# GOOD FORMS AND THE COUNT OF THE TARGET

Prerequisites. Characters of finite groups, central idempotents, induced representations and Mackey's formula [Se, §§2, 6, 7]; the characters of $\mathrm{GL}_2(\mathbb{F}_q)$ [FH, §5.2]. From Chapter 34, *Spaces, forms and twists*: the chosen places $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1$, the groups $\bar{G} = \bar{G}_0 \times \bar{G}_1$, labels, the characters $\Xi_c$, the idempotents $e_c$, good families and the idempotent $e_0$ (Section 34.1), with the integer $q_2$ of the data; the space $W_m$ and the standard lattice stabilizer $K^\circ(V)$ (Section 34.3); principal forms $H(V)$, good forms $H_g(V)$, the target $T$ of dimension $d_T$ (Section 34.4); $D_X$ (Section 34.5). From Chapter 36: the unipotent group $\mathcal{U}$ (Section 36.1) and the character $\psi_{\mathcal{U}}$ that the proof fixes on its points (Section 36.2), the scaled Poincaré map $\mathcal{P}^\circ$ and the Gram operators $\mathcal{K}_s$ (Definition 36.4.1), the group $\bar{R}_0$ and its representation $\rho_{\mathrm{Bes}}$ (Definition 36.4.2), the product deg of the degrees of the Bessel systems, the mass $\mathrm{mass}_r$, and admissible fillings with Weil data (Section 36.4). From Chapter 37: the block identity (Theorem 37.3.2), the sizes $D_{\mathrm{id}}$, $D_{\mathrm{main}}$ (Definition 37.3.2) and Problems 37.4.2 and 37.4.3. From Chapter 38: Theorem 38.3.3 and Problems 38.1.1, 38.3.9, 38.1.2 and 38.1.3. From Chapter 9: Hilbert–Schmidt norms $\|L\|^2 = \operatorname{tr} L^*L$, near isometries, Problem 9.3.1 and Theorem 9.3.1. From Chapter 21: class sets, masses, forms and Weyl's formula (4) (Section 21.3). From Chapter 57: Corollary 57.4.1.

Theorem B (Chapter 36) is deduced from three estimates for the scaled Poincaré map: a second moment over $\bar{G}$, the near isometry of $e_0\mathcal{P}^\circ$, and an upper bound for $d_T$. This sheet proves the last two from Chapters 37 and 38 and from equal multiplicities. Why does cutting $\mathcal{P}^\circ$ down to the good forms lose so little, and why is the target no larger than $\#\bar{G} \cdot \deg \cdot \mathrm{mass}_r$? The first answer is a trace against the Bessel representation, small because the conjugacy average of $\psi_{\mathcal{U}}$ is small; the second is a count of classes on the definite space $W_m$, where the level is free.

## 41.1. Central idempotents and their shares

For a finite group $H$, an element of $\mathbb{C}[H]$ is written $g = \sum_{t \in H} g(t)\, t$; in a representation $\rho$ it acts by $\rho(g) := \sum_t g(t)\rho(t)$. It is a *central idempotent* if $g^2 = g$ and $t \mapsto g(t)$ is a class function.

**Definition 41.1.1.** The *share* of a central idempotent $g \in \mathbb{C}[H]$ is its coefficient $g(1)$ at the identity.

In this sheet $q$ is the number of elements of a residue field.

**Problem 41.1.1.** The labels of degree 2 over $\mathbb{F}_3$ are $X^2+1$, $X^2+2$, $X^2+X+2$ and $X^2+2X+2$. The multiplicative orders of their roots in $\mathbb{F}_9^\times$ are $(4,4)$, $(1,2)$, $(8,8)$ and $(8,8)$. Three of them have no root 1, and two are irreducible with a root of order $8 = 3^2 - 1$.

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*Green's degrees.* For a label $c$ of degree $r$ over a finite field $k$ with $q = \#k$, the value $d := \Xi_c(1)$ of the character of Section 34.1 satisfies

$$\operatorname{Re} d \geq 0, \qquad \frac{|d|^2}{\#\mathrm{GL}_r(k)} \geq \frac{1 - r/q}{q^r}.$$

*Status*: classical; Green [Gr, Thms. 12–14]. For $r = 2$ the degree is $q+1$ for a label with two roots in $k$, $q-1$ for an irreducible one [FH, §5.2].

**Problem 41.1.2.** In $\mathrm{GL}_2(\mathbb{F}_3)$, of order 48, the idempotent $e_c$ of the label $X^2 + 2$ has share $1/3$, and that of each irreducible label has share $1/12 \geq (1-2/3)/9$. The sum of the $e_c$ over the three labels without root 1 is a central idempotent of share $1/4$.

**Problem 41.1.3.** Let $g$ be a central idempotent of $\mathbb{C}[H]$, $H$ finite. In every unitary representation of $H$, $g$ acts as an orthogonal projection, and $g(1)$ is the share $\dim(g\mathbb{C}[H])/\#H$ of the regular representation that $g$ keeps. Moreover $\sum_{t\in H}|g(t)|^2 = g(1)$, and $1 - g$ is a central idempotent.

The idempotents of the data. For $i = 0, 1$ let $g_i \in \mathbb{C}[\bar{G}_i]$ be the sum, over the families $(c_w)_{w\in\mathcal{M}_i}$ of labels such that no $c_w$ has a root $z \in k_w$ with $z^{q_2} = 1$ and some $c_w$ is irreducible with a root of order $p_i^r - 1$, of the tensor products $\bigotimes_w e_{c_w}$. These are central idempotents, and $g_0 \otimes g_1 \in \mathbb{C}[\bar{G}]$ acts on $H(Y_0)$ as $e_0$. Call a label over $k_w$ *allowed* if it has no root $z \in k_w$ with $z^{q_2} = 1$, and *primitive* if it is irreducible with a root of order $q^r - 1$; at one place $w \in \mathcal{M}_i$, with $q = p_i$, let $A$ be the set of allowed labels, $B$ the set of monic polynomials of degree $r$ over $k_w$ that are not primitive labels, and $m' := (1 - r/q)/q^r$. For $1 < r < q$ the proof shows that each tensor product $\bigotimes_w e_{c_w}$ has share at least $m'^f$ (by Green's degrees), that $\#A \cdot m' \geq 1 - \varepsilon'$ whenever $\varepsilon' \geq (1 + q_2 + 2r)/q$, and that $\#B \cdot m' \leq \eta := \exp\bigl(-1/(2r(2r\log q + 1))\bigr)$.

**Problem 41.1.4.** Let $f \geq 1$ be an integer, $A$, $B$ finite sets, $m' > 0$, $0 \leq \varepsilon' \leq 1$ and $\eta \geq 0$, with $\#A \cdot m' \geq 1 - \varepsilon'$ and $\#B \cdot m' \leq \eta$. Then $(1 - \varepsilon')^f \geq 1 - f\varepsilon'$ and $m'^f(\#A^f - \#B^f) \geq 1 - f\varepsilon' - \eta^f$. Hence, in the paragraph above, the families counted in $g_i$ include every family of allowed labels not all in $B$, and $\operatorname{Re}(1 - g_i)(1) \leq f\varepsilon' + \eta^f$.

With one place, $\eta$ is close to 1; only its $f$-th power is small. The proof takes $\varepsilon' = (1 + q_2 + 2r)/p_i$ and checks, for the data, $1 < r < p_i$, $\varepsilon' \leq 1$, $2r(2r\log p_i + 1) \cdot 9\log b \leq f$ and $f\varepsilon' + e^{-9\log b} \leq b^{-8}/2$. So $\operatorname{Re}(1 - g_i)(1) \leq b^{-8}/2$ for $i = 0, 1$.

**Problem 41.1.5.** If $p$ is an orthogonal projection, $q'$ any operator on $\mathcal{Y}$ and $L\colon \mathcal{X} \to \mathcal{Y}$, then $\|(1 - pq')L\|^2 \leq \|(1 - p)L\|^2 + \|(1 - q')L\|^2$. In the regular representation of $C_5 \times C_5$, with $p$ and $q'$ the projections removing one character of the first and of the second factor and $L = 1$, the two sides divided by 25 are $9/25 \leq 2/5$.

## 41.2. The trace on the Bessel subgroup

In this section, as in Theorem 38.3.3, consider an admissible filling over a CM field $E$ with Weil data, a unitary representation $\rho$ of $\bar{G}$ on a finite-dimensional complex inner product space, and a linear map $\mathcal{P}^\circ$ from the coordinate space of Definition 36.4.1 to it that satisfies the block identity of Theorem 37.3.2; $\mathcal{K}_s = \mathcal{P}^{\circ *}\rho(s)\mathcal{P}^\circ$, $\rho_{\mathrm{Bes}}$, $\bar{R}_0$, $D_X$ and $D_{\mathrm{id}} = \#\bar{R}_0 \cdot D_X$ are as there. In this setting, as for the data, $\rho_{\mathrm{Bes}}(s) = 0$ for $s \notin \bar{R}_0$, $\rho_{\mathrm{Bes}}$ is a unitary representation of $\bar{R}_0$, and $\mathcal{K}_{st} = \mathcal{K}_s\rho_{\mathrm{Bes}}(t)$, $\mathcal{K}_{ts} = \rho_{\mathrm{Bes}}(t)\mathcal{K}_s$ for $t \in \bar{R}_0$ (read from the proof; Problem 37.4.2).

In this sheet $\iota_i\colon \bar G_i \to \bar G$ is the inclusion with identity component in the other factor, and $\rho(g) := \sum_t g(t)\rho(\iota_i(t))$ for $g \in \mathbb{C}[\bar G_i]$.

**Problem 41.2.1.** (a) Let $\varphi\colon H' \to H$ be a homomorphism of finite groups, $g$ a central idempotent of $\mathbb{C}[H']$, and $F$ a function on $H$ with values in an inner product space. Then $\sum_{s\in H}\|\sum_{t\in H'} g(t)F(s\varphi(t))\|^2 \le \sum_{s\in H}\|F(s)\|^2$.

(b) Let $g \in \mathbb{C}[\bar G_i]$ be a central idempotent and $D_g := \sum_{t\in\bar G_i} g(t)\big(\mathcal{K}_{\iota_i(t)} - \rho_{\mathrm{Bes}}(\iota_i(t))\big)$. Then $\#\bar R_0 \cdot \|D_g\|^2 \le b^{-12}D_{\mathrm{id}}$, so $\|D_g\|^2 \le b^{-12}D_X$ and $|\operatorname{tr} D_g| \le b^{-6}D_X$.

There remains the trace of $\rho_{\mathrm{Bes}}$ against $g$. The proof shows: if $s \in \bar G$ has component 1 in $\bar G_0$ or in $\bar G_1$, then

$$\operatorname{tr}\rho_{\mathrm{Bes}}(s) = D_X\,\psi_{\mathcal{U}}(s),$$

where, in this sheet, $\psi_{\mathcal{U}}(s)$ is the value of that character at an integral finite adelic element of $\mathcal{U}$ reducing to $s$, which does not depend on the element, and $\psi_{\mathcal{U}}(s) := 0$ if there is none. (A Bessel term reducing to 1 at one prime has trivial definite part, by an argument like that of Problem 41.3.2; the unipotent part multiplies the identity block by $\psi_{\mathcal{U}}$.)

**Definition 41.2.1.** The *conjugacy average* of a function $\psi$ on a finite group $H$ is $\psi^\natural(s) := (\#H)^{-1}\sum_{y\in H}\psi(ysy^{-1})$.

**Problem 41.2.2.** In $S_3$ let $\varphi = 1$ at 1, $-1$ at (12) and 0 elsewhere. Its conjugacy average is 1, $-1/3$, 0 on the identity, the transpositions and the 3-cycles, and $\sum_s|\varphi(s)|^2 = 2$, $\sum_s|\varphi^\natural(s)|^2 = 4/3$.

**Problem 41.2.3.** For a character $\psi$ of a subgroup $R \subseteq H$, extended by zero to $H$, $\sum_{s\in H}|\psi^\natural(s)|^2 = \#R^2 \cdot \dim \operatorname{End}_H(\operatorname{Ind}_R^H \psi)/\#H$.

**Problem 41.2.4.** Let $U \subset \mathrm{GL}_2(\mathbb{F}_p)$ be the upper unitriangular group and $\psi_U(u) := e^{2\pi i u_{12}/p}$. By Problem 38.1.2, $\sum_s|\psi_U^\natural(s)|^2 = p^2/(p^2-1)$: 9/8 for $p = 3$, 25/24 for $p = 5$. For the upper unitriangular group of $\mathrm{GL}_3(\mathbb{F}_2)$ and $\psi_U(u) = (-1)^{u_{12}+u_{23}}$, Problem 38.1.3 gives 32/21.

*The conjugacy average of $\psi_{\mathcal{U}}$.* For an admissible filling with Weil data,

$$\sum_{s\in\bar G}|\psi_{\mathcal{U}}^\natural(s)|^2 \le 2.$$

*Status*: read from the proof. There the conjugacy average is written as an integral over the integral points at the chosen places and their conjugates, and bounded by a count of shells; this sheet does not treat it.

**Problem 41.2.5.** Let $g \in \mathbb{C}[\bar G_i]$ be a central idempotent with $\operatorname{Re} g(1) \le b^{-8}/2$. Then $\big|\sum_t g(t)\operatorname{tr}\rho_{\mathrm{Bes}}(\iota_i(t))\big| \le D_X\sqrt{2\operatorname{Re} g(1)} \le b^{-4}D_X$, and $\|\rho(g)\mathcal{P}^\circ\|^2 \le 2b^{-4}D_X$.

Recall from Chapter 9 that $L\colon \mathcal{X} \to \mathcal{Y}$ is a near isometry with error $\varepsilon$ if $\dim Z - \varepsilon\dim\mathcal{X} \le \|L\operatorname{pr}_Z\|^2$ for every subspace $Z \subset \mathcal{X}$, where $\operatorname{pr}_Z$ is the orthogonal projection onto $Z$, and that by Problem 9.3.1 this holds if $\varepsilon \ge 0$ and $\|L^*L - 1\|^2 \le \varepsilon^2\dim\mathcal{X}$.

**Problem 41.2.6.** If $L$ is a near isometry with error $\varepsilon$, $p$ is an orthogonal projection and $\|(1-p)L\|^2 \le \varepsilon'\dim\mathcal{X}$, then $pL$ is a near isometry with error $\varepsilon + \varepsilon'$.

For central idempotents $g_0 \in \mathbb{C}[\bar{G}_0]$, $g_1 \in \mathbb{C}[\bar{G}_1]$, the element $g_0 \otimes g_1 \in \mathbb{C}[\bar{G}]$ has coefficient $g_0(s_0)g_1(s_1)$ at $(s_0, s_1)$, and $\rho(g_0 \otimes g_1) = \rho(g_0)\rho(g_1)$, a product of two commuting orthogonal projections. Problems 41.1.3, 41.1.5 and 41.2.5 (with $g = 1 - g_i$) prove part (a) of the following theorem; Problem 38.3.9, whose proof (Theorem 38.3.3 on $\bar{R}_0$ and Problem 37.4.3) holds in the setting of this section, and Problems 9.3.1 and 41.2.6 then prove part (b). Part (c) follows from (b) and the paragraph on the idempotents of the data.

**Theorem 41.2.2** (good forms)**.** *In the setting of this section, let $g_i \in \mathbb{C}[\bar{G}_i]$ be central idempotents with $\mathrm{Re}(1 - g_i)(1) \le b^{-8}/2$ for $i = 0, 1$, and $e := \rho(g_0 \otimes g_1)$. Then*

(a) $\|(1 - e)\mathcal{P}^\circ\|^2 \le 4b^{-4}D_X$*;*
(b) $\|e\mathcal{P}^\circ \,\mathrm{pr}_Z\|^2 \ge \dim Z - 5b^{-4}D_X$ *for every subspace $Z$ of the coordinate space: $e\mathcal{P}^\circ$ is a near isometry with error $5b^{-4}$.*

*(c) For the data, with $\rho$ the action of $\bar{G}$ on $H(Y_0)$ and $\mathcal{P}^\circ$ the scaled Poincaré map, which satisfy the hypotheses of this section (Chapter 37), the idempotents $g_0, g_1$ of Section 41.1 satisfy these hypotheses and $e$ is the action of $e_0$: $e_0\mathcal{P}^\circ$ is a near isometry with error $5b^{-4}$.*

## 41.3. The count of the target

On a definite group a form is a vector of the weight at each class, fixed by the stabilizer of the class. First the smallest case.

**Problem 41.3.1.** Let $E = \mathbb{Q}(\mathrm{i})$, and let the unitary group of rank one over $E$, at the level $K$ of the lattice $\mathbb{Z}[\mathrm{i}]$, have class number one, stabilizer $\Gamma \cap K = \mu_4$ ($\Gamma$ the rational points) and mass $\mathrm{mass}_1 = 1/4$. At the split prime $13 = (3 + 2\mathrm{i})(3 - 2\mathrm{i})$ take the place $w = (3 + 2\mathrm{i})$, where $\mathrm{i} \equiv 5$. Then $\mu_4$ reduces to $\{1, 5, 12, 8\} \subset \mathbb{F}_{13}^\times$, no unit other than 1 reduces to 1, and the level of integral elements reducing to 1 at $w$ has $12/4 = 3$ classes, which is $\#\bar{G} \cdot \mathrm{mass}_1$ for $\bar{G} = \mathbb{F}_{13}^\times$.

In Problem 41.3.2, $n \ge 1$ is a size.

**Problem 41.3.2.** Let $E$ be a CM field, $\gamma$ a rational unitary matrix of size $n$ of the standard definite space over $E$, and $h$ a finite adelic unitary matrix such that $h^{-1}\gamma h$ is integral at every finite place and reduces to 1 at distinct finite places $w_1, \dots, w_l$ with $(2n)^{[E:\mathbb{Q}]} < \prod_i \#k_{w_i}$. Then $\gamma = 1$. In particular this holds if $(2n)^2 < p$, $[E : \mathbb{Q}] = 2f$, and $w_1, \dots, w_f$ have residue fields with $p$ elements.

The proof checks this inequality at the $f$ chosen places above $p_0$ for the rank $r$ of $W_m$. So every class of $W_m$ at the principal level has trivial stabilizer.

**Problem 41.3.3.** Let $\Gamma$ be the rational points of the finite adelic unitary group $G_f$ of a definite space, $K' \subseteq K$ compact open subgroups of $G_f$, and $\Gamma_h(K) := \Gamma \cap hKh^{-1}$. (a) $\sum_{[h] \in \Gamma\backslash G_f/K'} 1/\#\Gamma_h(K') = [K : K'] \sum_{[h] \in \Gamma\backslash G_f/K} 1/\#\Gamma_h(K)$. (b) If every $\Gamma_h(K')$ is trivial and $\pi$ is a representation of $\Gamma$ on a space $A$, the functions $\varphi\colon G_f \to A$ with $\varphi(\gamma hk) = \pi(\gamma)\varphi(h)$ for $\gamma \in \Gamma$, $k \in K'$ form a space of dimension $\#(\Gamma\backslash G_f/K') \cdot \dim A$. (c) For $W_m$, $K = K^\circ(W_m)$ and $K'$ the principal level, $[K : K'] = \#\bar{G}$, and, since the principal level is free and $\mathrm{mass}_r$ is the sum of (a) at the level $K$, the number of classes at the principal level is $\#\bar{G} \cdot \mathrm{mass}_r$.

*Harish-Chandra's formal degree.* Let $\varepsilon$ be signs of rank $r$ and $l = (l_1, \dots, l_r)$ distinct integers $\equiv r - 1 \pmod 2$ with $|l_i - l_j| \ge 2(500r^3 + 10r^5)$ whenever $\varepsilon_i \ne \varepsilon_j$. The

formal degree of the discrete series of the unitary group of the signs $\varepsilon$ with doubled parameter $l$ is

$$\prod_{l_i>l_j} \frac{l_i-l_j}{2} \Big/ \prod_{a=0}^{r-1} a!,$$

whatever the signs. *Status*: classical; Harish-Chandra [HC, Thm. 16]. For definite signs it is Weyl's formula (4) with $z = l/2 + (r-1)/2$.

**Problem 41.3.4.** For the doubled parameter $(6, 2, -2)$ the quotient is 8, the dimension of $\mathbb{S}_{(2,1,0)}(\mathbb{C}^3)$.

In the proof the degree of the Bessel system at each infinite place $\tau$ is this product for the doubled parameter $2\varsigma_\tau$, so deg is the product over $\tau$ of the formal degrees; and on the definite space $W_m$ the principal forms are the functions of Problem 41.3.3(b), at the principal level, with values in a representation of dimension deg.

**Problem 41.3.5.** Hence $\dim H(W_m) = \#\bar{G} \cdot \mathrm{mass}_r \cdot \deg$, and with Corollary 57.4.1 for $j = m$, $d_T \le \#\bar{G} \cdot \deg \cdot \mathrm{mass}_r$. In the toy of Chapter 37, where the 32 elements of $\Gamma_{I_2}$ have distinct reductions modulo $2+\mathrm{i}$ in $\mathrm{GL}_2(\mathbb{F}_5)$ (order 480), and the weight is $\mathbb{C}^2$: $30 = 480 \cdot \frac{1}{32} \cdot 2$.

Problems 41.3.2, 41.3.3 and 41.3.5, with Corollary 57.4.1, Harish-Chandra's formal degree and the facts read from the proof stated after Problems 41.3.2 and 41.3.4, prove the following.

**Theorem 41.3.1** (the count)**.** *For all data,* $d_T \le \#\bar{G} \cdot \deg \cdot \mathrm{mass}_r$.

The second moment, Theorem 40.3.1, reads

$$\sum_{s\in\bar{G}} \|\mathcal{K}_s\|^2 \le (1 + 9b^{-4})\, D_{\mathrm{main}}.$$

**Problem 41.3.6.** Then $d_T \sum_{s\in\bar{G}} \|\mathcal{K}_s\|^2 \le \#\bar{G}\,(1 + 9b^{-4})\, D_X^2$: with Theorem 41.2.2(c), this is the hypothesis of Theorem 9.3.1 for $\Gamma = \bar{G}$, $S$ the coordinate space, $B = \mathcal{P}^\circ$ and $p$ the action of $e_0$ on $H(Y_0)$, whose image $T_0$ is the target $T$, and any $\varepsilon \ge 9b^{-4}$.

This sheet has proved Theorem 41.2.2(a), (b) from Theorem 38.3.3, Problem 38.3.9 and two statements taken as given (the trace of $\rho_{\mathrm{Bes}}$ and the bound for the conjugacy average of $\psi_{\mathcal{U}}$), and Theorem 41.3.1 from equal multiplicities, freeness and class sums. Also taken as given: Green's degrees, Harish-Chandra's formal degree, the facts read from the proof after Problems 41.3.2 and 41.3.4 and in the paragraph on the idempotents of the data. Chapter 36 combines both theorems with the second moment to obtain Theorem B.

In the Lean formalization. In the Lean proof [GL] (`grh/src/`, namespace `New` omitted), Theorem 41.2.2(a), (b) are `Filling.HasAmplitudeBlocks.hilbertSchmidtNorm_one_sub_piTensorElem_comp_sq_le` and `Filling.HasAmplitudeBlocks.isNearIsometry_piTensorElem_comp`; (c) is `Data.bad_types_rare` with `Data.isNearIsometry_e₀_comp_scaledPoincare`. The facts read from the proof opening Section 41.2 are `Filling.besselTranslate_eq_zero`, `Filling.besselTranslate_one`, `Filling.besselTranslate_mul`, `Filling.adjoint_besselTranslate`,

`Filling.HasAmplitudeBlocks.adjoint_comp_rep_mul_right` and `Filling.HasAmplitudeBlocks.adjoint_comp_rep_mul_left`. The last bound of Problem 41.2.5 is `Filling.HasAmplitudeBlocks.hilbertSchmidtNorm_asAlgebraHom_comp_sq_le`; for its first bound `Filling.re_sum_coeff_mul_trace_besselTranslate_le` states the real part, and its proof bounds the absolute value. The share bounds are `re_coeff_one_sub_sum_tupleIdem_le` and `Data.re_coeff_one_sub_goodIdempotentAbove_le`, and the three facts of the paragraph on the idempotents of the data are `prod_le_re_tupleIdem_coeff_one`, `one_sub_le_card_allowed_mul` and `card_not_primitive_mul_le`; the trace of $\rho_{\mathrm{Bes}}$ is `Filling.trace_besselTranslate_eq_mul_reducedChar`; Definition 41.2.1 for $\psi_{\mathcal{U}}$ is `Filling.reducedCharMean`, and its bound is `Filling.sum_norm_reducedCharMean_sq_le`. Green's degrees are `labelChar_degree_bounds`, Harish-Chandra's formal degree the first part of `Orbital.harish_chandra_degree_and_orbital`. Theorem 41.3.1 is `Data.targetDim_le_card_mul_besselDegree_mul_mass`; freeness is `Variety.eq_one_of_reduceAt_eq_one`, stated there for $E$ with at least two infinite places, and applied to $W_m$ through `Variety.classAut_eq_bot` in `Data.classAut_eq_bot_of_le_levelOf_chosenPlace`, used by `Data.classAut_eq_bot_of_le_chosenLevel`; $\dim H(W_m)$ is `Data.finrank_principalForms_spaceW`, and the degree of the Bessel system is the field `deg_eq` of `Filling.IsDiscreteSeriesBessel`, at the parameter $2\varsigma_\tau$ by `Data.isDiscreteSeriesBessel_filling_besselSystem`.

## Hints

**§41.1.**

**41.1.1:** $\mathbb{F}_9 = \mathbb{F}_3[\mathrm{i}]$.

**41.1.2:** The share of $e_c$ is $\Xi_c(1)^2/\#\mathrm{GL}_2(\mathbb{F}_3)$. Which labels are irreducible?

**41.1.3:** Write $g$ as a sum of the idempotents $e_\chi$ of the irreducible characters. For the last identity compare the coefficients at 1 of $g^2$ and $g$.

**41.1.4:** Induction on $f$. The good tuples are the tuples of allowed labels minus those with no primitive member.

**41.1.5:** Split $1 - pq'$ into two orthogonal pieces.

**§41.2.**

**41.2.1:** (a) $H'$ acts unitarily on functions on $H$ by right translation through $\varphi$. (b) Apply (a), with $\varphi = \iota_i$ and the Hilbert–Schmidt inner product, to $\mathcal{K}_s - \rho_{\mathrm{Bes}}(s)$ on the set of Theorem 38.3.3 pinned in $\bar{G}_{1-i}$, and 0 elsewhere.

**41.2.3:** $\psi^\natural$ is a multiple of the induced character.

**41.2.4:** By Mackey, $\dim \mathrm{End}_H(\mathrm{Ind}_U^H \psi_U)$ is the number of double cosets supporting the functions of Problem 38.1.1.

**41.2.5:** Since $g$ is a class function, $\psi_{\mathcal{U}}$ may be replaced by $\psi_{\mathcal{U}}^\natural$. For the second bound, $\|\rho(g)\mathcal{P}^\circ\|^2 = \mathrm{tr}(\mathcal{P}^{\circ *}\rho(g)\mathcal{P}^\circ)$.

**§41.3.**

**41.3.1:** The classes are the orbits of $\mu_4$ on $\mathbb{F}_{13}^\times$.

**41.3.2:** Consider the norm from $E$ to $\mathbb{Q}$ of $\mathrm{tr}(\gamma) - n$.

**41.3.3:** (a) Each $K$-class splits into $K'$-classes; count by orbit and stabilizer. (c) $W_m$ has identity Gram matrix, so $\Gamma_h(K^\circ(W_m))$ is the automorphism

group of the lattice $h$ applied to the standard one. Reduction maps $K^\circ(W_m)$ onto $\bar{G}$: the $p_i$ split completely and $\mathcal{M}$ contains no two conjugate places.

**41.3.5:** The principal forms on $W_m$ contain the good ones.

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