WHICH TERMS SURVIVE: CHARACTERS, MINORS AND THE GAIN AT THE LEVEL

Prerequisites. From Chapter 37: amplitudes and amplitude sums $\mathrm{Amp}(\mathfrak{S}; c, s, c')$ over sets $\mathfrak{S}$ of rational points (Definition 37.3.1), the two sizes D_{id}, D_{main} (Definition 37.3.2), the block identity (Theorem 37.3.2), the full Bessel function W and its cut $W^\flat$ (Theorem 37.3.1), and Problems 37.4.3 and 37.4.4. From Chapter 36: the decomposition $Y_0 = \bigoplus_a Ev_a \oplus X_0 \oplus Ez \oplus \bigoplus_a Ev'_a$ with $h(z,z) = -N$, the groups $\mathcal{U}$ and $R = \mathcal{U} \rtimes U(X_0)$ (Section 36.1); the coordinate space with basis $e_{c,i}$, the scaled Poincaré map $\mathcal{P}^\circ$, the Gram operators $\mathcal{K}_s$ and the covolume ν (Definition 36.4.1); $\bar{R}_0$ and ρ_{Bes} (Definition 36.4.2); the lattice basis, in which integrality and reduction are taken. From Chapter 34: $\mathcal{M}_i$, $\bar{G} = \bar{G}_0 \times \bar{G}_1$ and Problem 34.1.1; from Definition 33.3.1: $b^{10} \le p_i \le b^{11}$, $M = p_0 p_1$; and $b \ge 1000$ (Chapter 36). From Chapter 10: Theorem 10.2.1 and Problem 10.3.2. The Bruhat decomposition of $\mathrm{GL}_n(\mathbb{F}_p)$ [Ca72, §8.2]. Hermitian forms and inner products are linear in the first variable; norms of operators are Hilbert–Schmidt norms.

This sheet serves the second moment in the proof of Theorem 36.4.2 of Chapter 36, *Theorem B: enough forms*. From three statements of the proof it derives the two estimates near the Bessel subgroup with which that proof begins: Theorem 38.3.3 and Problem 38.3.9. By Theorem 37.3.2, the $((c,i),(c',i'))$ entry of $\mathcal{K}_s$ is the complex conjugate of $\nu^{-1}\mathrm{Amp}(U(Y_0)(F); c, s, c')_{ii'}$, a sum over the rational points γ for which $h_c^{-1}\gamma_f h_{c'}$ is integral with reduction s. Which sets of terms have amplitude sum zero, and how large is the second moment of the terms outside $R(F)$ when s agrees with an element of $\bar{R}_0$ at one level prime?

38.1. Characters that disagree

Problem 38.1.1. Let U, U' be subgroups of a finite group H with characters ψ, ψ', and let $\phi\colon H \to \mathbb{C}$ satisfy $\phi(ugu') = \psi(u)\phi(g)\psi'(u')$ for $u \in U$, $u' \in U'$. If $ugu' = g$ for some u, u' with $\psi(u)\psi'(u') \ne 1$, then $\phi(g) = 0$.

Let $G = \mathrm{GL}_n(\mathbb{F}_p)$, and let $U \subset G$ be the upper unitriangular group, ψ a nontrivial character of $\mathbb{F}_p$, and $\psi_U(u) := \psi(\sum_i u_{i,i+1})$. A double coset UgU *supports* if $\psi_U(u)\psi_U(u') = 1$ whenever $ugu' = g$ (this does not depend on the representative g); by Problem 38.1.1 the functions ϕ with $\phi(ugu') = \psi_U(u)\phi(g)\psi_U(u')$ vanish off the supporting double cosets. By the Bruhat decomposition every double coset is $UwtU$ with w a permutation matrix and t diagonal.

Problem 38.1.2. In $\mathrm{GL}_2(\mathbb{F}_p)$, $U\,\mathrm{diag}(t_1,t_2)U$ supports if and only if $t_1 = t_2$, and every double coset in the big cell supports. So $(p-1) + (p-1)^2 = p(p-1)$ of the $2(p-1)^2$ double cosets support: 6 of 8 for $p = 3$, 20 of 32 for $p = 5$.

Problem 38.1.3. By machine: in $\mathrm{GL}_3(\mathbb{F}_2)$, 4 of the 6 double cosets support, of sizes $8, 32, 32, 64$ (the two simple reflections do not); in $\mathrm{GL}_3(\mathbb{F}_3)$, $18 = 2 + 8 + 8$ of the 48 support, of sizes 27, 243, 729 respectively.

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In all four cases the number of supporting double cosets is $p^{n-1}(p-1)$. By Mackey's formula it is $\dim \mathrm{End}_G(\mathrm{Ind}_U^G \psi_U)$, and since the Gelfand–Graev representation $\mathrm{Ind}_U^G \psi_U$ is multiplicity free [Ca85, §8.1], it is the number of its irreducible constituents.

Problem 38.1.4. On $\mathbb{F}_p^2$ put $h(y,y') := y_1y_2' + y_2y_1'$, and for $g \in \mathrm{GL}_2(\mathbb{F}_p)$ let $\langle g\rangle_1 := h(g\mathbf{e}_1, \mathbf{e}_1)$. Then $\langle g\rangle_1 = g_{21}$; $\mathbb{F}_p g\mathbf{e}_1 = \mathbb{F}_p\mathbf{e}_1$ if and only if $\langle g\rangle_1 = 0$; and the g with $\langle g\rangle_1 = 0$ whose double coset supports are exactly the scalar multiples of elements of U.

38.2. Frames, discriminants and links

In the proof the roles of U, ψ_U and $\mathbf{e}_1$ are played by $\mathcal{U}$, the character $\psi_{\mathcal{U}}$ and the flag of $v_1, \dots, v_\varrho$. The group $U(Y_0)(F)$ of rational points contains the rational Bessel subgroup $R(F)$.

Definition 38.2.1. The *frame* is $(v_1, \dots, v_\varrho, v_{\varrho+1})$ with $v_{\varrho+1} := z$. For $\gamma \in U(Y_0)(F)$ the matrix of *frame products* is $\big(h(\gamma v_b, v_a)\big)_{1\le a,b\le \varrho+1}$: for $a \le \varrho$ its entry is the coefficient of v_a' in γv_b, and for $a = \varrho+1$ it is $-N$ times the coefficient of z. Its leading minors are $\langle\gamma\rangle_j := \det\big(h(\gamma v_b, v_a)\big)_{1\le a,b\le j}$ for $1 \le j \le \varrho+1$, and $\langle\gamma\rangle_0 := 1$.

Definition 38.2.2. The *discriminant* of γ is $\mathrm{disc}(\gamma) := \langle\gamma\rangle_{\varrho+1}\overline{\langle\gamma\rangle_{\varrho+1}} - N^2\langle\gamma\rangle_\varrho\overline{\langle\gamma\rangle_\varrho} \in E$.

In this sheet $\mathrm{Fl} := Ev_1 + \cdots + Ev_\varrho$, and for $y \in Y_0$ we write $[y] \in X_0 \oplus Ez$ for its component in $X_0 \oplus Ez$; for $y \perp \mathrm{Fl}$ it represents the class of y in $\mathrm{Fl}^\perp/\mathrm{Fl}$.

Definition 38.2.3. The *link* of γ is the subspace $\mathrm{lk}(\gamma) := \{([y], [\gamma^{-1}y]) : y \perp \mathrm{Fl},\ y \perp \gamma\mathrm{Fl}\}$ of $(X_0 \oplus Ez)^2$.

The *characters agree* on γ if, for all $u, u' \in \mathcal{U}(F)$ with $u\gamma = \gamma u'$, the character that the proof fixes on the finite adelic points of $\mathcal{U}$ takes the same value at the finite components of u and u'. In this sheet

$$\Gamma^* := \{\gamma : \mathrm{disc}(\gamma) \ne 0\} \cup \{\gamma : \text{the characters agree on } \gamma,\ \mathrm{disc}(\gamma) = 0,\ \gamma\mathrm{Fl} \ne \mathrm{Fl}\}$$

is the set of *remaining terms*.

Definition 38.2.4. For $\zeta \in \mathcal{O}_E$ with $\zeta\bar\zeta = 1$, the *scalar coset* $\zeta\bar R_0$ is the set of $s \in \bar G$ for which some $t \in \bar R_0$ has $s_w = (\zeta \bmod w)\, t_w$ for every $w \in \mathcal{M}$. A set $\mathcal{B} \subseteq \bar G$ *avoids the scalar cosets* if it meets no $\zeta\bar R_0$ with $\zeta \ne 1$.

Theorem 38.2.1 (the terms that survive)**.** *If $\mathcal{B} \subseteq \bar G$ avoids the scalar cosets and $s \in \mathcal{B}$, then for all classes c, c', for W and for W^b,*

$$\mathrm{Amp}\big(U(Y_0)(F) \smallsetminus R(F); c, s, c'\big) = \mathrm{Amp}(\Gamma^*; c, s, c').$$

Status: read from the proof, where it is stated for an admissible filling (Section 36.4) over a CM field equipped with a CM type and containing a primitive 5040-th root of unity, which the data provide. Problem 38.1.1 is its finite model; the declaration is named in the remark at the end of the chapter.

Problem 38.2.1. In a three-dimensional model over E, let $\varrho = 1$ and $X_0 = 0$, with basis v_1, z, v_1', $h(v_1, v_1') = 1$, $h(v_1, v_1) = h(v_1', v_1') = 0$, $h(z,z) = -N$ and $z \perp v_1, v_1'$. The two maps $\gamma_\pm$ exchanging v_1 and v_1' with $z \mapsto \pm z$ are isometries of square 1

with $\langle\gamma_\pm\rangle_1 = 1$, $\langle\gamma_\pm\rangle_2 = \mp N$, $\mathrm{disc}(\gamma_\pm) = 0$ and $\gamma_\pm \mathrm{Fl} \neq \mathrm{Fl}$; the characters agree on them, whatever the character of $\mathcal{U}$, so both lie in Γ^*, and $\mathrm{lk}(\gamma_\pm)$ is spanned by $(z, \pm z)$.

Reductions act on the lattice modulo w in the lattice basis, in which the frame vectors are basis vectors. For $s \in \bar{G}$ and $w \in \mathcal{M}$, let s act modulo w by s_w, and modulo $\bar{w}$ by the adjoint inverse of s_w for the bilinear form whose matrix is the Gram matrix of the lattice basis reduced modulo w, which is invertible. For $w \in \mathcal{M}_i$, $\#k_w = p_i$ (Problem 34.1.1).

Definition 38.2.5. The element s *meets* at w (or at $\bar{w}$) if, modulo that place, the span of the frame and the span of its image under s share a nonzero vector. The *meet norm* $\mathrm{mt}(s)$ is the product, over $w \in \mathcal{M}$, of a factor $\#k_w$ if s meets at w and a further factor $\#k_w$ if s meets at $\bar{w}$.

Definition 38.2.6. The element s is *open* if at some $w \in \mathcal{M}_0$ the leading minors of orders $1, \dots, \varrho$ of the matrix $\big(h(s_w v_b, v_a)\big)_{a,b \le \varrho+1}$, formed with the reduced form, are all nonzero.

Definition 38.2.7. Let $D_0 \subseteq \mathcal{M}_0$, and for each $w \in D_0$ choose w or $\bar{w}$ and a nonzero vector y_w of the span of the frame modulo it. The *fixer* of these vectors is the subgroup of the $s \in \bar{G}$ whose action modulo the chosen place fixes y_w, for every $w \in D_0$.

For $j = 0, 1$, the *strip* $\mathcal{B}_j$ is the set of $s \in \bar{G}$ whose component in $\bar{G}_j$ equals that of an element of $\bar{R}_0$.

Problem 38.2.2. If $\zeta \in \mathcal{O}_E$, $\zeta\bar{\zeta} = 1$ and $\zeta \equiv 1$ modulo every $w \in \mathcal{M}_j$, then $\zeta = 1$. For example, in $\mathbb{Z}[\mathrm{i}]$ reduction modulo $2 + \mathrm{i}$ is injective on μ_4.

Problem 38.2.3. Every $s \in \mathcal{B}_j$ meets at w and at $\bar{w}$ for every $w \in \mathcal{M}_j$, so $\mathrm{mt}(s) \ge p_j^{2f} \ge (M/b)^f$; and $\mathcal{B}_j$ avoids the scalar cosets.

Problem 38.2.4. Every fixer contains the reduction of the finite adelic points of $U(X_0)$ that preserve the lattice, and each element of a fixer meets at the chosen place over every $w \in D_0$.

38.3. The gain at the level

On the strip $\mathcal{B}_j$ every reduction meets at the $2f$ places $w, \bar{w}$, $w \in \mathcal{M}_j$. In Petersson's formula the same mechanism appears when the terms are sorted by their reduction modulo p.

Problem 38.3.1. Let p be a prime and $k' \ge 2$, $m', n' \ge 1$ integers (primed, since k and m belong to the data); for even $k' \ge 4$ these are the weight and the indices of Theorem 10.2.1. If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is upper triangular modulo p and $\gamma_{21} \neq 0$, then $|\gamma_{21}| \ge p$ and $|J_{k'-1}(4\pi\sqrt{m'n'}/|\gamma_{21}|)| \le (2\pi\sqrt{m'n'}/p)^{k'-1}/(k'-1)!$.

Problem 38.3.2. Let $\mathbf{e}_1, \mathbf{e}_2$ be the standard basis of $\mathbb{Z}^2$. The integral columns $\mathbf{e}_1$ and $\mathbf{e}_1 + p\mathbf{e}_2$ agree modulo p; the lattice they span has determinant p and Gram determinant p^2 ($2, 4$ for $p = 2$; $3, 9$ for $p = 3$).

In this sheet $\mathrm{Amp}^\flat$ denotes the amplitude sums formed with $W^\flat$ in place of W, and $\|\cdot\|$ the Euclidean norm of the entries. The sums defining amplitude sums converge absolutely.

Definition 38.3.1. For $\mathfrak{S} \subseteq U(Y_0)(F)$ and $\mathcal{B} \subseteq \bar{G}$, the *second moment* of $\mathfrak{S}$ over $\mathcal{B}$ is $\nu^{-2}\sum_{c,c'}\sum_{s\in\mathcal{B}}\|\mathrm{Amp}(\mathfrak{S};c,s,c')\|^2$ (and likewise with $W^\flat$; without qualification, the second moment is formed with W). The *link second moment* of $\mathfrak{S}$ over $\mathcal{B}$ is $\sum_{c,c'}\sum_{s\in\mathcal{B}}\sum_{\Lambda}\|\mathrm{Amp}^\flat(\mathfrak{S}\cap\{\mathrm{lk}=\Lambda\};c,s,c')\|^2 \in [0,\infty]$, the inner sum over the subspaces Λ of $(X_0\oplus Ez)^2$.

Recall from Theorem 37.3.2 that, for all classes c,c', Bessel coordinates i,i' and $s\in\bar{G}$,

$$\langle \mathcal{P}^\circ e_{c,i}, \rho(s)\mathcal{P}^\circ e_{c',i'}\rangle = \nu^{-1}\mathrm{Amp}\big(U(Y_0)(F);c,s,c'\big)_{ii'},$$

and that the corresponding pairings for $\rho_{\mathrm{Bes}}(s)$ satisfy $\langle e_{c,i},\rho_{\mathrm{Bes}}(s)e_{c',i'}\rangle = \nu^{-1}\mathrm{Amp}(R(F);c,s,c')_{ii'}$.

Problem 38.3.3. $\|\mathcal{K}_s-\rho_{\mathrm{Bes}}(s)\|^2=\nu^{-2}\sum_{c,c'}\|\mathrm{Amp}(U(Y_0)(F)\smallsetminus R(F);c,s,c')\|^2$, the second moment of $U(Y_0)(F)\smallsetminus R(F)$ over $\{s\}$.

Problem 38.3.4. For vectors q_Λ of an inner product space, finitely many nonzero, $\|\sum_\Lambda q_\Lambda\|^2 \le \sum_\Lambda\|q_\Lambda\|^2+\sum_{\Lambda\neq\Lambda'}|\langle q_\Lambda,q_{\Lambda'}\rangle|$.

Theorem 38.3.1 (links at meeting reductions)**.** *If $z_0\ge 1$ is an integer with $z_0\le \mathrm{mt}(s)$ for every $s\in\mathcal{B}$, then the link second moment of $\{\gamma:\mathrm{disc}(\gamma)\neq 0\}$ over $\mathcal{B}$ is at most $(b^{2f}/z_0)^k D_{\mathrm{main}}\,\nu^2$.*

Theorem 38.3.2 (the cut)**.** *If $\mathcal{B}$ avoids the scalar cosets, $C_0\ge 0$, and the link second moment of $\{\gamma:\mathrm{disc}(\gamma)\neq 0\}$ over $\mathcal{B}$ is at most $C_0\nu^2$, then the second moment of $U(Y_0)(F)\smallsetminus R(F)$ over $\mathcal{B}$ has square root at most $\sqrt{C_0}+b^{-15}\sqrt{D_{\mathrm{id}}}$.*

Status: both are read from the proof, where they are stated for an admissible filling over a CM field equipped with a CM type and containing a primitive 5040-th root of unity. By Theorem 38.2.1, the blocks in Theorem 38.3.2 are those of Γ^*. In each cut block the terms of zero discriminant form one piece, and the terms of nonzero discriminant form one piece for each link. The proof bounds the norm of the block by the sum of the norms of these pieces and squares that bound. The squares of the pieces of nonzero discriminant, summed over the blocks, give the link second moment; after summation over the blocks and division by ν^2, all other products contribute at most $b^{-kf}D_{\mathrm{id}}$. For any set of terms, passing from $W^\flat$ to W costs at most $b^{-20}\sqrt{D_{\mathrm{id}}}$ in the square root of the second moment. The proof of Theorem 38.3.1 uses Hua's determinant integral over complex matrices [Hua, Ch. II], sums over sublattices and Liouville's inequality for Gram determinants; the next two problems give the one-row integral and a sublattice sum over $\mathbb{Z}^2$. For related lattice counts see [Sch].

Problem 38.3.5. For every integer $k\ge 1$, $\int_{\mathbb{C}^k}(1+|y|^2)^{-k-1}\,dy=\pi^k/k!$, for Lebesgue measure on $\mathbb{C}^k=\mathbb{R}^{2k}$; the values for $k=1,2,3,6,7$ are approximately 3.14, 4.93, 5.17, 1.34, 0.60, and the integral is at most 1 for $k\ge 7$.

Problem 38.3.6. For $n'\ge 1$, $\mathbb{Z}^2$ has $\sum_{e\mid n'}e$ sublattices of index n' (3 and 7 for $n'=2,4$); hence $\sum_{L'}[\mathbb{Z}^2:L']^{-\omega}=\zeta(\omega)\zeta(\omega-1)$ for real $\omega>2$, the sum over all sublattices L' of finite index.

Recall from Chapter 37 that $D_{\mathrm{main}}\le (M/b^6)^{kf}D_{\mathrm{id}}$ for the data; the proof also shows $kf\ge 40$.

Problem 38.3.7. With $z_0:=p_j^{2f}$, $(b^{2f}/z_0)^k D_{\mathrm{main}}\le b^{-3kf}D_{\mathrm{id}}\le b^{-30}D_{\mathrm{id}}$.

Problem 38.3.8. The second moment of $U(Y_0)(F) \smallsetminus R(F)$ over $\mathcal{B}_j$ has square root at most $2b^{-15}\sqrt{D_{\mathrm{id}}}$, and $4b^{-30} \le b^{-12}$.

Problems 38.2.2, 38.2.3, 38.3.3, 38.3.7 and 38.3.8, with Theorems 38.3.1 and 38.3.2, prove the following.

Theorem 38.3.3 (the strip)**.** *For the data and $j = 0, 1$,*

$$\sum_{s \in \mathcal{B}_j} \|\mathcal{K}_s - \rho_{\mathrm{Bes}}(s)\|^2 \le b^{-12} D_{\mathrm{id}}.$$

The proof states Theorem 38.3.3 for an admissible filling over a CM field equipped with a CM type and containing a primitive 5040-th root of unity, any representation ρ of its finite group on a finite-dimensional complex inner product space, unitary or not, and any linear map from the coordinate space that satisfies the block identity of Theorem 37.3.2. The data satisfy these hypotheses.

Recall Problem 37.4.3: under the hypotheses of Problem 37.4.2, which hold for the data, if $\sum_{t \in \bar{R}_0} \|\mathcal{K}_t - \rho_{\mathrm{Bes}}(t)\|^2 \le b^{-12} D_{\mathrm{id}}$, then

$$\|\mathcal{P}^{\circ *}\mathcal{P}^{\circ} - 1\|^2 \le b^{-12} D_X;$$

and Problem 37.4.4: then $\|\mathcal{K}_1\|^2 \le 2D_X$ for $b \ge 1000$.

Problem 38.3.9. For the data, $\|\mathcal{P}^{\circ *}\mathcal{P}^{\circ} - 1\|^2 \le b^{-12} D_X$ and $\|\mathcal{K}_1\|^2 \le 2D_X$.

What has been proved. Theorem 38.3.3 is the bound $b^{-12} D_{\mathrm{id}}$ for the sum of the squared Hilbert–Schmidt errors on each strip, and the first inequality of Problem 38.3.9 makes $\mathcal{P}^{\circ}$ a near isometry in the sense of Chapter 36. Taken as given: the block identity and the Bessel functions (Theorems 37.3.2 and 37.3.1), and Theorems 38.2.1, 38.3.1 and 38.3.2 and the inequalities $D_{\mathrm{main}} \le (M/b^6)^{kf} D_{\mathrm{id}}$ and $kf \ge 40$. Problem 38.2.3 gives a lower bound for the meet norm on each strip only. Chapters 39 and 40 use the open reductions of Definition 38.2.6 and the fixers of Definition 38.2.7 to bound $\sum_{s \in \bar{G}} \|\mathcal{K}_s\|^2$ by $(1 + 9b^{-4}) D_{\mathrm{main}}$ (Theorem 40.3.1).

In the Lean formalization. In the Lean proof [GL]: Theorem 38.3.3 is `Filling.HasAmplitudeBlocks.sum_hilbertSchmidtNorm_sub_besselTranslate_sq_le`; Theorem 38.2.1 is `Filling.AvoidsScalarCosets.amplitudeSum_compl_besselS_eq`; Theorem 38.3.1 is `Filling.linkSecondMoment_le_of_le_meetNorm`; Theorem 38.3.2 is `Filling.sqrt_secondMoment_compl_besselS_le`. The discriminant, the link, the meet norm, open reductions, fixers and the avoidance of scalar cosets are `Filling.termDisc`, `link`, `Filling.meetNorm`, `Filling.IsOpenElement`, `Filling.fixer` and `Filling.AvoidsScalarCosets`.

Hints

§38.1.

38.1.1: Evaluate $\phi(ugu')$ in two ways.

38.1.2: For $g = \mathrm{diag}(t_1, t_2)$ and $u' = \left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)$, find the u with $ugu' = g$. In the big cell $U \cap wUw^{-1} = 1$.

38.1.3: Sage or Python: run over $\mathrm{GL}_3(\mathbb{F}_p)$, form each UgU with the value $\psi_U(u)\psi_U(u')$ attached to ugu', and test consistency.

38.1.4: Use Problem 38.1.2; the g with $g_{21} = 0$ form the upper triangular group.

§38.2.

38.2.1: Compute the matrices of frame products of $\gamma_{\pm}$, and the intersection $\mathcal{U} \cap \gamma_{\pm}\mathcal{U}\gamma_{\pm}^{-1}$.

38.2.2: If $\zeta \neq 1$, the ideal $(\zeta - 1)$ is divisible by the f distinct primes $w \in \mathcal{M}_j$, of norm p_j; bound $|\sigma(\zeta) - 1| \leq 2$ for every embedding σ.

38.2.3: An integral element t of $R(\mathbb{A}_f)$ fixes v_1 and satisfies $h(ty, v_1) = h(y, v_1)$; reduce modulo w, and deduce that the adjoint inverse of t_w also fixes v_1. For a scalar coset, compare the entries at v_1 of s_w and t_w and use Problem 38.2.2. For $p_j^2 \geq M/b$ use $p_{1-j} \leq b\, p_j$.

38.2.4: Which vectors does $U(X_0)$ fix? Compare the definition of a fixer with Definition 38.2.5.

§38.3.

38.3.1: $p \mid \gamma_{21}$; then Problem 10.3.2.

38.3.3: Expand the norm in the basis $e_{c,i}$, and split $U(Y_0)(F)$ into $R(F)$ and its complement.

38.3.4: Expand the square.

38.3.5: Polar coordinates: the sphere S^{2k-1} has volume $2\pi^k/(k-1)!$.

38.3.6: Each sublattice is spanned by the columns of exactly one $\left(\begin{smallmatrix} g_1 & g_3 \\ 0 & g_2 \end{smallmatrix}\right)$ with $g_1 g_2 = n'$, $0 \leq g_3 < g_1$.

38.3.7: Problem 38.2.3.

38.3.8: Theorem 38.3.2 with $\mathcal{B} = \mathcal{B}_j$ and $C_0 = b^{-30} D_{\mathrm{id}}$, after Theorem 38.3.1 and Problems 38.2.3 and 38.3.7.

38.3.9: $\bar{R}_0 \subseteq \mathcal{B}_0$.

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