THE LEVEL PRIME AND THE RAMIFIED FORMS

Prerequisites. Chapter 33: the scales and ranks (Definition 33.3.1) and $m \log b \le \log N \le 30 m \log b$ (Problem 33.5.4). Chapter 34: the CM type Φ, the source forms S, their level, the operators T_w, the ramified forms S_{ram} and $D_X = \dim_{\mathbb{C}} S$ (Section 34.5), and the diagram (Definition 34.2.1). Chapter 48, Section 48.3: the gauge $\|\omega\|_g$ of a Weyl vector, "valuation at least", $\mathcal{O}_\eta$ and S_η, and Problem 48.3.1. Chapter 43: translations $\mathfrak{t} \in \mathcal{T}$, the translated vacua $v_j(\mathfrak{t})$ and the definite lifts (Definition 43.4.2); the pair gluing Gl_i, the glued tensor $\mathrm{gl}(\mathfrak{t})$, the restriction res to Y_m and the composite lift $B_{\vartheta,\Psi}$ (Definition 43.4.3); chain data (Definition 43.4.4). Chapter 50, Section 50.4: standard data of denominator δ (Definition 50.4.1), "multiplies gauges by at most" (Definition 50.4.2), the jet scale c_{jet}, statements (I), (II) and Problem 50.4.1. The colength of Chapter 5 (before Theorem 5.4.1), with the same definition over $\mathcal{O}_\eta$. Standard fillings and the Weyl modules of GL_k [Fu, §8].

At the level prime the gain is 0, but the glued pairs of the composite lift carry poles at N: their data translate the vacuum by $-y/N$. How much N-adic valuation does the composite lift lose there, and on how large a module of forms? This is the case $p = N$ of Theorem 48.7.1; Section 48.6 uses it in the form of Theorem 52.4.1 below. On ramified forms every glued pair becomes two direct lifts after an operator Up, and $\mathrm{Up}^{m/2}$ is integral, after a change of base point, on a submodule of small colength.

52.1. Ramified forms

Recall from Section 34.5 that for each prime $w \mid \pi_N$ of E right translation by the idele z_w is an operator T_w on S with $T_w^{N-1} = 1$, and $S_{\mathrm{ram}} = \bigcap_{w|\pi_N} \ker \sum_{i=0}^{N-2} T_w^i$.

Problem 52.1.1. Let $F_1, \dots, F_l$ be commuting operators on a finite-dimensional complex space with $F_i^n = 1$. Then $\bigcap_i \ker \sum_{s<n} F_i^s = \mathrm{im} \prod_i (1 - F_i)$. Hence $S_{\mathrm{ram}} = \mathrm{im} \prod_{w|\pi_N} (1 - T_w)$. For the cyclic shift F of $\mathbb{C}^4$, $Fe_s = e_{s+1}$, the vector $(1-F)e_0 = (1, -1, 0, 0)$ is killed by $\sum_{s<4} F^s$.

Problem 52.1.2. Let $N = 3$ and let F act on $\mathbb{C}$ through the sign character of $\mathbb{F}_3^\times$ at its generator. Then $1 + F = 0$, $1 - F = 2$, so the line is ramified and $|1 - F|^2 = 4$.

52.2. The places above N and the direct lifts

Fix a ring map $\eta\colon \mathbb{C} \to \mathbb{C}_N$. For $\tau \in \Phi$ let w_τ be the place of E above N at which $\eta \circ \Phi_\tau$ is continuous (the prime v_τ of Section 48.4, for $p = N$; we write w for places above N, as in Section 34.5). Let Σ_N (a notation of this chapter) be the set of the conjugates $\bar{w}_\tau$. As $N \equiv 1 \pmod{q_2^2}$, N splits completely in E; so $\tau \mapsto \bar{w}_\tau$ is a bijection from Φ onto Σ_N, which holds exactly one place of each conjugate pair above N; each residue field is $\mathbb{F}_N$. A *residue vector* on X_j is a family $y = (y_w)_{w \in \Sigma_N}$ of vectors $y_w \in \mathbb{F}_N^{n_j}$ in the basis of X_j; it is *primitive* if every $y_w \neq 0$; e_{last} is the last basis vector at every place.

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Definition 52.2.1. Let diag_N be the matrix $\mathrm{diag}(1,\dots,1,N)$ at every place of Σ_N. The *base point* δ_j is the finite adelic point of the unitary group of X_j, supported above N, whose component at $w \in \Sigma_N$ is $\mathrm{diag}(1,\dots,1,N)$ if $w \mid \pi_N$ and 1 otherwise. Put $\delta_j^\vee := \delta_j\,\delta_{j,\Sigma}^{-1}$, where $\delta_{j,\Sigma}$ is the point with component diag_N at Σ_N.

For a matrix g_1 over $\prod_{w\in\Sigma_N} E_w$ let $\mathrm{Rt}_j(g_1)$ be right translation of forms on X_j by $\delta_j g_1\,\delta_j^{-1}$: matrices at Σ_N are read through δ_j. For a function f of $n_{j+1}\times n_j$ matrices at Σ_N and a finite datum v of $\mathrm{Hom}(X_j,X_{j+1})$, let $\mathrm{Sch}(f)v$ put f on v at Σ_N in the Schrödinger model, and let $\Theta_j^X[f,v]$ (in this chapter) be the definite lift $X_j \to X_{j+1}$ for the twist ϑ (lift $j{+}1$ of Definition 43.4.2; here, as in Definition 48.3.3, a lift carries the index of its source space) whose datum is $\mathrm{Sch}(f)v$ translated by (δ_j,δ_{j+1}) under the Weil representation. A function of residue matrices is read on the integral matrices through reduction modulo N and extended by 0.

Definition 52.2.2. The *direct datum* dir is the function $\mathrm{col}_{e_{\mathrm{last}}}$, where $\mathrm{col}_y(T)=1$ if T is integral and its last column reduces to y, and $\mathrm{col}_y(T)=0$ otherwise. The *direct lift* j with datum v is $\Theta_j^X[\mathrm{dir},v]$.

Definition 52.2.3. Let $\psi_N(c) := \psi_0(-c/N)$ on $\prod_{w\in\Sigma_N}\mathbb{F}_N$, with ψ_0 the standard character of the finite adeles. For a residue vector y on X_0 whose last entry is a unit at every place, let $\mathfrak{g}_y$ be an integral invertible matrix whose last column and last row reduce to y. On forms on X_0,

$$\mathrm{Up} := \mathrm{Rt}_0(\mathrm{diag}_N^{-1})\sum_y \psi_N(y_{\mathrm{last}})\,\mathrm{Rt}_0(\mathfrak{g}_y),$$

the sum over these y.

A form φ on X_j is *ramified at* Σ_N (in this chapter) if for each $w\in\Sigma_N$ there are a form ψ and a unit scalar c at Σ_N, integral, with $c\equiv 1 \bmod N$ at the places other than w, such that $\varphi = \psi - \mathrm{Rt}_j(c)\psi$.

Problem 52.2.1. Every $\varphi \in S_{\mathrm{ram}}$ is ramified at Σ_N, with $\psi \in S$.

The proof reads the following *lift law* for $\Theta_j^X[\,\cdot\,,v]$, for every datum v. (i) For integral invertible matrices g_1 of size n_j and g_2 of size n_{j+1} at Σ_N, with reductions $\bar g_1$, $\bar g_2$, and every function f of residue matrices, $\mathrm{Rt}_{j+1}(g_2)\circ\Theta_j^X[T\mapsto f(\bar g_2 T\bar g_1),v]\circ\mathrm{Rt}_j(g_1) = \Theta_j^X[f,v]$. (ii) $\mathrm{Rt}_{j+1}(\mathrm{diag}_N^{-1})\circ\Theta_j^X[\mathrm{row},v] = \varepsilon'_j\,\Theta_j^X[\mathrm{dir},v]\circ \mathrm{Rt}_j(\mathrm{diag}_N^{-1})$, where row is the indicator of the integral matrices whose last row reduces to e_{last}, and ε'_j (in this chapter) is the value at diag_N on X_j of the character $g\mapsto\chi(\det g)|\det g|^{-n_{j+1}/2}$, divided by the value at diag_N on X_{j+1} of $g\mapsto\chi(\det g)|\det g|^{-n_j/2}$, with χ the base change to E of the character of the lift $X_j\to X_{j+1}$. Here and below $|\det g| := \prod_{w\in\Sigma_N}|\det g_w|_w$, with $|N|_w = N^{-1}$, and $\chi(\det g)$ reads $\det g$ as an idele.

Problem 52.2.2. Let $w\in\Sigma_N$ and let f be a function of residue matrices with $f(cT)=f(T)$ for every unit $c\in\prod_{\Sigma_N}\mathbb{F}_N^\times$ equal to 1 off w. Then $\Theta_j^X[f,v]\varphi = 0$ for every φ ramified at Σ_N. In particular $\Theta_j^X[\mathrm{col}_y,v]\varphi=0$ if y vanishes at some place.

Let $\mathrm{ch}_y(T) := \psi_N\big((\text{last row of } T)\cdot y\big)$ on integral matrices, 0 elsewhere.

Problem 52.2.3. $\mathrm{GL}_n(\mathbb{F}_N)$ is transitive on $\mathbb{F}_N^n\smallsetminus 0$, a set of N^n-1 elements; so there are $\#\mathrm{Prim}_n = (N^n-1)^f$ primitive residue vectors in rank n, a number prime

to N. For φ ramified at Σ_N and data v, v',

$$\sum_y \Theta^X_{j+1}[\mathrm{ch}_y, v']\,\Theta^X_j[\mathrm{col}_y, v]\,\varphi = \#\mathrm{Prim}_{n_{j+1}} \cdot \Theta^X_{j+1}[\mathrm{ch}_{e_{\mathrm{last}}}, v']\,\Theta^X_j[\mathrm{dir}, v]\,\varphi,$$

the sum over all residue vectors on X_{j+1}. For $N = 3$, $n = 2$, $f = 1$: eight vectors, one orbit.

All lifts become direct. The proof reads two further statements. First, on data fixed by the projection on the vacuum above N, the double lift $X_{2i} \to X_{2i+2}$ with the glued datum $\mathrm{Gl}_i(v \otimes v')$ is $\varepsilon_i\,\mathrm{Rt}_{2i+2}(\mathrm{diag}_N^{-1}) \sum_y \Theta^X_{2i+1}[\mathrm{ch}_y, v']\,\Theta^X_{2i}[\mathrm{col}_y, v]$, where ε_i (in this chapter) is the value of the character of the lift $X_{2i+1} \to X_{2i+2}$, formed with $|\det|^{-n_{2i+1}/2}$, at $\mathrm{diag}(1, \dots, 1, N)$ on X_{2i+2} at the places of Σ_N prime to π_N, times the value of the character of $X_{2i} \to X_{2i+1}$, also formed with $|\det|^{-n_{2i+1}/2}$, at the same matrix on X_{2i} at the places dividing π_N. Second, the following, whose proof sorts the terms of $\Theta^X_{2i+1}[\mathrm{ch}_{e_{\mathrm{last}}}, v']$ by the last row and uses the lift law: *there is $c_{\mathrm{dir}} \in \mathbb{C}$ (in this chapter) such that for all data v_j and all $\varphi \in S_{\mathrm{ram}}$, the chain of the $m/2$ double lifts with the glued data of the projections of the v_j on the vacuum above N sends φ to c_{dir} times the chain of the m direct lifts with data v_j applied to* $\mathrm{Up}^{m/2}\varphi$. *Here*

$$c_{\mathrm{dir}} = \prod_{i<m/2} \Big(\varepsilon_i\, \#\mathrm{Prim}_{n_{2i+1}} \prod_{l<2i+2} \varepsilon'_l \Big).$$

Status: read from the proof. Problems 52.2.2 and 52.2.3 are its first steps. We also read: for g equal to $\mathrm{diag}(1, \dots, 1, N)$ at a set $\Sigma' \subseteq \Sigma_N$ and 1 at the other places of Σ_N, the value at g of the character of any of the lifts, formed with $|\det|^{-r/2}$ for any $r \geq 0$, has N-adic absolute value at most $N^{2b\,\#\Sigma'}$ after η.

Problem 52.2.4. $|\eta(\varepsilon_i)| \leq N^{2fb}$, $|\eta(\varepsilon'_l)| \leq 1$ and $|\eta(\#\mathrm{Prim}_n)| \leq 1$; hence $|\eta(c_{\mathrm{dir}})| \leq N^{mfb}$.

52.3. The colength of the forms integral after $\mathrm{Up}^{m/2}$

A form φ is *integral at* a point h if $\|\varphi(g)\|_{gh} \leq 1$ for every g; so S_η is the module of source forms integral at 1. Write $S_\eta(h)$ (in this chapter) for the $\mathcal{O}_\eta$-module of source forms integral at h; so $S_\eta = S_\eta(1)$. By Section 34.5 the values of source forms lie in $\bigotimes_{\tau \in \Phi_a} \mathrm{Sym}^a \mathbb{C}^k$, whose basis is the families $I = (z^{\alpha_\tau})_{\tau \in \Phi_a}$ of monomials of degree a in k variables; let $e_I := \sum_\tau \alpha_{\tau,k}$ be the degree of I in the last variable.

Problem 52.3.1. Let $k = a = 2$, one place, $N = 3$. The monomials $z_1^2, z_1 z_2, z_2^2$ have degrees $0, 1, 2$ in z_2; the change of lattice multiplies them by $\mathrm{diag}(1, 3, 9)$, of determinant 3^3, and the colength of the span of $3^{\alpha_2} z^\alpha$ in the span of the z^α over $\mathbb{Z}_3$ divides 3^3. The mean degree in z_2 is $1 = a/k$.

A source form is determined by its values at representatives of the double cosets of its level, and these values are arbitrary: the proof reads that the stabilizers of the double cosets are trivial. So the pairs (c, I) of a double coset and a basis vector are D_X in number. Let $D_{\mathrm{last}} := \sum_{(c,I)} e_I$.

Problem 52.3.2. The degree in the u-th variable, summed over the monomials of degree a in k variables, does not depend on u, and k times it is a times the number of these monomials.

Problem 52.3.3. $k\,D_{\mathrm{last}} = \#\Phi_a\, a\, D_X \leq f a\, D_X$.

Definition 52.3.1. S_N is the $\mathcal{O}_\eta$-module of the forms φ on X_0 such that $\mathrm{Up}^{m/2}\varphi$ is a source form integral at δ_0.

The proof reads that every double coset has a representative r_c such that $r_c\delta_0^\vee$ and $\delta_{0,\Sigma}^{-1} r_c \delta_0$ are integral above N, that is, lie in $\mathrm{GL}_k(\mathcal{O}_{E_w})$ at every place w above N.

Problem 52.3.4. $S_\eta \subseteq S_\eta(\delta_0^\vee)$. The coordinates of $\varphi(r_c)$ are integral for $\varphi \in S_\eta(\delta_0^\vee)$, and a source form whose coordinate (c, I) at r_c is divisible by N^{e_I} for all (c, I) lies in $S_\eta(\delta_0)$. Hence the colength of $S_\eta(\delta_0)$ in $S_\eta(\delta_0^\vee)$ divides $N^{D_{\mathrm{last}}}$.

Problem 52.3.5. $(\mathrm{Up}\varphi)(g) = \sum_y \psi_N(y_{\mathrm{last}})\, \varphi(g\, \delta_0^\vee\, \mathfrak{g}_y \delta_0^{-1})$; hence Up maps $S_\eta(\delta_0)$ into forms integral at $\delta_0^\vee$. As Up preserves S (read from the proof), the colength in S_η of the forms φ with $\mathrm{Up}^i\varphi \in S_\eta(\delta_0)$ divides $N^{(i+1)D_{\mathrm{last}}}$, for every $i \geq 0$.

Problem 52.3.6. $(m/2+1)\, a \leq mbk$, and so $(m/2+1)D_{\mathrm{last}} \leq mfb\, D_X$.

Problems 52.3.3, 52.3.5 with $i = m/2$, and 52.3.6 prove part (a) of Theorem 52.4.1 below, with $e = (m/2+1)D_{\mathrm{last}}$.

52.4. Direct lifts lose little

The Fourier expansion of the direct datum has poles at N, which an element of K must clear.

Problem 52.4.1. Let $K = \mathbb{Q}(\sqrt{-7})$, $\theta_K = (1+\sqrt{-7})/2$, $N = 11$. Then $z^2 - z + 2$ has the roots $5, 7$ modulo 11; with $c_0 = 7$, $(1-2c_0)^2 \equiv -7$ and $(1-2c_0)\cdot 7 \equiv 8 \pmod{11}$. So $\delta := \theta_K - c_0$ lies in the prime $\mathfrak{p} = (11, \delta)$, and $\bar{\delta}\cdot 7$ is a unit at $\mathfrak{p}$.

Problem 52.4.2. Let $\theta_K = (1+\sqrt{-d})/2$, let $c_0 \in \mathbb{Z}$ represent θ_K modulo the prime $\mathfrak{p}$ of K below w_τ (the same for all τ), and $\delta := \theta_K - c_0$. Then $\delta \in w_\tau$ for every τ, and $|\eta(\bar{\delta}d)| = 1$.

The proof reads: *for $j < m$, every translation $\mathfrak{t}$ and every $\delta \in \mathcal{O}_K$ lying in all the w_τ, the datum $N^{2fn_jn_{j+1}}\,\mathrm{Sch}(\mathrm{dir})v_j(\mathfrak{t})$ is standard of denominator δ*; the translates of the vacuum in the expansion of dir have simple poles at the w_τ. Recall from Section 50.4 that the kernel of the lift $X_j \to X_{j+1}$ is the theta kernel at scale $\pi_M c_{\mathrm{jet}}$ and $|\eta\pi_M| \leq 1$. A datum *multiplies gauges from δ_j to δ_{j+1} by at most c* if the inequality of Definition 50.4.2 holds with $\|\cdot\|_{g'\delta_{j+1}}$ and $\|\cdot\|_{g\delta_j}$ in place of $\|\cdot\|_{g'}$ and $\|\cdot\|_g$.

Problem 52.4.3. Let $0 \neq \delta \in \mathcal{O}_K$ with $|\eta\pi_M| \leq |\eta(\bar{\delta}d)|$. Every standard datum of denominator δ multiplies the gauges of the kernel of the lift $X_j \to X_{j+1}$ by at most 1. Hence, for $j < m$, the datum of the direct lift j with $v_j(\mathfrak{t})$ multiplies gauges from δ_j to δ_{j+1} by at most $N^{2fn_jn_{j+1}} \leq N^{fb}$.

Three more statements are read from the proof. For $\mathfrak{t}$ in the support of a chain datum, $v_j(\mathfrak{t})$ is fixed by the projection on the vacuum above N. The direct datum commutes with the action of the levels of X_j and X_{j+1}; so for a chain datum Ψ for ϑ, if every $\mathrm{Sch}(\mathrm{dir})v_j(\mathfrak{t})$, translated by (δ_j, δ_{j+1}), multiplies gauges from δ_j to δ_{j+1} by at most N^{r_j}, then $\sum_{\mathfrak{t}} \Psi(\mathfrak{t})$ times the chain of the m direct lifts with data $v_j(\mathfrak{t})$ maps source forms integral at δ_0 to forms with $\|\cdot(g)\|_{g\delta_m} \leq N^{r_0+\cdots+r_{m-1}}$ for every g, as in Problem 48.3.1. And if a form on X_m satisfies $\|\varphi(g)\|_{g\delta_m} \leq C$ for every g, then its restriction $\mathrm{res}\,\varphi$ to Y_m has valuation at least $-\log_N C$: the branching map

is integral, and its scalar has the absolute value of the determinant factor of δ_m. The branching rule is classical [GW, Thm. 8.1.1]; the integrality of the branching map and the rest are read from the proof.

Problem 52.4.4. Let Ψ be a chain datum for a twist ϑ and $\varphi \in S_{\mathrm{ram}} \cap S_N$. Then $B_{\vartheta,\Psi}\varphi = c_{\mathrm{dir}}\,\mathrm{res}\big(\sum_{\mathfrak{t}} \Psi(\mathfrak{t})\,(\text{chain of direct lifts with } v_j(\mathfrak{t}))(\mathrm{Up}^{m/2}\varphi)\big)$, and it has valuation at least $-2mfb$. Moreover $2mfb\log N \le 4fa(\log b)^2 \log x_1$.

Problems 52.2.4, 52.4.3 and 52.4.4 prove part (b).

Theorem 52.4.1. *Let $\eta\colon \mathbb{C} \to \mathbb{C}_N$ be a ring map.*

(a) *There is an integer $0 \le e \le mfb\,D_X$ such that the colength of S_N in S_η divides N^e.*
(b) *For every twist ϑ, every chain datum Ψ for ϑ and every $\varphi \in S_{\mathrm{ram}} \cap S_N$, the form $B_{\vartheta,\Psi}\varphi$ has valuation at least $-2mfb$.*

What has been established. Theorem 52.4.1, the case $p = N$ of Theorem 48.7.1, with $S' = S_N \cap S_\eta$ (the combinations in (a) of elements of S_η stay in S_η), $l = e$ and $\mathrm{loss}(N) = 2mfb$; Problem 52.4.4 bounds its term in the sum of the errors. Taken as given: the statements read from the proof, listed in the remark, and the branching rule.

In the Lean formalization. In the Lean proof [GL] (namespace `New.` omitted), Theorem 52.4.1(a) is `Data.exists_colengthDvd_upIntegralForms_pow` and (b) is `Data.gaugeLe_compositeLift_at_N`. Definitions 52.2.1, 52.2.2, 52.2.3, 52.3.1 are `Data.diagPointN` (with `Data.dualPointN`), `IsSplitLevel.directDatum`, `IsSplitLevel.upOperator` and `Data.upIntegralForms`; Problem 52.1.1 is `Data.ramifiedSourceForms_eq_range`, Problem 52.3.4 is `Data.colengthDvd_integralFormsAt_diagPointN`, and the statement that all lifts become direct is `Data.doubleLiftChain_gluedDoubleLiftDatum_eq_smul_chain` (one pair: `IsSplitLevel.LiftLaw.glued_eq_direct_direct_upOperator`); the integral branching map is `exists_branching_matrix`. The other statements read in the text are: the lift law, `Data.liftLaw_liftAtN` (ε'_j is `Data.upCommutationFactor`); the gluing at N, `Data.liftInPairs_gluedDoubleLiftDatum` (ε_i is `Data.gluedFactor`); the bound on the character values, `Data.norm_detCharAt_diagN_le`; the trivial stabilizers, `Data.classAut_levelX_zero_eq_bot`; the representatives r_c, `Data.exists_mul_dualPointN_mem_integralAbove`; that Up preserves S, `Data.upOperator_mem_sourceForm`; the Fourier step, a step of the proof of `Data.schrodingerOpAt_directDatum_mem_raises`; the translated vacua at N, `Data.vacuumProjFinPart_translateDatum`; the invariance and the bound along the chain, `Data.omega_schrodingerOpAtN_directDatum` and `Data.IsWeakChainDatum.gaugeLe_chain`; the restriction at δ_m, `Data.GaugeLe.restrictToY_of_diag`.

Hints

§52.1.

52.1.1: As in Problems 43.5.1 and 43.5.2: $1 - F^n = (1-F)\sum_{s<n} F^s$; decompose into joint eigenspaces.

§52.2.

52.2.1: Problem 52.1.1 with one factor. For $w \in \Sigma_N$ let u be the one of $w, \bar{w}$ that divides π_N; z_u is 1 off $u, \bar{u}$, and a scalar commutes with δ_0.

52.2.2: Lift law (i) with g_1 the scalar c and $g_2 = 1$.

52.2.3: Write $y = \bar{g}\, e_{\text{last}}$ with g integral invertible, and apply the lift law (i) to both lifts; Problem 52.2.2 for the other y.

52.2.4: Each τ has $\bar{w}_\tau$ in exactly one of the two sets in ε_i. For ε_l': $\chi(\det \operatorname{diag}_N)$ is the same on both spaces, and $|\eta(\sqrt{N})| = N^{-1/2}$; $n_l \le n_{l+1}$.

§52.3.

52.3.1: Sym^2 of $\operatorname{diag}(1,3)$. Over a valuation ring one condition "this coordinate is divisible by z" costs a colength dividing z: use a member of the family whose coordinate divides the others'.

52.3.2: Exchanging two variables permutes the monomials; sum over u.

52.3.3: Sum Problem 52.3.2 over the places of Φ_a, one at a time.

52.3.4: At τ, $\delta_0^\vee$ acts through $\operatorname{diag}(1,\dots,1,N^{-1})$ or 1, and $\delta_{0,\Sigma}$ through $\operatorname{diag}(1,\dots,1,N)$; the last row of an element of the level is $\equiv (0,\dots,0,1) \bmod \pi_N$, so $\delta_0^{-1} k \delta_0$ is integral above N. Every g is $\gamma r_c k$ with γ rational and k in the level, and $\|\gamma\omega\|_{\gamma h} = \|\omega\|_h$. For the colength, impose the divisibility of the (c, I) coordinates one at a time, as in the hint to Problem 52.3.1.

52.3.5: $\mathfrak{g}_y$ is integral above N, so $\|\omega\|_{h\mathfrak{g}_y} = \|\omega\|_h$; and $|\psi_N| = 1$. Induction on i; colengths multiply along a chain of modules.

52.3.6: $a = xb$, $\sqrt{x} \le m$, $k = r + 1 - m \ge x^5 - 2\sqrt{x}$; Problem 52.3.3.

§52.4.

52.4.1: $\theta_K^2 - \theta_K + 2 = 0$; $\bar{\delta} = 1 - \theta_K - c_0$.

52.4.2: Φ_τ restricts to one embedding of K; $(1 - 2\theta_K)^2 = -d$, and $N \nmid d$.

52.4.3: Statements (I), (II) of Section 50.4 and Problem 50.4.1, with Problem 52.4.2; the entries of $\operatorname{schur}_\beta(z)$ are homogeneous. Then $|\eta(N)| = N^{-1}$, $n_j \le n_m$, and the scale inequalities give $2n_m^2 \le b$.

52.4.4: The chain of double lifts is linear in the glued tensor; Problems 52.2.4 and 52.4.3; the values $\Psi(\mathfrak{t})$ are integral at N. For the last claim use Problem 33.5.4, $m \le 2\sqrt{x}$ and $(\log b)\log x_1 \ge 60$.

References

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