Poisson summation, one entry at a time: the zero frequencies give an integral, and the other frequencies are small. The bound has no small instance: in the proof $\varrho \geq 10^5$. We meet the steps separately: one isometry over $\mathbb{Q}$, six squares, and the groups $\mathbb{Z}/5$ and $\mathbb{Z}/3$.

## 39.1. Head, last entry and residual isometry

Let $V = \bigoplus_{a \leq \varrho} Ev_a \oplus X \oplus Ez \oplus \bigoplus_{a \leq \varrho} Ev'_a$ be a hermitian space of the shape of Section 36.1, with $X$ of rank $k$ and totally positive definite, and $h(z,z) = -N$. A *residual isometry* is an isometry $\omega$ of $V$ with $\omega v_a = v_a$ and $\omega v'_a = v'_a$ for all $a$. It preserves the orthogonal complement $X \oplus Ez$ of the $v_a$, $v'_a$.

**Problem 39.1.1.** As a model with $E = \mathbb{Q}$ (trivial involution), let $\varrho = 0$, $N = 5$ and $X = \mathbb{Q}\mathbf{e}$ of rank one: $V = \mathbb{Q}\mathbf{e} \oplus \mathbb{Q}z$ with $h(\mathbf{e},\mathbf{e}) = 1$, $h(z,z) = -5$. The matrix $\omega = \left(\begin{smallmatrix} 3/2 & 5/2 \\ 1/2 & 3/2 \end{smallmatrix}\right)$ in the basis $\mathbf{e}, z$ (columns are the images) is an isometry, $h(z, \omega z) = -15/2$, and the coefficient $5/2$ of $\mathbf{e}$ in $\omega z$ satisfies $5 \cdot (5/2)^2 = (15/2)^2 - 25$.

**Problem 39.1.2.** Let $\omega$ be a residual isometry, and write $\omega z = y + c_\omega z$ with $y \in X$, $c_\omega \in E$. Then $|h(z, \omega z)|^2 = N^2 |c_\omega|^2$ and

$$N\, h(y,y) = |h(z,\omega z)|^2 - N^2.$$

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Hence $|\tau h(z,\omega z)| \geq N$ at every infinite place $\tau$, with strict inequality at every $\tau$ if $y \neq 0$ and equality at every $\tau$ if $y = 0$.

Now let $V = Y_0$, with the frame $(v_1, \dots, v_\varrho, z)$ and the leading minors $\langle\gamma\rangle_j$, $0 \leq j \leq \varrho+1$, of Definition 38.2.1, so that $\langle\gamma\rangle_0 = 1$ and $\langle\gamma\rangle_{\varrho+1}$ is the determinant of the matrix of frame products.

**Definition 39.1.1.** Let $\gamma \in U(Y_0)(F)$ have $\langle\gamma\rangle_j \neq 0$ for $1 \leq j \leq \varrho$. Its *head* is $\mathrm{hd}(\gamma) := \big(\langle\gamma\rangle_{a+1}/\langle\gamma\rangle_a\big)_{0\leq a<\varrho} \in (E^\times)^\varrho$. If also $\langle\gamma\rangle_{\varrho+1} \neq 0$, its *last entry* is $\mathrm{le}(\gamma) := \langle\gamma\rangle_{\varrho+1}/\langle\gamma\rangle_\varrho$. A *residual isometry of the last floor* is a residual isometry of $Y_0$.

By Definition 38.2.2, $\mathrm{disc}(\gamma) = |\langle\gamma\rangle_\varrho|^2\big(|\mathrm{le}(\gamma)|^2 - N^2\big)$ whenever $\mathrm{le}(\gamma)$ is defined. In the proof, if $s$ is open, then every term $\gamma$ over $(c,s,c')$ has $\langle\gamma\rangle_j \neq 0$ for $j \leq \varrho$. Moreover $\gamma = n^{-1}\mathfrak{w}\,\omega\,n'$ with $n, n' \in \mathcal{U}(F)$, $\omega$ a residual isometry of $Y_0$, and $\mathfrak{w}$ a Weyl element that exchanges each line $Ev_a$ with $Ev'_a$, fixes $X_0 \oplus Ez$ and depends only on $\mathrm{hd}(\gamma)$. The $(z,z)$ entry $h(\omega z, z)$ of the matrix of frame products of $\omega$ is $\mathrm{le}(\gamma)$, so $|\mathrm{le}(\gamma)| = |h(z,\omega z)|$. By Problem 39.1.2 the last entry is therefore always defined there, and $\mathrm{disc}(\gamma) \neq 0$ exactly when the $X_0$-part $y$ of $\omega z$ is nonzero.

Consider a block of the link second moment over an open reduction $s$ whose sum is nonzero. In the proof one residual isometry $\omega$ names the link, and the heads of the contributing terms lie in a finite set $\mathcal{E}^\varrho$, where $\mathcal{E} \subset E^\times$ consists of quotients of integers of bounded size. The block is the sum, over $\mathfrak{h} \in \mathcal{E}^\varrho$, of the sums of the amplitudes of the $n^{-1}\mathfrak{w}\,\omega\,n'$ over $(c,s,c')$, for $n, n' \in \mathcal{U}(F)$, where $\mathfrak{w}$ is the Weyl element of $\mathfrak{h}$. Different nonzero blocks give different $(c, c', s, \omega)$. So the heads are summed inside the square, and the residual isometries outside it. By Problem 39.1.2, the sum over the $\omega$ with a given last entry is controlled by a sum over vectors $y \in X_0$ with $N\,h(y,y) = |\mathrm{le}|^2 - N^2$, one for each class.

## 39.2. Siegel's formula for six squares

The smallest rank to which Braun's formula (3) applies is 3, and for $\mathbb{Z}[\mathrm{i}]^3$ the class set is $\{I_3\}$ (Problem 21.1.4). So the mean over the class set is a count: $r_{I_3}(j) := \#\{y \in \mathbb{Z}[\mathrm{i}]^3 : y^*y = j\}$ equals $\alpha_\infty(j)\prod_p \alpha_p(j)$, with the densities of Section 22.2 for the form $Q(y) = y^*y$, a sum of six squares on $\mathbb{Z}^6$. Let $\chi_{-4}$ be the nontrivial character modulo 4.

**Problem 39.2.1.** By machine, $r_{I_3}(1) = 12$, $r_{I_3}(2) = 60$ and $r_{I_3}(5) = 312$.

**Problem 39.2.2.** For odd $p$, the number of $x \in (\mathbb{Z}/p)^6$ with $\sum_i x_i^2 = 1$ is $p^5 - \chi_{-4}(p)p^2$, and the same holds with 1 replaced by any $j$ prime to $p$: by machine 252, 3100, 16856 for $p = 3, 5, 7$. By machine, there are $61236 = 3^5 \cdot 252$ solutions modulo 9, and $3 \cdot 2^{5e}/4$ solutions modulo $2^e$ for $e = 2, 3$ (768, 24576). For $e \geq 3$ the count modulo $2^{e+1}$ is $2^5$ times the count modulo $2^e$. Hence $\alpha_p(1) = 1 - \chi_{-4}(p)p^{-3}$ for odd $p$, and $\alpha_2(1) = 3/4$.

**Problem 39.2.3.** For real $j > 0$, $\mathrm{vol}\{y \in \mathbb{C}^3 : |y|^2 \leq j\} = \pi^3 j^3/6$, so $\alpha_\infty(j) = \pi^3 j^2/2$.

**Problem 39.2.4.** With $L(3, \chi_{-4}) = \pi^3/32$: $\frac{\pi^3}{2} \cdot \frac{3}{4} \cdot \prod_{p \text{ odd}}\big(1 - \chi_{-4}(p)p^{-3}\big) = 12 = r_{I_3}(1)$. For $j = 5$ the factor at 5 is $\alpha_5(5) = 3224/3125$, the factor at infinity is $\alpha_\infty(5) = 25\pi^3/2$, the other factors are those of $j = 1$, and the product is $312 = r_{I_3}(5)$.

The general statement is for a CM field with Weil data: $E$ contains a primitive 5040th root of unity, and there are a $j_E \in E$ with $\bar{j}_E = -j_E$, $\operatorname{Im}\tau(j_E) > 0$ at every $\tau$ and $(1 - j_E^{-1})/2$ an algebraic integer, a finite idele that makes $\hat{\mathcal{O}}_E$ self-dual for the pairing defined by $j_E$ and the additive character $\psi_f$ of the finite adeles, and a splitting character that is $-1$ at the chosen uniformizer of every finite place fixed by complex conjugation [GL]. Here $\operatorname{vol}(\hat{\mathcal{O}}_E) = 2^f |d_E|^{-1/2}$, with $d_E$ the discriminant of $E$.

**Theorem 39.2.1** (Siegel's formula for one vector)**.** *Let $E$ be a CM field with Weil data, $k \geq 3$, $K'$ an open subgroup of the stabilizer of $\hat{\mathcal{O}}_E^k$ in the finite adelic unitary group of $y^*y$, $\phi \geq 0$ a locally constant, compactly supported, $K'$-invariant function on $\mathbb{A}_{E,f}^k$, and $\mathfrak{n} \in E$ with $\bar{\mathfrak{n}} = \mathfrak{n}$ and $\tau\mathfrak{n} > 0$ at every infinite place. Then each sum $\sum_{y \in E^k,\, y^*y = \mathfrak{n}} \phi(g^{-1}y)$ converges, and*

$$\sum_{[g]} \frac{1}{\#\mathrm{Aut}(g)} \sum_{\substack{y \in E^k \\ y^*y = \mathfrak{n}}} \phi(g^{-1}y)$$
$$= \Big(\sum_{[g]} \frac{1}{\#\mathrm{Aut}(g)}\Big) |d_E|^{1/4} \prod_{\tau} \frac{\pi^k (\tau\mathfrak{n})^{k-1}}{(k-1)!} \iint \phi(u)\, \psi_f\big(\tfrac{1}{2}\zeta(u^*u - \mathfrak{n})\big)\, du\, d\zeta,$$

*where $[g]$ runs over the classes of level $K'$, $u$ over $\mathbb{A}_{E,f}^k$ with $\hat{\mathcal{O}}_E$ of volume $\operatorname{vol}(\hat{\mathcal{O}}_E)$ in each coordinate, and $\zeta$ over the self-adjoint finite adeles with the codifferent of volume 1.*

*Status*: classical; due to Siegel [Si] for quadratic forms and Braun [Br] for hermitian forms, and a case of Weil's theorem [We65, Thm. 5]. The last integral is the product of the local densities, left as one adelic integral; its normalization is not compared here with that of Chapter 22. If there is one class, or every class has trivial automorphism group, the formula is a statement about counts; Problem 39.2.1 is the case of one class, over a field without Weil data.

In the proof, Theorem 39.2.1 is applied, for each last entry, to the vectors $y \in X_0$ with $N\, h(y,y) = |\mathrm{le}|^2 - N^2$, that is at $\mathfrak{n} = (\mathrm{le}\ \overline{\mathrm{le}} - N^2)/N$. It is used as an upper bound, with an extra factor $1 + b^{-6}$ from the local densities.

## 39.3. Poisson summation, one entry at a time

In the proof, each double-coset sum of Section 39.1 is a Kloosterman sum at finitely many finite places times an orbital integral at the infinite places. Squaring the sum over heads gives a sum over pairs of heads, and the pairs of different heads are kept. They are summed by Poisson summation in one entry at a time. For a function $\mathfrak{f}$ on $\mathbb{Z}/p$ let $\hat{\mathfrak{f}}(l) := p^{-1} \sum_j \mathfrak{f}(j)\, \mathrm{e}(-lj/p)$, so that $\mathfrak{f}(j) = \sum_l \hat{\mathfrak{f}}(l)\, \mathrm{e}(lj/p)$.

**Problem 39.3.1.** On $\mathbb{Z}/5$, $\mathfrak{f}(j) = 2 + \cos(2\pi j/5)$ has $\hat{\mathfrak{f}} = (2, \frac{1}{2}, 0, 0, \frac{1}{2})$. So $\mathfrak{f}(0) = 3$, while the zero frequency alone gives 2.

**Problem 39.3.2.** Let $\mathfrak{f}$ be a function on a finite abelian group $\mathcal{G}$, and $\hat{\mathfrak{f}}(\psi') := \#\mathcal{G}^{-1} \sum_{j \in \mathcal{G}} \mathfrak{f}(j)\overline{\psi'(j)}$ for the characters $\psi'$ of $\mathcal{G}$. Then $|\mathfrak{f}(j)| \leq \sum_{\psi'} |\hat{\mathfrak{f}}(\psi')|$ for every $j$, and $\hat{\mathfrak{f}}(\psi_0)$ is the mean of $\mathfrak{f}$, for the trivial character $\psi_0$. If $\mathfrak{f}$ is real and $\sum_{\psi' \neq \psi_0} |\hat{\mathfrak{f}}(\psi')| \leq c$, then $\mathfrak{f}(j) \leq \hat{\mathfrak{f}}(\psi_0) + c$.

For $j \in \mathbb{Z}/p$ let $\mathrm{Kl}(j) := \sum_{l \in (\mathbb{Z}/p)^\times} \mathrm{e}\big((l + jl^{-1})/p\big)$, a Kloosterman sum.

**Problem 39.3.3.** For $p = 5$ the values $\mathrm{Kl}(0), \dots, \mathrm{Kl}(4)$ are

$$-1, \quad \tfrac{3-\sqrt{5}}{2}, \quad -1-\sqrt{5}, \quad \sqrt{5}-1, \quad \tfrac{3+\sqrt{5}}{2},$$

and $\sum_j |\mathrm{Kl}(j)|^2 = 20$.

**Problem 39.3.4.** $\sum_{j \in \mathbb{Z}/p} |\mathrm{Kl}(j)|^2 = p(p-1)$. In the expansion of the square, the terms with $l \neq l'$ cancel after the sum over $j$: the matrix $\big(\sum_j \mathrm{e}(j(l^{-1} - l'^{-1})/p)\big)_{l,l'}$ is $p$ times the identity.

So once a free variable is summed, different heads are orthogonal. At a fixed nonzero frequency they are not orthogonal, but the frequency gains a factor in each free direction.

**Problem 39.3.5.** Let $\mathfrak{A}_l := \big(\mathrm{e}(ljj'/3)\big)_{j,j' \in \mathbb{Z}/3}$. Then $\mathfrak{A}_1 \mathfrak{A}_1^* = 3 \cdot 1$, and $\mathfrak{A}_0 \mathfrak{A}_0^*$ is the matrix with all entries 3. Their squared operator norms are 3 and 9, and those of the $k$-th tensor powers are $3^k$ and $9^k$.

In the proof, the rank $k$ of $X_0$ pays for the nonzero frequencies in the same way. At each infinite place the bound at a nonzero frequency decays at least like the $(k-(\varrho+1)^7)$-th power of its inverse size. We do not derive these local bounds here.

**Problem 39.3.6.** The entrywise product of two positive semidefinite matrices of the same size is positive semidefinite.

In the proof, the form on pairs of heads with a common last entry is the entrywise product of positive semidefinite kernels, one at each infinite place and one at the finite places. So it is positive semidefinite, by Problem 39.3.6, and the densities obtained from it by integrating out entries are nonnegative. Nonnegativity is what Theorem 39.3.1 below needs.

**Problem 39.3.7.** Let $\pi$ be an irreducible unitary representation of a unimodular group $\mathcal{G}'$ with square-integrable matrix coefficients and formal degree $d_\pi$, so that $\int_{\mathcal{G}'} \langle \pi(g)u_1, u_3\rangle \overline{\langle \pi(g)u_2, u_4\rangle}\, dg = d_\pi^{-1} \langle u_1, u_2\rangle \overline{\langle u_3, u_4\rangle}$. For every vector $u$ with $\|u\|^2 = D'$,

$$\int_{\mathcal{G}'} |\langle \pi(g)u, u\rangle|^2\, dg = \frac{D'^2}{d_\pi}.$$

For the representation of Problem 10.4.5 and $D' = 1$, this is the reciprocal of the formal degree found there.

Identity (c) of Theorem 37.3.1 at each infinite place $\tau$ is the relation of Problem 39.3.7 for the single vector $W_\tau$, whose squared norm is $\#I_\tau$, with $d_\pi = d_\tau$. In the proof, at the zero frequency the integral of a form over a common entry is exactly the form of one length less, and the form of length zero has modulus at most $(1 + b^{-10})/d_\tau$ at the infinite place $\tau$. This is the factor $\prod_\tau 1/d_\tau = 1/\deg$ of $c_{\mathrm{bot}}$ below. Poisson summation is used in the following one-sided form.

**Theorem 39.3.1** (one-sided Poisson summation)**.** *Let $E$ be a totally complex number field with chosen complex embeddings, $\psi$ the standard character of $\mathbb{A}_E/E$, and $du$ the Haar measure on $\mathbb{A}_E$ for which $E$ has covolume 1. Let $\mathfrak{f}\colon \mathbb{A}_E \to \mathbb{R}$ be nonnegative and integrable, and lower semicontinuous at each point of a finite set $\mathcal{Z} \subset E$, and put $\hat{\mathfrak{f}}(l) := \int \mathfrak{f}(u)\psi(-lu)\, du$. If $\sum_{l \in \mathcal{Z}'} |\hat{\mathfrak{f}}(l)| \leq c_{\mathfrak{f}}$ for every finite $\mathcal{Z}' \subset E$, then $\sum_{u \in \mathcal{Z}} \mathfrak{f}(u) \leq c_{\mathfrak{f}}$.*

*Status*: an inequality form of Poisson summation on $\mathbb{A}_E/E$ [Ta], for functions that need not be continuous. We use it as the proof proves it, where the test function is smoothed by a bump on the compact quotient.

**Theorem 39.3.2** (the sieve)**.** *In the proof, fix $\mathcal{D}$ and the vectors of Theorem 39.3.3. There are real numbers $\Sigma_0, \dots, \Sigma_{\varrho+1}$, the sums of the densities of the pairs of heads of length $i$ (tuples of $i$ entries) with entries in $\mathcal{E}$ and a common last entry (for $i = 0$, the density of the empty pair), and $\lambda_0, \dots, \lambda_\varrho \geq 0$, the sums of the weights of the heads of length $i$ with entries in $\mathcal{E}$. They satisfy, for $0 \leq i \leq \varrho$,*

$$\Sigma_0 \leq c_{\mathrm{bot}}, \qquad \Sigma_{i+1} \leq c_i\big(\Sigma_i + 2c_{\mathrm{fr}}\lambda_i^2\big), \qquad \lambda_i^2 \leq c_{\mathrm{wt}} \prod_{i'<i} c_{i'},$$

*and $\Sigma_{\varrho+1}$ bounds the sum, over the last entries, of the top form over all pairs of heads. Here $c_i > 0$ is the constant of Poisson summation in the entry $i$: $\mathrm{vol}(\hat{\mathcal{O}}_E)$ divided by the Jacobian, at the infinite places, of the rescaling that the proof applies to that entry. Further $c_{\mathrm{bot}} = \prod_\tau \big((1 + b^{-10})/d_\tau\big) \cdot (1 + b^{-6})\, b^{-6k\#\mathcal{D}}$, $c_{\mathrm{fr}} = b^{2002(\varrho+1)^9 f} c_{\mathrm{dec}}$ with an explicit constant $c_{\mathrm{dec}} > 0$, and $c_{\mathrm{wt}} = b^{400(\varrho+1)^9 f}$.*

*Status*: read from the proof. Each step of the recurrence is Theorem 39.3.1 in the entry $i$. The zero frequency gives $c_i$ times the sum of the densities over all pairs of heads of length $i$; the pairs with a common last entry give $c_i\Sigma_i$, and those whose last entries differ give the error $c_i c_{\mathrm{fr}}\lambda_i^2$. The nonzero frequencies give the other error $c_i c_{\mathrm{fr}}\lambda_i^2$. The local bounds behind $c_{\mathrm{fr}}$ are not derived here.

**Problem 39.3.8.** Under Theorem 39.3.2, $\Sigma_i \leq \big(\prod_{i'<i} c_{i'}\big)\big(c_{\mathrm{bot}} + 2i\, c_{\mathrm{wt}} c_{\mathrm{fr}}\big)$ for $0 \leq i \leq \varrho + 1$.

In the proof, Theorem 39.2.1 bounds the open link moment, up to one normalizing constant, by $(1 + b^{-6})$ times the constant of Siegel's step times $\Sigma_{\varrho+1}$; this constant is the constant of Siegel's formula times the number of classes, the order of the reduction of the integral points of $\mathcal{U}$, and $\#\bar{R}_0$ (Chapter 37). Apply Problem 39.3.8 with $i = \varrho+1$. The product of the $c_i$, of the constant of Siegel's step and of $c_{\mathrm{bot}}$ is at most $(1 + b^{-9})^2(1 + b^{-6})\, b^{-6k\#\mathcal{D}} D_{\mathrm{main}}\nu^2$, up to the same normalizing constant. The factors are: $1 + b^{-9}$ for the constants, $(1 + b^{-10})^f \leq 1 + b^{-9}$ for the degrees (in the proof $f \leq b/1000$), and $1 + b^{-6}$ from $c_{\mathrm{bot}}$. The error terms are at most $b^{-10} \cdot b^{-6k\#\mathcal{D}} D_{\mathrm{main}}\nu^2$, because the large powers of $b$ in $c_{\mathrm{fr}}$ and $c_{\mathrm{wt}}$ are paid by $b^{11kf} D_{\mathrm{id}} \leq D_{\mathrm{main}}$ (Chapter 37). Since $b \geq 1000$, the following inequality collects these factors.

**Problem 39.3.9.** For $b \geq 1000$: $(1 + b^{-6})\big((1 + b^{-9})^2(1 + b^{-6}) + b^{-10}\big) \leq 1 + b^{-4}$.

Problems 39.1.2, 39.3.2, 39.3.6, 39.3.7, 39.3.8 and 39.3.9 model the steps of the following theorem. The proof proves it by these steps, with Theorems 39.2.1, 39.3.1 and 39.3.2 and the statements read from the proof in this sheet.

**Theorem 39.3.3.** *For the data, let $\mathcal{D} \subseteq \mathcal{M}_0$ be a set of chosen places above $p_0$, and for each $w \in \mathcal{D}$ choose $w$ or $\bar{w}$ and a nonzero vector of the span of the frame modulo it, as in Definition 38.2.7 with $D_0 = \mathcal{D}$; let the fixer of $\mathcal{D}$ be the fixer of these vectors. Then the link second moment of the terms $\gamma$ with $\mathrm{disc}(\gamma) \neq 0$, over the reductions that are open and lie in the fixer of $\mathcal{D}$, is at most*

$$(1 + b^{-4})\, b^{-6k\#\mathcal{D}}\, D_{\mathrm{main}}\, \nu^2.$$

*Status*: read from the proof, where it is stated for an admissible filling over a CM field with Weil data, which the data provide. The local bounds at nonzero frequencies and the matching of the constants are not derived here; see the remark below.
What has been proved. A term over an open reduction is a head, a last entry and a residual isometry (Section 39.1). Its last entry has $|\mathrm{le}|^2 - N^2 = N\,h(y,y)$ (Problem 39.1.2), so the residual isometries are counted by Siegel's formula (Theorem 39.2.1, modelled in Problems 39.2.1–39.2.4). The heads are summed by Poisson summation one entry at a time: zero frequencies give integrals and the degrees $d_\tau$, and the rank $k$ pays for the rest (Problems 39.3.1–39.3.7, Theorems 39.3.1 and 39.3.2). Taken as given: Theorems 39.2.1, 39.3.1 and 39.3.2, and the statements marked "in the proof". Chapter 40 combines Theorem 39.3.3 with the cut and the absolute bound of Chapter 38, and bootstraps over fixers, to bound the second moment over all of $\bar{G}$.

*In the Lean formalization.* Theorem 39.3.3 is `New.Filling.linkSecondMoment_isOpenElement_le` in the Lean proof [GL]. Theorem 39.2.1 is `New.classSum_eq_mul_normDensity`, Theorem 39.3.1 is `New.IsDualFamily.sum_le_of_fourier`, and Theorem 39.3.2 is read from `New.Filling.sieve`. Problem 39.3.8 is `densitySum_le`, Problem 39.1.2, for $V = Y_0$, is `mul_star_dotProduct_eq_last_frameProducts` with the strict inequality, Problem 39.3.6 is used as `posSemidef_prod_mul`, and the matching of the constants is `sieve_constant_le`. The block as a sum over heads is `exists_amplitudeSum_eq_sum_cosetAmplitude`, and the Kloosterman factorization is `IsFillingData.cosetAmplitude_eq_kloostermanSum_smul`. The other statements marked "in the proof" are at these places of [GL], commit `72fead1`: minors of open terms, the normal form and its minors, distinct blocks, Siegel's step for the blocks, Siegel's formula as an upper bound, the product of kernels, nonnegative densities, the zero frequency, the decay, the recurrence, $\varrho \geq 10^5$, $f \leq b/1000$, and $b^{11kf} D_{\mathrm{id}} \leq D_{\mathrm{main}}$.

## Hints

**§39.1.**

**39.1.1:** Check that $\omega$ preserves the Gram matrix $\mathrm{diag}(1,-5)$.
**39.1.2:** Expand $h(\omega z, \omega z) = h(z,z)$, using $y \perp z$.

**§39.2.**

**39.2.1:** In Sage or Python, enumerate the vectors of $\mathbb{Z}^6$ with entries in $[-2,2]$.
**39.2.2:** By machine: convolve the histogram of the squares modulo $m$ six times. For the formula, detect the equation with $\mathrm{e}(\cdot/p)$ and use the quadratic Gauss sum: every nonzero frequency gives the same sixth power. Solutions modulo $p$ with a unit coordinate lift by Hensel's lemma; compare the counts modulo 3 and 9. Every solution has an odd coordinate; for $e \geq 3$, adding $2^{e-1}$ to it changes the sum of squares by $2^e$ modulo $2^{e+1}$.
**39.2.3:** The unit ball of $\mathbb{R}^6$ has volume $\pi^3/6$.
**39.2.4:** $\prod_{p\ \mathrm{odd}} (1 - \chi_{-4}(p) p^{-3}) = L(3,\chi_{-4})^{-1}$. For $j = 5$, compute modulo 25 and 125 at 5, and at 2 use that $y \mapsto (2+\mathrm{i})y$ is a bijection modulo $2^e$ from the solutions of $y^* y \equiv 1$ to those of $y^* y \equiv 5$.

**§39.3.**

**39.3.2:** Fourier inversion and the triangle inequality.

**39.3.4:** Expand $|\mathrm{Kl}(j)|^2$ and sum over $j$ first.

**39.3.5:** The rows of $\mathfrak{A}_1$ are the characters of $\mathbb{Z}/3$; the operator norm is multiplicative under tensor products.

**39.3.6:** The entrywise product is a principal submatrix of the Kronecker product; compare [HJ, Thm. 7.5.3].

**39.3.7:** Apply the orthogonality relation with $u_1 = u_2 = u_3 = u_4 = u$.

**39.3.8:** Induction on $i$.

**39.3.9:** Each of $b^{-6}$, $b^{-9}$, $b^{-10}$ is at most $b^{-4}/100$.

## References


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