EXCEPTIONAL PRIMES: A FULL MARK

Prerequisites. Chapter 33: the data and the scale inequalities (Definition 33.3.1, Problem 33.3.3). Chapter 34: the spaces X_j of rank n_j and the diagram β of the data (Definition 34.2.1). Chapter 48: c_p, I_p, gain(p) (Section 48.1); $C_p(\Phi')$, o_p, r_p, exceptional primes, L_p (Section 48.2); gauges, valuations, S_η, weak chain data, translations $\mathfrak{t}$ with translated vacua $v_j(\mathfrak{t})$, and the restricted chain $R_{\vartheta,\Psi}$ (Section 48.3); the primes v_σ, marks, full marks, $\Phi_{\mathfrak{M}}$, a_v, T_v, $Y_v = p^{a_v}T_v$ and $Y_{\mathfrak{M}}$, the product of the Y_v in a fixed order (Section 48.4). Chapter 49: $\beta_{>1}$ (Definition 49.2.1), charges (Definition 49.3.1) and Theorem 49.3.2. Chapter 50: standard data (Definition 50.4.1), "multiplies gauges by at most" (Definition 50.4.2), the stage bound $\mathcal{D}_j(\Phi')$ (Definition 50.1.1), Theorem 50.4.4, statements (III) and (IV) of Section 50.4, Problems 50.1.1, 50.4.3, 50.4.4 and 50.5.1. Hecke operators at a split place (Chapter 24, Section 24.5). Decomposition groups in abelian extensions [Ne, Ch. I, §9]; elementary divisors over a discrete valuation ring.

At an exceptional prime $p < x_1$ the collision sum can exceed the unit of loss L_p (Problem 48.2.5). Chapter 48 therefore applies the scaled Hecke operator $Y_{\mathfrak{M}}$ of a full mark before the chain, and uses the bound of Theorem 53.4.1 below. How does a Hecke operator at primes above p remove the collisions, and what does it cost?

53.1. The primes above an exceptional prime

Let p be a prime with $\chi_K(p) = -1$ and $\eta\colon \mathbb{C} \to \mathbb{C}_p$ a ring map. Then $p \neq d$, and $p \nmid q_1$ by Problem 48.1.1; so p is unramified in E. For $\sigma \in \Phi$, v_σ is the prime of E induced by $\eta \circ \Phi_\sigma$, and $\bar{v}$ is the complex conjugate of v.

Definition 53.1.1. The *residue degree above* p is $f_p := \operatorname{lcm}(2, o)$, where o is the order of $[-p]$ in $\Phi \cong (\mathbb{Z}/q_1)^\times$.

Problem 53.1.1. For $\sigma, \tau \in \Phi$: $v_\sigma = v_\tau$ if and only if $\sigma = [-p]^i\tau$ with i even, and $v_\sigma = \bar{v}_\tau$ if and only if $\sigma = [-p]^i\tau$ with i odd. The residue field of every $v \mid p$ has p^{f_p} elements.

Problem 53.1.2. Let $q_0 = 18480$, $d = 138599$ and $m = 16384$. The prime 479 is inert in K, with $o_{479} = 15$ and $r_{479} = 2$; it is not exceptional. The prime $55439 \equiv 6287 \pmod m$ is inert, with $o_{55439} = r_{55439} = 1$; it is exceptional.

Problem 53.1.3. If $p \not\equiv \pm 1 \pmod m$, then m divides the order of $[-p]$; so f_p is this order, it is even, and $f_p \geq m$. [Check, with $m = 2^{14}$ and $s = 308$: the order of -55439 modulo q_1 has 2-adic valuation 308.]

Problem 53.1.4. Let p be exceptional. For $\tau \in \Phi_a$, $[-p^{o_p}]\tau \in \Phi_a$ and $v_{[-p^{o_p}]\tau} = \bar{v}_\tau$. No prime $v \mid p$ equals $\bar{v}$.

Problem 53.1.5. If $p < x_1$ is exceptional, a full mark at (p, η) exists.

For a full mark, $C_p(\Phi_{\mathfrak{M}}) = 0$ (Problem 48.4.1). Let $\bar{\mathfrak{M}} := \{\bar{v} : v \in \mathfrak{M}\}$ and $\Phi_{\bar{\mathfrak{M}}} := \{\sigma : v_\sigma \in \bar{\mathfrak{M}}\}$.

Date: November 26, 2026.

Problem 53.1.6. Let $\mathfrak{M}$ be a mark. No $\sigma \in \Phi_{\mathfrak{M}} \cup \Phi_{\bar{\mathfrak{M}}}$ is a b-place, so $\kappa(\sigma) = a/x_1$ there. For $v \in \mathfrak{M}$, $\sigma \mapsto [-p]\sigma$ maps $\{\sigma : v_\sigma = v\}$ bijectively onto $\{\sigma : v_\sigma = \bar{v}\}$ and preserves κ.

The *model* is that of Problem 50.5.1: $\Phi = (\mathbb{Z}/4)^\times = \{1, 3\}$, both places are a-places, $a = 24$, and $p = 5$ is treated as inert, so that $[-5^e]$ exchanges 1 and 3 for every $e \in I_5$, the odd $e \geq 1$. Think of 1 as inducing v and 3 as inducing $\bar{v}$, and put $\Phi_{\mathfrak{M}} := \{1\}$. There are two rows, 0 and 1, and β has 24 columns of height one at each place and no others. The model is bookkeeping, not data.

Problem 53.1.7. In the model, $C_5(\varnothing) = 10$ and $C_5(\{1\}) = 0$; $\mathcal{D}(\beta) = 0$, while $\mathcal{D}(\beta \smallsetminus \{1\}) + \gamma_5 \cdot 24 = 10 = 48\gamma_5$, with $\gamma_5 = 5/24$.

Without the mark the collision takes the whole credit $48\gamma_5$; with it, nothing is lost. The erasure is not free: at a neighbour the charges and the diagonal cost a_v (Section 53.2), and Y_v carries the factor p^{a_v} that pays for them.

53.2. One lift at a neighbour

Let $\mathfrak{M}$ be a mark and $j \geq 0$. For $v \in \mathfrak{M}$ the pair $\{v, \bar{v}\}$ is split (Problem 53.1.4), and the unitary group of X_{j+1} at it is $\mathrm{GL}_{n_{j+1}}(E_v)$: g at v and $(\bar{g}^{\,t})^{-1}$ at $\bar{v}$. As p is unramified, it is a uniformizer of E_v.

Definition 53.2.1. A *neighbour of index at most* r_p at $\mathfrak{M}$ is an element of the finite adelic unitary group of X_{j+1} which is 1 away from $\mathfrak{M} \cup \bar{\mathfrak{M}}$ and whose component g_v at each $v \in \mathfrak{M}$ lies in the $\mathrm{GL}_{n_{j+1}}(\mathcal{O}_v)$-double coset of $\mathrm{diag}(p1_i, 1)$ for some $i \leq r_p$.

Problem 53.2.1. An element of the double coset of $\mathrm{diag}(p1_i, 1)$ in $\mathrm{GL}_n(E_v)$ is $g_1 \,\mathrm{diag}(p1_{\mathcal{I}}, 1)\, g_2$ with $g_1, g_2 \in \mathrm{GL}_n(\mathcal{O}_v)$, for every set $\mathcal{I}$ of i rows, $\mathrm{diag}(p1_{\mathcal{I}}, 1)$ having p at the rows of $\mathcal{I}$ and 1 elsewhere.

Fix sets $\mathcal{I}_v$ of rows, $\#\mathcal{I}_v \leq r_p$, for $v \in \mathfrak{M}$. For τ with $v_\tau \in \mathfrak{M}$ or $\bar{v}_\tau \in \mathfrak{M}$ let $\mathcal{I}(\tau) := \mathcal{I}_v$ for that $v \in \mathfrak{M}$; otherwise $\mathcal{I}(\tau) := \varnothing$. The *neighbour diagonal* $g_{\mathcal{I}}$ is the neighbour (Definition 53.2.1) with component $\mathrm{diag}(p1_{\mathcal{I}_v}, 1)$ at each $v \in \mathfrak{M}$.

Problem 53.2.2. (i) At τ, the extension to finite adeles of $u \mapsto \eta(\Phi_\tau(\bar{u}))$ sends $g_{\mathcal{I}}$ to the diagonal matrix with entries $p^{-\epsilon_\tau(i)}$, where $\epsilon_\tau(i) = 1$ if $v_\tau \in \mathfrak{M}$ and $i \in \mathcal{I}(\tau)$, $\epsilon_\tau(i) = -1$ if $\bar{v}_\tau \in \mathfrak{M}$ and $i \in \mathcal{I}(\tau)$, and $\epsilon_\tau(i) = 0$ otherwise; so a box in row i at τ has its gauge multiplied by $|p^{-\epsilon_\tau(i)}| = p^{\epsilon_\tau(i)}$. (ii) For every integer c, $\sum_{\tau:\,\kappa(\tau)=c} \sum_i \epsilon_\tau(i) = 0$.

The following extends statement (II) of Section 50.4 from 1 to a diagonal; it is read from the proof. *(C)* Let ϕ be a standard datum of denominator δ on $\mathrm{Hom}(X_j, X_{j+1})$, let g_1, g_2 be elements of the finite adelic unitary group of X_{j+1} whose components at the places above p are integral with integral inverses, and let g_0 be an element of the same group sent at each τ, as in Problem 53.2.2, to a diagonal matrix with entries $y_\tau(i)$. For a filling $\mathbf{i}$ put $\mathrm{dw}(\mathbf{i}) := \prod_\tau |\det y_\tau|^{e_\tau} \prod_{\text{boxes}} |y_\tau(\text{row})|$, where e_τ, a function of $\kappa(\tau)$ and j, is the exponent of the determinant in the weight at τ. If $\mathrm{dw}(\mathbf{i}')$ times the absolute value of the entry $(\mathbf{i}', \mathbf{i})$ of the Schur jet matrix of the jet series, at the scale of the kernel, of every standard datum of denominator δ is at most $\mathfrak{b}$, then $\omega(1, g_1 g_0 g_2)\phi$ multiplies gauges by at most $\mathfrak{b}$.

For $\tau \in \Phi$, $h \geq 1$ and rows i let $\mathfrak{c}(\tau, h, i) := 1$ if either $v_\tau \in \mathfrak{M}$, $h = 1$ and $i \notin \mathcal{I}(\tau)$, or $\bar{v}_\tau \in \mathfrak{M}$ and $i \in \mathcal{I}(\tau)$; and $\mathfrak{c}(\tau, h, i) := 0$ otherwise.

Problem 53.2.3. The numbers $\mathfrak{c}$ are charges for $(\beta, \Phi_{\mathfrak{M}})$, with β read at heights below n_{j+1}. [With two rows and $\mathcal{I}_v = \{0\}$: $\mathfrak{c}(\tau, 1, \cdot) = (0, 1)$ if $v_\tau \in \mathfrak{M}$, $\mathfrak{c}(\tau, h, \cdot) = (1, 0)$ for every h if $\bar{v}_\tau \in \mathfrak{M}$, and in each pairing exempt by Definition 49.3.1 the two charges add up to 1 in each row.]

Problem 53.2.4. Let ϕ be a standard datum of denominator 1 on $\mathrm{Hom}(X_j, X_{j+1})$ and g a neighbour of index at most r_p at $\mathfrak{M}$. Then $\omega(1, g)\phi$ multiplies gauges by at most

$$p^{\sum_{v\in\mathfrak{M}} a_v + r_p \sum_{\sigma\in\Phi_{\mathfrak{M}}} \beta_{>1}(\sigma) - \mathcal{D}_j(\Phi_{\mathfrak{M}})},$$

with β read at heights below n_{j+1}.

In the model, with $\mathcal{I} = \{0\}$ and one input coordinate, let $F := \exp\bigl(\sum_{i=0,1}(z_{1i}^5 z_{3i} + z_{3i}^5 z_{1i})/5\bigr)$, which has the normal form with $W = 1$. A filling puts a_i boxes in row i at place 1 and a_i' at place 3 (in this sheet), $a_0 + a_1 = a_0' + a_1' = 24$; its jet is $\partial_{z_{10}}^{a_0}\partial_{z_{11}}^{a_1}\partial_{z_{30}}^{a_0'}\partial_{z_{31}}^{a_1'}F(0)$, its charges total $a_1 + a_0'$, and the diagonal weights it by $5^{a_0 - a_0'}$ in absolute value.

Problem 53.2.5. (m) By machine: of the 625 fillings, 25 have a nonzero jet; every nonzero jet has valuation 0; and $\mathrm{val}(\mathrm{jet}) + a_1 + a_0' \geq 8$, with equality only at $(a_0, a_0') = (20, 4)$.

So p^{24} times the weighted jets of the neighbour have valuation at least 8, against 0 for the plain kernel: the mark recovers 8 of the 10, and the missing 2 is the digit loss in the factorials, which the logarithmic term of Theorem 49.3.2 pays for. The model is a direct computation; the error term of that theorem is about 54 here.

53.3. Moving the Hecke operators into the data

For $v \in \mathfrak{M}$, $q_v := p^{f_p}$, and $0 \leq i \leq n_j$, let $T_{v,i}^{(j)}$ be the Hecke operator at v of the double coset of $\mathrm{diag}(p1_i, 1)$ on forms on X_j, and $\mathbf{T}_j(X) := \sum_{i\leq n_j}(-X)^i q_v^{\binom{i}{2}} T_{v,i}^{(j)}$ for an indeterminate X; so $T_v = T_{v,r_p}^{(0)}$.

(A) The split unramified theta correspondence. Let w be a split place of norm q, let hermitian spaces of ranks n and $n+k'$ carry characters unramified at w, let ϕ_0 be a vacuum at w, and let $\Theta(\phi; \varphi)$ be bilinear in data ϕ and forms φ and equivariant at ϕ_0. For φ spherical at w, $\Theta(\phi_0; \varphi)$ is spherical at w and $\mathbf{T}'(X)\Theta(\phi_0; \varphi) = \prod_{l<k'}(1 - q^l \mathrm{fr}\, X)\,\Theta\bigl(\phi_0; \mathbf{T}(q^{k'}\mathrm{fr}\,\mathrm{fr}'^{-1}X)\varphi\bigr)$, where $\mathbf{T}$, $\mathbf{T}'$ are the Hecke polynomials of the two spaces and fr, fr′ the Frobenius values of the two characters at w. *Status*: classical [Ra, Ku]; the proof proves it.

(B) For the chain the proof reads (A) as follows. Let Ψ be a weak chain datum for ϑ, $p \nmid \mathrm{Nm}\, Q_\vartheta$, $v \in \mathfrak{M}$ and $j \geq 0$. Let $\chi_{v,j}$ be the value at a uniformizer of E_v of the base change of the character of the lift $X_j \to X_{j+1}$; then $|\eta\chi_{v,j}| = 1$, and for every $\mathfrak{t}$ in the support of Ψ the lift Θ from X_j to X_{j+1} with datum $v_j(\mathfrak{t})$ maps forms spherical at v to forms spherical at v, and on them

$$\mathbf{T}_{j+1}(X)\,\Theta = \mathcal{Q}_j(X)\,\Theta\,\mathbf{T}_j\bigl(p^{(t+1)f_p/2}X\bigr), \qquad \mathcal{Q}_j(X) := \prod_{l=0}^{t}\bigl(1 - q_v^l\, p^{n_j f_p/2}\chi_{v,j}X\bigr).$$

Problem 53.3.1. Model: $q_v = 25$, $p^{f_p/2} = 5$, $\chi_{v,0} = 1$, ranks 1 and 2 ($t = 0$), and $T_{v,1}^{(0)}$ acting by 2. Then $\mathcal{Q}_0(X)\,(1 - 2\cdot 5X) = (1 - 5X)(1 - 10X) = 1 - 15X + 50X^2$; so $T_{v,1}^{(1)}$ acts by 15 and $T_{v,2}^{(1)}$ by 2 on the image of Θ, and $5\,\Theta T_{v,1}^{(0)} = (T_{v,1}^{(1)} - 5)\Theta$.

Problem 53.3.2. Let $\mathcal{Q} \in \mathcal{O}_\eta[X]$ with $\mathcal{Q}(0) = 1$, and let linear maps Θ, T_i, T_i' $(0 \le i \le n)$ satisfy $\sum_i (-X)^i q_v^{\binom{i}{2}} T_i' \Theta = \mathcal{Q}(X)\,\Theta \sum_i (-p^{(t+1)f_p/2}X)^i q_v^{\binom{i}{2}} T_i$ as polynomials in X. Then for every i there are $c_{ii'} \in \mathcal{O}_\eta$, depending only on $\mathcal{Q}$ and q_v, with $p^{(t+1)if_p/2} q_v^{\binom{i}{2}}\,\Theta T_i = \sum_{i' \le i} c_{ii'} T_{i'}' \Theta$.

Problem 53.3.3. For $i \le r_p$, $p^{(t+1)if_p/2} q_v^{\binom{i}{2}}$ divides p^{fr_pt}; and $(j+1)fr_pt \le r_p f(n_m + r_p)$ for $j < m$.

Iterating Problem 53.3.2 along the first $j+1$ lifts, the proof writes $p^{r_p f(n_m+r_p)}$ times the chain of $j+1$ lifts after $T_{v,r_p}^{(0)}$ as $\sum_{i \le r_p} c_i T_{v,i}^{(j+1)}$ after the chain, with c_i independent of the translation, and $c_i \in \mathcal{O}_\eta$ for $j < m$ (Problem 53.3.3); Theorem 25.3.1 computes such coefficients in a model. A Hecke operator of the group of X_{j+1} acting on the datum passes to the lifted form, and is the sum of the translates $\omega(1,u)$ over the cosets u of its double coset. So the datum of lift j is replaced, for each $v \in \mathfrak{M}$ in turn, by $\sum_i c_i p^{a_v - r_p f(n_m+r_p)}\, T_{v,i}^{(j+1)}$ applied to it; the chain with the new data, applied to φ, is the chain applied to $Y_{\mathfrak{M}}^m \varphi$. For $j < m$, by Problem 53.2.4 and as $|\eta(c_i p^{a_v - r_p f(n_m+r_p)})| \le p^{r_p f(n_m+r_p) - a_v}$, the new datum of lift j multiplies gauges by at most p to the power $\#\mathfrak{M} \cdot r_p f(n_m + r_p) + r_p \sum_{\Phi_{\mathfrak{M}}} \beta_{>1} - \mathcal{D}_j(\Phi_{\mathfrak{M}})$: the a_v cancel. The new data still commute with the levels, the exponents add as in Problem 48.3.1, and restriction to Y_m at $p \ne N$ does not increase gauges. The proof obtains, for a mark $\mathfrak{M}$, a weak chain datum Ψ for ϑ with $p \nmid \operatorname{Nm} Q_\vartheta$, and $\varphi \in S_\eta$:

$(\star)$ $R_{\vartheta,\Psi}(Y_{\mathfrak{M}}^m \varphi)$ has valuation at least

$$-\sum_{j<m} \Big(\#\mathfrak{M} \cdot r_p f(n_m + r_p) + r_p \sum_{\sigma \in \Phi_{\mathfrak{M}}} \beta_{>1}(\sigma) - \mathcal{D}_j(\Phi_{\mathfrak{M}}) \Big),$$

β read at heights below n_{j+1} in the jth term.

53.4. Paying for the transfer

Let $\mathfrak{M}$ be a mark and $0 \le j < m$, with β read at heights below n_{j+1}.

Problem 53.4.1. For $\sigma \in \Phi_{\mathfrak{M}}$, $r_p \beta_{>1}(\sigma) \le a/(pm) + \mathbf{1}_{\Phi_a}(\sigma)\, a/x_1$.

Problem 53.4.2. $\#\mathfrak{M} \cdot r_p f(n_m + r_p) + r_p \sum_{\sigma \in \Phi_{\mathfrak{M}}} \beta_{>1}(\sigma) \le L_p/m + \#(\Phi_a \cap \Phi_{\mathfrak{M}})\, a/x_1$.

Problem 53.4.3. If $\mathfrak{M}$ is full, then $\#\mathfrak{M} \cdot r_p f(n_m + r_p) + r_p \sum_{\sigma \in \Phi_{\mathfrak{M}}} \beta_{>1}(\sigma) - \mathcal{D}_j(\Phi_{\mathfrak{M}}) \le (2L_p - \operatorname{gain}(p))/m$.

In the model of Problem 50.2.7 at $p = 7$, the set $\{1, 9\}$ plays the part of $\Phi_{\mathfrak{M}}$: $C_7(\{1,9\}) = 0$, and the left side there is $46865/48$, against $\operatorname{gain}(7)/m = 1022$ (Problems 50.2.8 and 50.2.3). Summing Problem 53.4.3 over $j < m$ in $(\star)$ gives the following theorem, read from the proof.

Theorem 53.4.1. *Let p be a prime, $\eta\colon \mathbb{C} \to \mathbb{C}_p$ a ring map, ϑ a twist, Ψ a weak chain datum for ϑ with $p \nmid \operatorname{Nm} Q_\vartheta$, and $\mathfrak{M}$ a full mark at (p,η) (so $p < x_1$ is exceptional). For every $\varphi \in S_\eta$, $R_{\vartheta,\Psi}(Y_{\mathfrak{M}}^m \varphi)$ has valuation at least* $\operatorname{gain}(p) - 2L_p$.

What has been established. Theorem 53.4.1, from ($\star$) and Problems 53.4.1–53.4.3; ($\star$) rests on Problem 53.2.4 (from Problems 53.2.1–53.2.3, (C), Theorem 49.3.2 and (III), (IV) of Chapter 50), on (A) and (B), and on Problems 53.3.2 and 53.3.3. A full mark exists (Problem 53.1.5). Taken as given: the split unramified theta correspondence (A), classical; its form (B) for the chain and the kernel bound (C), read from the proof; the steps of Section 53.3 that lead from Problems 53.3.2 and 53.3.3 to ($\star$), read from the proof; and, from the earlier chapters, Theorem 49.3.2 and statements (III), (IV) of Chapter 50. Problem 48.4.3 adds $p^{j_p}S_\eta$ to the image; what remains for the exceptional primes is the colength of $Y^m_{\mathfrak{M}}S_\eta + p^{j_p}S_\eta$ in S_η (Section 48.4).

In the Lean formalization. In the Lean proof [GL] (paths relative to `grh/src/`; the namespace `New.` is omitted from names), Theorem 53.4.1 is `Data.IsWeakChainDatum.gaugeLe_restrictedChain_of_isFullMark` (`New/UpperBound/Divisibility/LocalDivisibility.lean`); ($\star$) is `Data.IsWeakChainDatum.gaugeLe_scaledHeckeOperator_pow` and Problem 53.2.4 is `Data.IsMarked.ω_ofSplitPlaces_mem_raises` (`New/UpperBound/Divisibility/ValuationAtNonSplitPrimes.lean`); Problems 53.1.5 and 53.4.2 are `Data.exists_isFullMark` and `Data.IsMarked.card_mul_transferExponent_add_sum_lowColumns_le` (`New/UpperBound/Divisibility/MarkedPrimes.lean`). Statement (A) is `SiegelWeil.hecke_transfer_of_pairing`, (B) is `Data.IsWeakChainDatum.exists_heckePolyOp_lift`, which states the existence of a polynomial with integral coefficients and constant term 1 ($\mathcal{Q}_j$ is the one its proof constructs), and (C) is `Parameters.omega_mem_raises_of_schurJet_le`. In the derivation of ($\star$) in Section 53.3, the transfer through the first $j+1$ lifts is `Data.IsWeakChainDatum.exists_hecke_transfer`; the Hecke operator on the datum is `Data.liftKernel_lift_heckeSplit` and `Data.IsMarked.ω_heckeSplit_mem_raises`; the new data are `Data.IsWeakChainDatum.exists_lift_eq_chain_scaledHeckeOperator`; the addition of the exponents and the restriction to Y_m are `Data.IsWeakChainDatum.gaugeLe_chain` and `Data.GaugeLe.restrictToY` (`New/UpperBound/Divisibility/ChainValuation.lean`). The double coset of Definition 53.2.1 is `basicDoubleCoset`, with `ofSplitPlaces`; f_p occurs in `Parameters.q_eq_pow_lcm_of_inert`.

Hints

§53.1.

53.1.1: $\mathrm{Gal}(E/\mathbb{Q}) \cong \mathrm{Gal}(K/\mathbb{Q}) \times (\mathbb{Z}/q_1)^\times$; the Frobenius at p is (nontrivial, p) and complex conjugation is (nontrivial, -1). Compare v_σ and v_τ through the decomposition group.

53.1.2: Orders modulo 18480; r_p is the least h with $m \le p^h$.

53.1.3: $2^{s+1} \mid q_1$ and $m = 2^e$, $s = 22e$ (Definition 33.3.1). Lift the exponent in $(-p)^o \equiv 1 \pmod{2^{s+1}}$.

53.1.4: Classes modulo q_0; Problems 53.1.1 and 53.1.3.

53.1.5: Choose one prime of each pair $\{v_\tau, \bar{v}_\tau\}$, $\tau \in \Phi_a$, by a total order; Problem 53.1.4 for why both are induced by a-places.

53.1.6: The class of $[-p]^i$ modulo q_0 lies in a group of odd order o_p, which does not contain -1.

53.1.7: Geometric series in $1/25$.

§53.2.

53.2.1: Elementary divisors over $\mathcal{O}_v$, and permutation matrices.

53.2.2: (i) The map is continuous for $\bar{v}_\tau$. (ii) Problem 53.1.6.

53.2.3: For $e \in I_p$, $v_{[-p^e]\tau} = \bar{v}_\tau$ and $\mathcal{I}([-p^e]\tau) = \mathcal{I}(\tau)$. Treat the rows in and out of $\mathcal{I}(\tau)$ separately.

53.2.4: Problem 53.2.1 and (C); then, as in Problem 50.4.4, Theorem 49.3.2 with Problem 53.2.3. Count $\epsilon + \mathfrak{c}$ box by box at $\Phi_{\mathfrak{M}}$ and at $\Phi_{\bar{\mathfrak{M}}}$; a column has distinct rows.

53.2.5: Only the monomials $(z_{1i}^5 z_{3i})^k (z_{3i}^5 z_{1i})^l$ contribute; exact rational arithmetic.

§53.3.

53.3.1: Compare the coefficients of X and X^2.

53.3.2: $\mathcal{Q}$ is a unit in $\mathcal{O}_\eta[[X]]$.

53.3.3: $f_p \le f$ and $r_p \le t$; $mt \le n_m$.

§53.4.

53.4.1: Problems 50.1.1 and 53.1.6; $r_p \le s$. If $p > m$, then $r_p = o_p = 1$.

53.4.2: $\#\mathfrak{M} \le f$; the scale inequalities make $f^2 r_p (n_m + r_p)p + fa/m \le fa(\log b)^2/m$.

53.4.3: Theorem 50.4.4(b) and Problem 53.4.2.

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