UNFOLDING TWICE: GRAM OPERATORS AS SUMS OVER RATIONAL POINTS

Prerequisites. From Chapter 36: Sections 36.1–36.3, in particular Problems 36.2.3 and 36.3.3, and the scaled Poincaré map $\mathcal{P}^\circ$, its Gram operators $\mathcal{K}_s = \mathcal{P}^{\circ *}\rho(s)\mathcal{P}^\circ$, the covolume ν, the group $\bar{R}_0 \subseteq \bar{G}$ and its representation ρ_{Bes}, from Section 36.4 (Definitions 36.4.1 and 36.4.2), and Theorem 36.4.2. From Chapter 9: Hilbert–Schmidt norms (Problem 9.2.1) and Problems 9.2.2 and 9.2.5. From Chapter 10: Poincaré series on $\mathrm{SL}_2(\mathbb{Z})$, Problems 10.1.1, 10.2.1 and 10.2.4 with its proof, and Theorem 10.2.1. From Chapter 21: Problems 21.1.2 and 21.3.2. From Chapters 33 and 34: the data, $\bar{G}$, the source S and $D_X = \dim S$. Schur orthogonality for finite groups [Se, §2.2]. As in Chapter 36, inner products are linear in the first variable, norms of operators are Hilbert–Schmidt norms, and a function Θ on a group is of *positive type* if $\Theta(s^{-1}t) = \langle u_t, u_s\rangle$ for some vectors u_s.

This sheet serves the second moment in the proof of Theorem 36.4.2 of Chapter 36. There the scaled Poincaré map $\mathcal{P}^\circ$ is translated by the finite group $\bar{G}$, and the sizes of the Gram operators $\mathcal{K}_s$ decide how many translates span the target. The proof computes $\mathcal{K}_s$ by unfolding twice, as Petersson's formula computes the inner product of two Poincaré series: each entry of $\mathcal{K}_s$ becomes a sum over the rational points whose reduction is s. What does this sum look like when the group is finite, which part of it is forced, and what is the least possible second moment $\sum_s \|\mathcal{K}_s\|^2$?

37.1. A finite model

Over a definite unitary group with one class, at the principal level of an ideal, a form is a function on the finite group $\bar{G}$ of residues, equivariant for the integral points. We take this as the model.

Definition 37.1.1. A *finite model* consists of a finite group $\bar{G}$, a subgroup $\Gamma \subseteq \bar{G}$ (the integral points), a unitary representation π of Γ on a space V_π, a subgroup $\Gamma_X \subseteq \Gamma$, a unitary representation σ_X of Γ_X on a space V_X, and a Γ_X-equivariant linear map $\jmath\colon V_X \to V_\pi$ with $\jmath^*\jmath = c_0$ for a constant $c_0 > 0$. Its *forms* are the functions $\varphi\colon \bar{G} \to V_\pi$ with $\varphi(\gamma g) = \pi(\gamma)\varphi(g)$ for $\gamma \in \Gamma$; they form a space $\mathcal{Y}$ with $\langle \varphi, \varphi'\rangle := \sum_{g\in\Gamma\backslash\bar{G}} \langle \varphi(g), \varphi'(g)\rangle$, on which $\bar{G}$ acts by $(\rho(s)\varphi)(g) := \varphi(gs)$. Its *Poincaré map* $P\colon V_X \to \mathcal{Y}$ sends w to the function equal to $c_0^{-1/2}\pi(g)\jmath w$ at $g \in \Gamma$ and to 0 off Γ.

In this sheet P without subscript is always this map. In a finite model we write $\mathcal{K}_s := P^*\rho(s)P$, $\bar{R}_0 := \Gamma_X$, $\rho_{\mathrm{Bes}}(s) := \sigma_X(s)$ for $s \in \Gamma_X$ and $\rho_{\mathrm{Bes}}(s) := 0$ for $s \notin \Gamma_X$, and $D_X := \dim V_X$: they play the parts of the objects of the same names in Chapter 36.

Problem 37.1.1. $Pw \in \mathcal{Y}$, and $\langle \varphi, Pw\rangle = c_0^{-1/2}\langle \varphi(1), \jmath w\rangle$ for $\varphi \in \mathcal{Y}$: the map P represents the period $\varphi \mapsto c_0^{-1/2}\jmath^*\varphi(1)$.

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Problem 37.1.2. $\mathcal{K}_s = c_0^{-1} \jmath^* \pi(s) \jmath$ for $s \in \Gamma$, and $\mathcal{K}_s = 0$ for $s \notin \Gamma$. In particular P is an isometry.

Problem 37.1.3. $\mathcal{K}_s = \rho_{\mathrm{Bes}}(s)$ for $s \in \bar{R}_0$. For $t \in \bar{R}_0$ and $s \in \bar{G}$, $\rho(t)P = P\rho_{\mathrm{Bes}}(t)$ and $\mathcal{K}_{st} = \mathcal{K}_s \rho_{\mathrm{Bes}}(t)$.

Problem 37.1.4. Let $\Gamma_X = \Gamma$, $V_X = V_\pi$ and $\jmath = 1$. Then $\dim \mathcal{Y} = [\bar{G} : \Gamma] \dim V_\pi$, $\sum_{s \in \bar{G}} \|\mathcal{K}_s\|^2 = \#\Gamma \dim V_\pi = \#\bar{G} (\dim V_\pi)^2 / \dim \mathcal{Y}$, and if J is a set of representatives of the cosets $s\Gamma$, then $\mathcal{Y}$ is the orthogonal sum of the $\rho(s)P(V_\pi)$, $s \in J$.

Now the smallest case. Let Γ_{I_2} be the group of monomial 2×2 matrices with entries in μ_4, of order 32 by Problem 21.1.2. In $\mathbb{F}_5$ the element 3 is a square root of -1, and reduction modulo $2 + \mathrm{i}$ is the ring map $\mathbb{Z}[\mathrm{i}] \to \mathbb{F}_5$, $\mathrm{i} \mapsto 3$.

Problem 37.1.5. Reduction modulo $2 + \mathrm{i}$ is an injective homomorphism $\Gamma_{I_2} \to \mathrm{GL}_2(\mathbb{F}_5)$, and its image Γ has index 15 in $\bar{G} := \mathrm{GL}_2(\mathbb{F}_5)$, a group of order 480. With π the standard representation of Γ_{I_2} on $V_\pi = \mathbb{C}^2$, transported to Γ, one has $\dim \mathcal{Y} = 30 = \#\bar{G} \cdot \mathrm{mass}_2 \cdot \dim V_\pi$, where $\mathrm{mass}_2 = 1/32$ is the mass of rank two, as in Chapter 36.

In this model take $\Gamma_X := \{\mathrm{diag}(u, 1) : u \in \mu_4\}$ (reduced modulo $2 + \mathrm{i}$), the stabilizer of $\mathbf{e}_2$ as in Section 36.3, $V_X := \mathbb{C}\mathbf{e}_1$ with $\sigma_X(\mathrm{diag}(u, 1)) = u$, and $\jmath$ the inclusion, so that $c_0 = 1$ and $D_X = 1$.

Problem 37.1.6. $\|\mathcal{K}_s\|^2 = 1$ for the 16 diagonal $s \in \Gamma$, and $\mathcal{K}_s = 0$ for the other 464 elements of $\bar{G}$. Hence $\sum_{s \in \bar{G}} \|\mathcal{K}_s\|^2 = 16$: the four elements of $\bar{R}_0$ contribute $\#\bar{R}_0 \cdot D_X = 4$, with $\mathcal{K}_s = \rho_{\mathrm{Bes}}(s)$ there, and the three cosets $\bar{R}_0 \,\mathrm{diag}(1, u)$, $u \neq 1$, contribute 4 each.

At level one the same elements land on the identity.

Problem 37.1.7. In the toy of Section 36.3, let $a \geq 4$ be divisible by 4, let the source be the orthogonal direct sum of the lines $\mathbb{C}\xi_j$ for $4 \mid j$ and $0 \leq 2j < a$, with their form norms, and let D_{toy} be its dimension. Let $\mathcal{K}$ be the operator whose matrix in the unit basis $(\xi_j / \|\xi_j\|)_j$ has the entries $\mathcal{K}_{jj'}$ of Problem 36.3.3. Then $\|\mathcal{K} - 1\|^2 = 9 D_{\mathrm{toy}} > b^{-12} D_{\mathrm{toy}}$ for every $b \geq 1$: the conclusion of Problem 37.4.3 below fails there.

By Problem 36.3.3 the excess $3\delta_{jj'}$ comes from the classes of $\mathrm{diag}(1, u)$, $u \neq 1$, which preserve the line of $z = \mathbf{e}_2$ without fixing z. In Problem 37.1.6 these elements reduce to $\mathrm{diag}(1, u) \notin \bar{R}_0$; they contribute to $\mathcal{K}_s$ for $s \neq 1$, and $\mathcal{K}_1 = 1$. The level has separated them from the Bessel subgroup.

37.2. The least second moment

By Problem 9.2.5, if p is the orthogonal projection onto a $\bar{G}$-stable subspace $\mathcal{Y}_0 \subseteq \mathcal{Y}$ and pP is an isometry, then $\#\bar{G}\, D_X^2 \leq \dim \mathcal{Y}_0 \cdot \sum_{s \in \bar{G}} \|\mathcal{K}_s\|^2$. When is this an equality?

Problem 37.2.1. If π is irreducible, then $\sum_{s \in \bar{G}} \|\mathcal{K}_s\|^2 = \#\Gamma\, D_X^2 / \dim V_\pi$, and equality holds in Problem 9.2.5 with $\mathcal{Y}_0 = \mathcal{Y}$. In Problem 37.1.6: $32 \cdot 1/2 = 16$ and $30 \cdot 16 = 480 = \#\bar{G}$.

Put $\deg := \dim V_\pi$, the formal degree of π for the Haar measure of total mass one, and $\mathrm{mass} := 1/\#\Gamma$. Then Problem 37.2.1 says that the second moment

equals $D_X^2/(\deg\cdot\mathrm{mass})$, the quantity D_{main} of Definition 37.3.2 below. Without irreducibility it is at least as large, by Problem 9.2.5, and it can be larger.

Problem 37.2.2. (a) Let $V_\pi = \mathrm{Sym}^a(\mathbb{C}^2)$, $a \geq 1$, with the inner product of Problem 21.3.2, the other data as in Problem 37.1.6 but $V_X = \mathbb{C}\,\mathbf{e}_1^j\mathbf{e}_2^{a-j}$, $0 \leq j \leq a$, with $\sigma_X(\mathrm{diag}(u,1)) = u^j$ and the norm induced from V_π. Then $\sum_s \|\mathcal{K}_s\|^2 = 16$, or 32 if $2j = a$, while $\#\Gamma\, D_X^2/\dim V_\pi = 32/(a+1)$; for $a = 8$, $j = 0$ the ratio is $9/2$. (b) Let $\bar{G} = \Gamma = S_3$ act on $V_\pi = \{x \in \mathbb{C}^3 : x_1 + x_2 + x_3 = 0\}$ by permuting coordinates, $\Gamma_X = 1$, $V_X = \mathbb{C}$ and $\jmath(1) = (\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$. In the lexicographic order of the permutations, written as $(s(1), s(2), s(3))$, the scalars $\mathcal{K}_s$ are $1, \frac{1}{2}, -1, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}$, and $\sum_s |\mathcal{K}_s|^2 = 3 = 6/2$. Here π is irreducible, and this is the equality of Problem 37.2.1.

So D_{main} is the value for integral points that are equidistributed against the coefficients of π; in Problem 37.2.2(a) a finite group has too few. For the data the real group is not compact, and infinitely many rational γ lie over each s.

37.3. The block identity

In Chapter 10, unfolding $\langle P_n, P_m\rangle$ and expanding P_n group the terms by double cosets of Γ_∞.

Problem 37.3.1. In this problem and the next paragraph k, m, n, c and P_n are the weight, the two indices, the lower left entry and the Poincaré series of Chapter 10. By Problems 10.1.1 and 10.2.4, $\langle P_n, P_m\rangle = \Gamma(k-1)(4\pi m)^{1-k}p_n(m)$. In the series for P_n, the m-th Fourier coefficient of the sum of the terms with $\gamma \in \pm\Gamma_\infty$ is $\delta(n,m)$, and for each $c \geq 1$ that of the sum of the terms with lower left entry $\pm c$ is $2\pi\mathrm{i}^{-k}(m/n)^{(k-1)/2}c^{-1}S(n,m;c)J_{k-1}(4\pi\sqrt{mn}/c)$.

In Theorem 37.3.2 below, the rational points of the Bessel subgroup $R = \mathcal{U} \rtimes U(X_0)$ replace $\pm\Gamma_\infty$, a Bessel function at each infinite place replaces $e(mz)$, a pair of a class of $U(X_0)$ and a Bessel coordinate replaces m, and the sum over the rational γ outside the Bessel subgroup with reduction s replaces the Kloosterman terms $S(n,m;c)J_{k-1}$; the term $\delta(n,m)$ becomes $\rho_{\mathrm{Bes}}(s)$. The reduction $s \in \bar{G}$ has no counterpart at level one. The Bessel functions are the following.

Theorem 37.3.1 (Bessel functions)**.** *For the data, at each infinite place τ there are a finite nonempty set I_τ, a number $d_\tau > 0$ equal to the formal degree of the discrete series of the real unitary group G_τ of Y_0 with doubled Harish-Chandra parameter $2\varsigma_\tau$, an irreducible unitary representation of $U(X_0)_\tau$ on $\mathbb{C}^{I_\tau}$, and continuous functions $W_\tau, W_\tau^\flat \colon G_\tau \to \mathbb{C}^{I_\tau}$ such that:*

(a) *$W_\tau(xy) = \psi_{\mathcal{U}}(x)W_\tau(y)$ for $x \in \mathcal{U}_\tau$, and $W_\tau(hy)$ is the representation applied to $W_\tau(y)$ for $h \in U(X_0)_\tau$; the same holds for $W_\tau^\flat$;*

(b) *$\int_{\mathcal{U}_\tau\backslash G_\tau} W_{\tau,i}\overline{W_{\tau,j}} = \delta_{ij}$ for $i, j \in I_\tau$;*

(c) *$\int_{G_\tau} \big|\int_{\mathcal{U}_\tau\backslash G_\tau} \langle W_\tau(y), W_\tau(yg)\rangle\, dy\big|^2 dg = (\#I_\tau)^2/d_\tau$;*

(d) *by construction, $W_\tau^\flat$ is W_τ times a function of the Bessel length of y (a measure of the size of y modulo the Bessel subgroup on the left) with values in $[0,1]$, equal to 1 up to half the radius b^{6n+2}, $n = \varrho + 1$, and to 0 beyond the radius; and $\int_{\mathcal{U}_\tau\backslash G_\tau} \|W_\tau - W_\tau^\flat\|^2 \leq b^{-2r^3 f}\,\#I_\tau$;*

with further bounds that later sheets use.

Status: read from the proof, which constructs W_τ as an average of a discrete-series coefficient over the Bessel subgroup and shows by Laplace's method that the average is not zero; see the remark at the end of the chapter. We know no reference for the statement in this form. The formal degree and Schur orthogonality for discrete series are due to Harish-Chandra [HC66], [HC70]; (b) follows from the normalization $\int \|W_\tau\|^2 = \#I_\tau$ and the irreducibility, by Schur's lemma.

Write $W_i(y) := \prod_\tau W_{\tau,i_\tau}(y_\tau)$ for a Bessel coordinate $i = (i_\tau)_\tau$. The classes c with representatives h_c, the coordinates $e_{c,i}$ of the source (an orthonormal basis), the product G_∞ of the real groups in the rescaled bases with its product measure dy, the components γ_∞, γ_f of a rational $\gamma \in U(Y_0)(F)$, and integrality and reduction in the lattice basis are those of Definition 36.4.1. The *class cutoff* $u_c \geq 0$ is a measurable function on G_∞ whose translates by the rational points u of $\mathcal{U}$ with $h_c^{-1} u_f h_c$ in the principal level sum to 1.

Definition 37.3.1. For classes c, c' and $\gamma \in U(Y_0)(F)$, the *amplitude* of γ is the matrix with entries

$$\mathrm{Amp}_\gamma(c,c')_{ij} := \int_{G_\infty} u_c(y)\, u_{c'}(\gamma_\infty^{-1} y)\, W_i(y)\, \overline{W_j(\gamma_\infty^{-1} y)}\, dy.$$

For a set $\mathfrak{S} \subseteq U(Y_0)(F)$ and $s \in \bar{G}$, the *amplitude sum* $\mathrm{Amp}(\mathfrak{S}; c, s, c')$ is the sum of $\mathrm{Amp}_\gamma(c,c')$ over the $\gamma \in \mathfrak{S}$ such that $h_c^{-1}\gamma_f h_{c'}$ is integral with reduction s.

The following is the analogue of Problems 37.1.2 and 36.2.3.

Theorem 37.3.2 (the block identity)**.** *For all data, all classes c, c', Bessel coordinates i, j and $s \in \bar{G}$,*

$$\big\langle \mathcal{P}^\circ e_{c,i},\, \rho(s)\mathcal{P}^\circ e_{c',j} \big\rangle = \nu^{-1}\, \mathrm{Amp}\big(U(Y_0)(F); c, s, c'\big)_{ij}.$$

Equivalently, the $((c,i),(c',j))$ entry of $\mathcal{K}_s$ is the complex conjugate of $\nu^{-1}\mathrm{Amp}(U(Y_0)(F); c, s, c')_{ij}$.

Theorem 37.3.2 is not proved here; see the remark. Its proof unfolds one Poincaré series against the translate of the other, as in Problem 37.1.1; the coordinate $e_{c,i}$ selects the class c, and the finite indicator in the second series selects the γ of reduction s, as Γ does in Problem 37.1.2. Recall from Definition 36.4.2 that $\rho_{\mathrm{Bes}}(s)$ has the same entries with the sum restricted to the rational Bessel subgroup, that $\rho_{\mathrm{Bes}}(s) = 0$ for $s \notin \bar{R}_0$, and that ρ_{Bes} is a unitary representation of $\bar{R}_0$. Moreover, for the data, $\mathcal{K}_{st} = \mathcal{K}_s \rho_{\mathrm{Bes}}(t)$ for all $s \in \bar{G}$ and $t \in \bar{R}_0$, as in Problem 37.1.3.

Definition 37.3.2. $D_{\mathrm{id}} := \#\bar{R}_0 \cdot D_X$ and $D_{\mathrm{main}} := D_X^2/(\deg \cdot \mathrm{mass}_r)$, with deg the product of the d_τ and mass_r the mass of Chapter 36.

For the data, $b^{11kf} D_{\mathrm{id}} \leq D_{\mathrm{main}} \leq (M/b^6)^{kf} D_{\mathrm{id}}$; these two inequalities are read from the proof (see the remark).

Problem 37.3.2. $\|\mathcal{K}_s\|^2 = \nu^{-2} \sum_{c,c'} \|\mathrm{Amp}(U(Y_0)(F); c, s, c')\|^2$, where the norm of a matrix is the square root of the sum of the squared absolute values of its entries.

37.4. The Bessel subgroup and near isometry

The translates by $\bar{R}_0$ keep the special role they have in Problem 37.1.3. In Problems 37.4.1–37.4.5, ρ is a unitary representation of $\bar{G}$, and $\mathcal{P}^\circ$, $\mathcal{K}_s$, ρ_{Bes} need only satisfy the stated hypotheses.

Problem 37.4.1. If $\mathcal{K}_{st} = \mathcal{K}_s \rho_{\mathrm{Bes}}(t)$ for all $s \in \bar{G}$ and $t \in \bar{R}_0$, then $\rho(t)\mathcal{P}^\circ = \mathcal{P}^\circ \rho_{\mathrm{Bes}}(t)$ for $t \in \bar{R}_0$.

Problem 37.4.2. If in addition ρ_{Bes} is a unitary representation of $\bar{R}_0$, then $\mathcal{K}_{ts} = \rho_{\mathrm{Bes}}(t)\mathcal{K}_s$ for $t \in \bar{R}_0$, $s \in \bar{G}$, and $\|\mathcal{K}_t - \rho_{\mathrm{Bes}}(t)\| = \|\mathcal{P}^{\circ *}\mathcal{P}^\circ - 1\|$ for $t \in \bar{R}_0$. Moreover $\sum_{t \in \bar{R}_0} \|\rho_{\mathrm{Bes}}(t)\|^2 = D_{\mathrm{id}}$.

Problem 37.4.3. Under the hypotheses of Problem 37.4.2, if $\sum_{t \in \bar{R}_0} \|\mathcal{K}_t - \rho_{\mathrm{Bes}}(t)\|^2 \le b^{-12} D_{\mathrm{id}}$, then $\|\mathcal{P}^{\circ *}\mathcal{P}^\circ - 1\|^2 \le b^{-12} D_X$.

Problem 37.4.4. If $\|\mathcal{P}^{\circ *}\mathcal{P}^\circ - 1\|^2 \le b^{-12} D_X$ and $b \ge 1000$, then $\|\mathcal{K}_1\|^2 \le 2D_X$.

Problem 37.4.5. $s \mapsto \|\mathcal{K}_s\|^2$ is of positive type, and $\|\mathcal{K}_s\|^2 \le \|\mathcal{K}_1\|^2$ for every $s \in \bar{G}$.

Problem 37.4.6. In Problem 37.1.6, $D_{\mathrm{id}} = 4$ and $D_{\mathrm{main}} = 16$ (with deg and mass as in Section 37.2), so $D_{\mathrm{id}}/D_{\mathrm{main}} = 1/4$. For the data $D_{\mathrm{id}}/D_{\mathrm{main}} \le b^{-11kf}$: if $\|\mathcal{K}_1\|^2 \le 2D_X$, the elements of $\bar{R}_0$ contribute at most $2b^{-11kf} D_{\mathrm{main}}$ to $\sum_s \|\mathcal{K}_s\|^2$, so a second moment of size about D_{main} must come almost entirely from outside $\bar{R}_0$.

The hypothesis of Problem 37.4.3 follows, by dropping the terms with $s \notin \bar{R}_0$, from the estimate over the strip of Chapter 38. There the sum of $\|\mathcal{K}_s - \rho_{\mathrm{Bes}}(s)\|^2$ is written, by Theorem 37.3.2 and Problem 37.3.2, as a sum over the rational γ outside the Bessel subgroup, which that chapter estimates.

We have proved, in the finite model, the unfolding of the Gram operators, the role of the Bessel subgroup and the least second moment; for the data we have stated the block identity (Theorem 37.3.2) and the Bessel functions (Theorem 37.3.1), both taken from the proof, and derived from the block identity the norm formula of Problem 37.3.2, and from $\mathcal{K}_{st} = \mathcal{K}_s \rho_{\mathrm{Bes}}(t)$ the passage from the Bessel subgroup to near isometry and to $\|\mathcal{K}_1\|^2 \le 2D_X$. Also taken as given: the two properties of ρ_{Bes} recalled from Definition 36.4.2, the identity $\mathcal{K}_{st} = \mathcal{K}_s \rho_{\mathrm{Bes}}(t)$ for the data, and the two inequalities between D_{id} and D_{main}. What remains is the sum over the γ outside the Bessel subgroup: over a strip around $\bar{R}_0$ (Chapter 38) and then over all of $\bar{G}$, where it must be compared with D_{main}.

In the Lean formalization. In the Lean proof [GL], Theorem 37.3.2 is `Data.hasAmplitudeBlocks_scaledPoincare`, which is proved from `Filling.Fits.petersson_poincare`; the Lean inner product is conjugate-linear in its first variable, so its $\langle \rho(s)\mathcal{P}^\circ e_{c',j}, \mathcal{P}^\circ e_{c,i} \rangle$ is the left side of the display. The amplitude and the amplitude sum of Definition 37.3.1 are `Filling.amplitude` and `Filling.amplitudeSum`, the class cutoff is `Filling.classCutoff`, the sizes of Definition 37.3.2 are `Filling.identitySecondMoment` and `Filling.mainSecondMoment`, and their two inequalities for the data are fields of `Filling.IsAdmissible`. Theorem 37.3.1 is `Data.exists_besselSystem`, with the properties of the Bessel system it produces. The identity $\mathcal{K}_{st} = \mathcal{K}_s \rho_{\mathrm{Bes}}(t)$ is `Filling.HasAmplitudeBlocks.adjoint_comp_rep_mul_right`.

Hints

§37.1.

37.1.1: Only the coset Γ contributes to the inner product.

37.1.2: $\langle \rho(s)Pw, Pw' \rangle$ has at most one nonzero term, at the coset of 1.

37.1.3: $\jmath$ is equivariant and $\jmath^*\jmath = c_0$.
37.1.4: Problem 37.1.2; the support of $\rho(s)Pw$ is Γs^{-1}.
37.1.5: An element of Γ_{I_2} is determined by its residues, since μ_4 maps injectively into $\mathbb{F}_5^\times$.
37.1.6: Problem 37.1.2: $\mathcal{K}_s$ is the $(1,1)$ entry of the lift of s to Γ_{I_2}.
37.1.7: Problem 36.3.3.

§37.2.

37.2.1: Schur orthogonality: $\sum_{\gamma\in\Gamma}|\langle\pi(\gamma)v, v'\rangle|^2 = \#\Gamma\|v\|^2\|v'\|^2/\dim V_\pi$; apply it to an orthonormal basis of V_X and use Problem 37.1.2. For the equality, compute $\dim \mathcal{Y}$.
37.2.2: A monomial matrix sends a monomial to a multiple of a monomial.

§37.3.

37.3.1: Follow the proof of Problem 10.2.4, with Problem 10.2.1; for $c \neq 0$ the terms with lower left entry $\pm c$ are indexed by d modulo c and an integer translation, and Γ_∞ contains ± 1.
37.3.2: Problem 9.2.1 with the coordinate basis; complex conjugation does not change absolute values.

§37.4.

37.4.1: Compute $\mathcal{P}^{\circ *}\rho(s^{-1})d$ for $d := \rho(t)\mathcal{P}^\circ x - \mathcal{P}^\circ\rho_{\mathrm{Bes}}(t)x$.
37.4.2: Take adjoints in $\rho(t^{-1})\mathcal{P}^\circ = \mathcal{P}^\circ\rho_{\mathrm{Bes}}(t^{-1})$. Multiplication by a unitary does not change a Hilbert–Schmidt norm, and $\mathcal{K}_t = \mathcal{K}_1\rho_{\mathrm{Bes}}(t)$.
37.4.3: Problem 37.4.2: every term of the sum equals $\|\mathcal{P}^{\circ *}\mathcal{P}^\circ - 1\|^2$.
37.4.4: $\mathcal{K}_1 = \mathcal{P}^{\circ *}\mathcal{P}^\circ$, $\|1\|^2 = D_X$, and the triangle inequality.
37.4.5: Problem 9.2.2 and the Cauchy–Schwarz inequality; $\|\mathcal{P}^\circ\mathcal{P}^{\circ *}\| = \|\mathcal{P}^{\circ *}\mathcal{P}^\circ\|$.

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